Does a single spin-1/2 pure quantum state have a counterpart in physical reality?

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We discuss that a single spin observable σ_x in a quantum state does not have a counterpart in physical reality. We consider whether a single spin-1/2 pure state has a counterpart in physical reality. It is an eigenvector of Pauli observable σ_z or an eigenvector of Pauli observable σ_x. We assume a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We assume also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. The value of transition probability $|\langle +_z|+_x\rangle|^2$ is 1/2. Surprisingly, the existence of a single classical probability space for the transition probability within the formalism of von Neumann's projective measurement does not coexist with the value of the transition probability $|\langle +_z|+_x\rangle|^2 = 1/2$. We have to give up the existence of such a classical probability space for the state $|+_z\rangle$ or for the state $|+_x\rangle$, as they define the transition probability. It turns out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality. A single spin-1/2 pure state (e.g., $|+\rangle\langle+|$) is a single one-dimensional projection operator. In other word, a single one-dimensional projector does not have a counterpart in physical reality, in general.

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I. INTRODUCTION

As a famous physical theory, the quantum theory (cf. [1–5]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

On the other hand, from the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [6], a hidden-variable interpretation of the quantum theory has been an attractive topic of research [2, 3]. There are two main approaches to study the hidden-variable interpretation of the quantum theory. One is the Bell-EPR theorem [7]. This theorem says that the quantum predictions violate the inequality following from the EPR-locality condition in the Hilbert space formalism of the quantum theory. The EPR-locality condition tells that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system.

The other is the no-hidden-variables theorem of Kochen and Specker (KS theorem) [8]. The original KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators. Kochen and Specker constructed [8] a hidden-variable theory in two-dimensional space formalism of the quantum theory within von Neumann's projective measurement theory. In general, the quantum theory does not accept the KS type of hidden-variable theory. The proof of the original KS theorem relies on intricate geometric argument. Greenberger, Horne, and Zeilinger discover [9, 10] the so-called GHZ theorem for four-partite GHZ state. And, the KS theorem becomes very simple form (see also Refs. [11–15]).

Mermin considers the Bell-EPR theorem in a multipartite state. He derives multipartite Bell inequality [16]. The quantum predictions by n-partite GHZ state violate the Bell-Mermin inequality by an amount that grows exponentially with n. And, several multipartite Bell inequalities are reported [17–25]. They also say that the quantum predictions violate local hidden-variable theories by an amount that grows exponentially with n.

As for the KS theorem, it is begun to research the validity of the KS theorem by using inequalities (see Refs. [26–29]). To find such inequalities to test the validity of the KS theorem is particularly useful for experimental investigation [30]. The KS theorem is related to the algebraic structure of a set of quantum operators. The KS theorem is independent of a quantum state under study. One of authors derives an inequality [29] as tests for the validity of the KS theorem. The quantum predictions violate the inequality when the system is in an uncorrelated state. An uncorrelated state is defined in Ref. [31]. The quantum predictions by n-partite uncorrelated state violate the inequality by an amount that grows exponentially with n.

Leggett-type nonlocal hidden-variable theory [32] is experimentally investigated [33–35]. The experiments report that the quantum theory does not accept Leggett-type nonlocal hidden-variable theory. These experiments are done in four-dimensional space (two parties) in order to study nonlocality of hidden-variable theories.

Many researches address non-classicality of observables. And non-classicality of quantum state itself is not investigated very much (however see [36]). Here we ask: Does a single spin-1/2 pure quantum state have a counterpart in physical reality? Surprisingly quantum state also does not have such a counterpart in physical reality, in general.

We see a single spin-1/2 pure state is used in quantum computation, quantum cryptography and so on. As for quantum computation, we are inputting non-classical information into quantum computer. As for quantum cryptography, we are exchanging non-classical information. Further, in various quantum information processing, we control quantum state by means of Pauli observables, which are non-classical. This manuscript gives new and important insight to quantum information theory, which can be implemented only by non-classical devices.

In this paper, first, we discuss that a single spin observable σ_x in a quantum state does not have a counterpart in physical reality. Next, we consider whether a single spin-1/2 pure state has a counterpart in physical reality. It is an eigenvector of Pauli observable σ_z or an eigenvector of Pauli observable σ_x. We assume a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We assume also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. The value of transition probability $|\langle +_z|+_x\rangle|^2$ is $1/2$. Surprisingly, the existence of a single classical probability space for the transition probability within the formalism of von Neumann's projective measurement does not coexist with the value of the transition probability $|\langle +_z|+_x\rangle|^2 = 1/2$. We have to give up the existence of such a classical probability space for the state $|+_z\rangle$ or for the state $|+_x\rangle$, as they define the transition probability. It turns out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality. A single spin-1/2 pure state (e.g., $|+\rangle\langle+|$) is a single one-dimensional projection operator. In other word, a single one-dimensional projector does not have a counterpart in physical reality, in general.

