PATH-DEPENDENT FUNCTIONS

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Various path-dependent functions are described in a uniform manner by means of a series expansion of Taylor type. For this, "path integrals" and "path tensors" are introduced. They are systems of multicomponent quantities whose values are defined for an arbitrary path in a coordinated region of space in such a way that they carry sufficient information about the shape of the path. These constructions are regarded as elementary path-dependent functions and are used instead of the power monomials of an ordinary Taylor series. The coefficients of such expansions are interpreted as partial derivatives, which depend on the order of differentiation, or as nonstandard covariant derivatives, called two-point derivatives. Examples of path-dependent functions are given. We consider the curvature tensor of a space whose geometrical properties are specified by a translator of parallel transport of general type (nontransitive). A covariant operation leading to "extension" of tensor fields is described.

1. Introduction

Path-dependent functions are widely used in physics. It is true that they are frequently called functionals, but we shall use the word "function" because it is simpler. A good example of a path-dependent function is the operator (or matrix) of parallel transport from one point x^ν of a curved space to another point* $x^{\nu'}$. We call this operator the translator of parallel transport and usually denote it by the Greek letter Θ. The equation $V^{\alpha'}=\Theta^{\alpha'}{}_\beta \cdot V^\beta$ expresses the fact that by applying the translator $\Theta^{\alpha'}{}_\beta$ to the vector V^β at the point x we obtain the vector $V^{\alpha'}$ at the point x'. As argument of the translator we shall sometimes indicate the points that are the beginning and end of the path: $\Theta^{\alpha'}{}_\beta(x', x)$. The result of this operation depends on the transport path, and $\Theta^{\alpha'}{}_\beta$ can be determined in some manner or other in a space with given affine connection $\Gamma_{\beta\nu}{}^\alpha(x)$. Recently, Einstein's equation of the general theory of relativity for empty space has been expressed directly as an equation for this translator [4].

Another example of the use of path-dependent functions is given in [5], in which an extended group of coordinate transformations is considered with a view to the construction of a unified quantum field theory. The idea is that in place of the ordinary one-to-one coordinate transformation $x \to x'$ one introduces a multiply valued transformation under which the image of the point x consists of different points x' depending on the path P among those that lead to the point x which is taken into consideration. Thus, x' is a function of the path P: $x'\{P\}$ (notation of [5]).

In the well-known paper [6], a path-dependent function is used as a field variable in quantum electrodynamics. Similarly, in [7] a path-dependent gauge phase factor in the theory of gauge fields is considered.

A very simple example of a path-dependent function is the work in a nonpotential force field F_ν, which can be represented by the integral

$$W(x', x)= \int_x^{x'} F_{\bar\nu} \cdot d\bar{x}^{\bar\nu}. \tag{1.1}$$

We restrict ourselves to these examples of path-dependent functions, fully aware of the extreme incompleteness of our list, and point out that at the present time we lack a unified constructive definition of

*ν, ν' = 1, 2, ..., N; N is the dimension of space. We use the system of notation employed in the books [1, 2, 3] and elsewhere according to which primes and other symbols used to distinguish different points are transferred to the coordinate index, i.e., we write $x^{\nu'}$ instead of x'^{ν} to denote a point or $\varphi_{\nu'}$ instead of $\varphi_\nu(x')$ to denote a covector. The differentiation is denoted similarly: $\partial_{\nu'}$ instead of $\partial/\partial x'^{\nu}$.

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the various path-dependent functions such as is provided, for example, by the Taylor series (or Fourier expansion) for ordinary functions. The aim of the present paper is to fill this gap by creating a construction which we may naturally call Taylor expansion for path-dependent functions. In accordance with this, we shall call the functions considered here (real) analytic path-dependent functions.*

The Taylor expansion for two-point functions that do not depend on the path can be readily constructed by analogy with the ordinary Taylor expansion using partial derivatives at the point of expansion (see Eq. (2.3) below). Such an expansion, but using covariant derivatives, was given in [8]. It is very important that in [8] the path-independence of the expanded functions made it possible to use only the symmetric parts of the covariant derivatives. The information contained in the nonsymmetric parts was superfluous. This last circumstance suggests that the use of all the information that may be contained in the nonsymmetric derivatives (covariant and, as we shall see, partial) makes it possible to construct an expansion of path-dependent functions. The present paper is devoted to the realization of this idea. The material of the paper was partly published in [9, 10, 11]. It is presented in detail in [12].

