Quantum measurement theory improves the no-cloning theorem

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(Dated: July 24, 2013)

We improve the no-cloning theorem that relies on the property of the quantum theory. Usually, the no-cloning theorem allows for a cloning two orthogonal quantum states, simultaneously. Here we take into account specific quantum measurement theory. We result in the fact that we cannot allow for a cloning two orthogonal quantum states, simultaneously. Especially, we systematically describe our assertion based on more mathematical analysis using raw data.

PACS numbers: 03.65.Ta, 03.67.-a, 03.65.Ca

I. INTRODUCTION

As a famous physical theory, the quantum theory (cf. [1–5]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

The no-cloning theorem is a result of quantum mechanics that forbids the creation of identical copies of an arbitrary unknown quantum state. It was stated by Wootters and Zurek [6] and Dieks [7] in 1982, and has profound implications in quantum computing and related fields.

The state of one system can be entangled with the state of another system. For instance, one can use the Controlled NOT gate and the Walsh-Hadamard gate to entangle two qubits. This is not cloning. No well-defined state can be attributed to a subsystem of an entangled state. Cloning is a process whose result is a separable state with identical factors. According to Asher Peres [8] and David Kaiser [9], the publication of the no-cloning theorem was prompted by a proposal of Nick Herbert [10] for a superluminal communication device using quantum entanglement.

A literature concerning quantum cloning topic can be seen in Ref. [11].

In this paper, we improve the no-cloning theorem that relies on the property of the quantum theory. Usually, the no-cloning theorem allows for a cloning two orthogonal quantum states, simultaneously. Here we take into account specific quantum measurement theory. We result in the fact that we cannot allow for a cloning two orthogonal quantum states, simultaneously. Especially, we systematically describe our assertion based on more mathematical analysis using raw data.

We review the no-cloning theorem as follow:

$$U|\phi\rangle_A|e\rangle_B = |\phi\rangle_A|\phi\rangle_B. \tag{1}$$

U is the time evolution operator. Alice has a quantum state $|\phi\rangle_A$. Bob has a quantum state $|e\rangle_B$. Bob's state changes into $|\phi\rangle_B$ by using the time evolution operator. Thereby Alice's state is cloned into Bob's state. Let us consider inner product. Thus,

$$\begin{aligned}\langle e|_B\langle\phi|_A|\psi\rangle_A|e\rangle_B &= \langle e|_B\langle\phi|_A U^\dagger U|\psi\rangle_A|e\rangle_B \\ &= \langle\phi|_B\langle\phi|_A|\psi\rangle_A|\psi\rangle_B.\end{aligned} \tag{2}$$

Thus,

$$\langle\phi|\psi\rangle_A = \langle\phi|\psi\rangle_A\langle\phi|\psi\rangle_B. \tag{3}$$

By omitting subscript A and B, we have

$$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2. \tag{4}$$

We derive the following proposition:

$$\langle\phi|\psi\rangle^2 = 0 \vee \langle\phi|\psi\rangle^2 = 1. \tag{5}$$

Therefore the no-cloning theorem allows for a cloning two orthogonal quantum states or for a cloning two identical quantum states, simultaneously. However, we cannot assume

$$\langle\phi|\psi\rangle^2 = 0 \tag{6}$$

when we take into account specific quantum measurement theory. Therefore new no-cloning theorem does not allow for a cloning two orthogonal quantum states, simultaneously.

II. NEW TYPE OF NO-CLONING THEOREM

A. Orthogonal case

We consider a quantum expected value as

$$\langle\phi|\psi\rangle^2 = 0. \tag{7}$$

The above quantum expected value is zero if $|\phi\rangle$ and $|\psi\rangle$ are orthogonal.

We derive a necessary condition for the quantum expected value given in (7). We derive the following proposition

$$\langle\phi|\psi\rangle^2 = 0. \tag{8}$$

B. Specific quantum measurement theory forbids orthogonal case

On the other hand, a mean value E satisfies specific quantum measurement theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m} \tag{9}$$

where l denotes a label and r is the result of specific quantum measurement. We assume the value of r is ± 1.

In what follows, we show that we cannot assign the truth value "1" for the proposition (8).

Assume the quantum mean value given in (9) admits the quantum measurement theory. One has the following proposition concerning the quantum measurement theory

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m}. \tag{10}$$

We can assume as follows by Strong Law of Large Numbers,

$$\langle\phi|\psi\rangle^2(+\infty) = \langle\phi|\psi\rangle^2. \tag{11}$$

Assume the proposition (10) is true. By changing the label l into l' and by changing the label m into m', we have same quantum mean value as follows

$$\langle\phi|\psi\rangle^2(m') = \frac{\sum_{l'=1}^{m'} r_{l'}(\langle\phi|\psi\rangle^2)}{m'}. \tag{12}$$

An important note here is that the value of the right-hand-side of (10) is equal to the value of the right-hand-side of (12) because we only change the labels. We have

$$\begin{aligned}
&\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m') \\
&= \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m} \times \frac{\sum_{l'=1}^{m'} r_{l'}(\langle\phi|\psi\rangle^2)}{m'} \\
&= \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{\sum_{l=1}^{m}} \times \frac{\sum_{l'=1}^{m'} r_{l'}(\langle\phi|\psi\rangle^2)}{\sum_{l'=1}^{m'}} \times \frac{\delta_{ll'}}{\delta_{ll'}} \\
&= \frac{\sum_{l=1}^{m}}{m} \cdot (r_l(\langle\phi|\psi\rangle^2))^2 \\
&= \frac{\sum_{l=1}^{m}}{m} = 1.
\end{aligned} \tag{13}$$

Here $\delta_{ll'}$ is a delta function. We use the following fact

$$(r_l(\langle\phi|\psi\rangle^2))^2 = 1 \tag{14}$$

and

$$\frac{\delta_{ll'}}{\delta_{ll'}} = 1. \tag{15}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that the quantum measurement theory is true, that is

$$\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m') = 1. \tag{16}$$

From Strong Law of Large Numbers, we have

$$\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2 = 1. \tag{17}$$

Hence we derive the following proposition concerning the quantum measurement theory

$$\langle\phi|\psi\rangle^4 = 1. \tag{18}$$

Thus,

$$\langle\phi|\psi\rangle^2 = 1. \tag{19}$$

This implies that we cannot assume

$$\langle\phi|\psi\rangle^2 = 0 \tag{20}$$

and we can assume only the following case

$$\langle\phi|\psi\rangle = 1 \wedge \langle\phi|\psi\rangle^2 = 1. \tag{21}$$

This implies we can assume only the following case

$$|\phi\rangle = |\psi\rangle. \tag{22}$$

Therefore new no-cloning theorem allows only for a cloning one kind quantum state, in this case.

III. CONCLUSIONS

In conclusion, we have improve the no-cloning theorem that relies on the property of the quantum theory. Usually, the no-cloning theorem has allowed for a cloning two orthogonal quantum states, simultaneously. Here we have taken into account specific quantum measurement theory. We have resulted in the fact that we cannot allow for a cloning two orthogonal quantum states, simultaneously. Especially, we have systematically described our assertion based on more mathematical analysis using raw data.

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