Therefore at this stage we are in the following situation.

1. A single spin observable in a quantum state does not have a counterpart in physical reality.

2. A single spin-1/2 pure quantum state does not have a counterpart in physical reality, in general.

We consider a single classical probability space. A single classical probability space is enough to investigate a hidden-variable interpretation of the quantum theory. We can consider the direct product of many spaces $(\Omega_1 \times \Omega_2 \times \Omega_3 \times \cdots)$ as a single space.

II. DOES PAULI OBSERVABLE IN A QUANTUM STATE HAVE A COUNTERPART IN PHYSICAL REALITY?

We assume an implementation of double-slit experiment. There is a detector just after each slit. Interference figure does not appear, and we do not consider such a pattern. Let (σ_z, σ_x) be Pauli vector. We assume that a source of spin-carrying particles emits them in a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z.

We consider a quantum expected value $\langle\sigma_x\rangle$ as

$$\langle\sigma_x\rangle = \langle +_z|\sigma_x|+_z\rangle = 0. \tag{1}$$

We introduce a hidden variables theory for the quantum expected value of Pauli observable σ_x. Then, the quantum expected value given in (1) can be

$$\langle\sigma_x\rangle = \int_{\omega\in\Omega} \mu(\mathrm{d}\omega) f(\omega). \tag{2}$$

The possible values of $f(\omega)$ are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is -1.

In what follows, we discuss that we cannot assign the truth value "1" for the proposition (2). Assume the proposition (2) is true. We have same proposition

$$\langle\sigma_x\rangle = \int_{\omega'\in\Omega} \mu(\mathrm{d}\omega') f(\omega'). \tag{3}$$

An important note here is that the value of the right-hand-side of (2) is equal to the value of the right-hand-side of (3) because we only change the label.

We derive a necessary condition for the quantum expected value given in (2). We derive the possible values of the product $\langle\sigma_x\rangle^2\delta(\omega-\omega')$ of the quantum expected value and a delta function. The quantum expected value is $\langle\sigma_x\rangle$

given in (2). We have

$$\begin{aligned}
&\langle \sigma_x \rangle^2 \delta(\omega - \omega') \\
&= \int_{\omega \in \Omega} \mu(\mathrm{d}\omega) f(\omega) \times \int_{\omega' \in \Omega} \mu(\mathrm{d}\omega') f(\omega') \delta(\omega - \omega') \\
&= \int_{\omega \in \Omega} \mu(\mathrm{d}\omega) \int_{\omega' \in \Omega} \mu(\mathrm{d}\omega') f(\omega) f(\omega') \delta(\omega - \omega') \\
&= \int_{\omega \in \Omega} \mu(\mathrm{d}\omega) (f(\omega))^2 \\
&= \int_{\omega \in \Omega} \mu(\mathrm{d}\omega) = 1.
\end{aligned} \tag{4}$$

Here we use the fact

$$(f(\omega))^2 = 1 \tag{5}$$

since the possible values of $f(\omega)$ are ± 1. Hence we derive the following proposition if we assign the truth value "1" for a hidden variables theory for Pauli observable σ_x

$$\langle \sigma_x \rangle^2 \delta(\omega - \omega') = 1. \tag{6}$$

We derive a necessary condition for the quantum expected value for the system in a pure spin-1/2 state $|+_z\rangle$ given in (1). We derive the possible value of the product

$$\langle \sigma_x \rangle \times \langle \sigma_x \rangle \times \delta(\omega - \omega') = \langle \sigma_x \rangle^2 \delta(\omega - \omega'). \tag{7}$$

$\delta(\omega - \omega')$ is the delta function. $\langle \sigma_x \rangle$ is the quantum expected value given in (1). We have the following proposition since $\langle \sigma_x \rangle = 0$

$$\langle \sigma_x \rangle^2 \delta(\omega - \omega') = 0. \tag{8}$$

We do not assign the truth value "1" for two propositions (6) and (8), simultaneously. We are in the contradiction. We have to give up a hidden variables theory for the expected value of Pauli observable σ_x. The measured observable σ_x in the state does not have a counterpart in physical reality.

III. DOES A SINGLE SPIN-1/2 PURE QUANTUM STATE HAVE A COUNTERPART IN PHYSICAL REALITY?

Let (σ_z, σ_x) be Pauli vector. We assume a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We assume also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. We consider a quantum expected value (transition probability) $|\langle +_z | +_x \rangle|^2$ as

$$|\langle +_z | +_x \rangle|^2 = 1/2. \tag{9}$$

We introduce a hidden variables theory for the quantum state $|+_z\rangle$ and for the quantum state $|+_x\rangle$. Then, the quantum expected value given in (9) can be

$$|\langle +_z | +_x \rangle|^2 = \int_{\omega \in \Omega} \mu(\mathrm{d}\omega) f(\omega). \tag{10}$$

The possible values of $f(\omega)$ are $+1, 0$ (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is 0.

In what follows, we discuss that we cannot assign the truth value "1" for the proposition (10). Assume the proposition (10) is true. We have same proposition

$$|\langle +_z | +_x \rangle|^2 = \int_{\omega' \in \Omega} \mu(\mathrm{d}\omega') f(\omega'). \tag{11}$$

An important note here is that the value of the right-hand-side of (10) is equal to the value of the right-hand-side of (11) because we only change the label.

We derive a necessary condition for the quantum expected value given in (10). We derive the possible values of the product $|\langle +_z|+_x\rangle|^4\delta(\omega-\omega')$ of the quantum expected value and a delta function. The quantum expected value is $|\langle +_z|+_x\rangle|^2$ given in (10). We have

$$\begin{aligned}
&|\langle +_z|+_x\rangle|^4\delta(\omega-\omega')\\
=&\int_{\omega\in\Omega}\mu(\mathrm{d}\omega)f(\omega)\times\int_{\omega'\in\Omega}\mu(\mathrm{d}\omega')f(\omega')\delta(\omega-\omega')\\
=&\int_{\omega\in\Omega}\mu(\mathrm{d}\omega)\int_{\omega'\in\Omega}\mu(\mathrm{d}\omega')f(\omega)f(\omega')\delta(\omega-\omega')\\
=&\int_{\omega\in\Omega}\mu(\mathrm{d}\omega)(f(\omega))^2\\
=&\int_{\omega\in\Omega}\mu(\mathrm{d}\omega)f(\omega)=1/2.
\end{aligned}\tag{12}$$

Here we use the fact

$$(f(\omega))^2=f(\omega)\tag{13}$$

since the possible values of $f(\omega)$ are $+1,0$. Hence we derive the following proposition if we assign the truth value "1" for a hidden variables theory for the quantum state $|+_z\rangle$ and for the quantum state $|+_x\rangle$

$$|\langle +_z|+_x\rangle|^4\delta(\omega-\omega')=1/2.\tag{14}$$

We derive a necessary condition for the quantum expected value given in (9). We derive the possible values of the product

$$|\langle +_z|+_x\rangle|^2\times|\langle +_z|+_x\rangle|^2\times\delta(\omega-\omega')=|\langle +_z|+_x\rangle|^4\delta(\omega-\omega').\tag{15}$$

$\delta(\omega-\omega')$ is the delta function. $|\langle +_z|+_x\rangle|^2$ is the quantum expected value given in (9). We have the following proposition since $|\langle +_z|+_x\rangle|=1/2$

$$|\langle +_z|+_x\rangle|^4\delta(\omega-\omega')=1/4\delta(\omega-\omega').\tag{16}$$

We do not assign the truth value "1" for two propositions (14) and (16), simultaneously. We are in the contradiction. We have to give up a hidden variables theory for the quantum state $|+_z\rangle$ or for the quantum state $|+_x\rangle$. It turns out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality.

IV. CONCLUSIONS

In conclusion, first, we have discussed that a single spin observable σ_x in a quantum state does not have a counterpart in physical reality. Next, we have considered whether a single spin-1/2 pure state has a counterpart in physical reality. It has been an eigenvector of Pauli observable σ_z or an eigenvector of Pauli observable σ_x. We have assumed a state $|+_z\rangle$, which can be described as an eigenvector of Pauli observable σ_z. We have assumed also a state $|+_x\rangle$, which can be described as an eigenvector of Pauli observable σ_x. The value of transition probability $|\langle +_z|+_x\rangle|^2$ has been 1/2. Surprisingly, the existence of a single classical probability space for the transition probability within the formalism of von Neumann's projective measurement does not have coexisted with the value of the transition probability $|\langle +_z|+_x\rangle|^2=1/2$. We have had to give up the existence of such a classical probability space for the state $|+_z\rangle$ or for the state $|+_x\rangle$, as they define the transition probability. It has turned out that the single spin-1/2 pure state $|+_z\rangle$ or the single spin-1/2 pure state $|+_x\rangle$ does not have counterparts in physical reality. A single spin-1/2 pure state (e.g., $|+\rangle\langle+|$) has been a single one-dimensional projection operator. In other word, a single one-dimensional projector does not have had a counterpart in physical reality, in general.

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