2. Path-Independent Functions

As is well known, an analytic (real) function $\Phi(x')$ defined at the points $x^{\nu'}$ of an "ordinary" N-dimensional space [1] can be defined by the set of values of its partial derivatives:

$$\varphi_{,\nu_1\ldots\nu_n} \overset{\text{def}}{=} [\partial_{\nu_n'\ldots\nu_1'}\Phi(x')]_{x'=x} = [\Phi_{,\nu_1'\ldots\nu_n'}(x')]_{x'=x}$$

at a certain point x if they are used as coefficients of the power series

$$\Phi(x') = \sum_{n=0}^{\infty} \varphi_{,\nu_1\ldots\nu_n}\cdot(x'-x)^{\nu_1}\cdot\ldots\cdot(x'-x)^{\nu_n}\cdot\frac{1}{n!}. \tag{2.1}$$

We have here introduced notation for the difference between the coordinates of the points $x^{\nu'}$ and x^{ν}: $(x'-x)^{\nu} = x^{\nu'}\cdot\delta_{\nu'}{}^{\nu} - x^{\nu}$. Note also that in the sum (2.1), as in all such expressions in this paper, the indices ν_n are annihilated for $n = 0$, this happening moreover frequently together with the radical symbol (this will be clear from the context). For example, the zeroth term of the sum (2.1) is simply $\varphi(x)$. Because of the (unavoidable) complexity of the employed notation, multiplication in the formulas is always indicated by a raised dot.

If we change the point x of the expansion (2.1) but require that the value of $\Phi(x')$ should remain the same, i.e., require the derivatives of the series (2.1) at the point x to be equal to zero, we can readily obtain a relationship between the partial derivatives $\varphi_{,\nu_1\ldots\nu_n}(x)$ as functions of x:

$$\varphi_{,\nu_1\ldots\nu_n}(x) = \partial_{\nu_n\ldots\nu_1}\varphi(x), \quad \varphi(x) = \Phi(x). \tag{2.2}$$

We point out that this (ordinary) relation is a consequence of (2.1) being a single-point function. One can give up this requirement, and then the series (2.1) determines a certain two-point function:

$$\Phi(x', x) = \sum_{n=0}^{\infty} \varphi_{,\nu_1\ldots\nu_n *}(x)\cdot(x'-x)^{\nu_1}\cdot\ldots\cdot(x'-x)^{\nu_n}\cdot\frac{1}{n!}. \tag{2.3}$$

The coefficients $\varphi_{,\nu_1\ldots\nu_n *}(x)$ of the series (2.3), which are functions not related to each other, will be called the functions of the determining system of the two-point function $\Phi(x', x)$ when it is expanded at the point of the right-hand argument, this last circumstance being indicated by the placing to the right of the indices $\nu_1, \ldots, \nu_n$ of the asterisk; for now it is possible to expand the same two-point function at the point of the left-hand argument or, putting it briefly, at the left point:

$$\Phi(x', x) = \sum_{n=0}^{\infty} \varphi_{*,\nu_1'\ldots\nu_n'}(x')\cdot(x-x')^{\nu_1'}\cdot\ldots\cdot(x-x')^{\nu_n'}\cdot\frac{1}{n!}, \tag{2.4}$$

a different determining system being now used. We shall always denote a two-point function by a capital Greek letter, and the functions of its determining systems by corresponding lower-case letters.

*The path length l, for example, does not belong to functions of this kind because its increment obviously cannot be represented in the form $dl = l_{\nu}\cdot dx^{\nu}$, since the path length increases for any direction of the infinitesimal displacement dx^{ν} of the end of the path.

Differentiating multiply the series (2.3) at the point x', and the series (2.4) at the point x with the subsequent equating $x' = x$, we find that the determining functions are "coincidence limits" of partial derivatives:

$$\varphi_{,\nu_1\ldots\nu_n *}(x) = [\partial_{\nu'_n\ldots\nu'_1}\Phi(x', x)]_{x'=x}, \tag{2.5}$$

$$\varphi_{*,\nu_1\ldots\nu_n}(x) = [\partial_{\nu_n\ldots\nu_1}\Phi(x', x)]_{x'=x}. \tag{2.6}$$

3. Path Integrals

The series (2.3) and (2.4) determine the function $\Phi(x', x)$ by expanding it with respect to the elementary two-point functions

$$(x'-x)^{\nu_1}\cdot\ldots\cdot(x'-x)^{\nu_n}\cdot\frac{1}{n!}, \tag{3.1}$$

which depend only on the points x' and x and do not contain the concept of the path between these points. To go over to path-dependent functions, we replace the functions (3.1) by path integrals $(x', x)^{\nu_1\ldots\nu_n}$, which are defined by the recursive formula

$$(x', x)^{\nu_1\ldots\nu_n} = \int_x^{x'} \delta_{\bar{\nu}}^{\nu_1}\cdot(\bar{x}, x)^{\nu_2\ldots\nu_n}\cdot d\bar{x}^{\bar{\nu}}, \tag{3.2}$$

in which the integration is over the path which serves as the argument of the function. By $x^{\bar{\nu}}$ we here denote the intermediate "running" point of the path, while x and x' are the initial and final points of the path. Formula (3.2), which relates path integrals of neighboring orders, arises naturally by analogy with the relation

$$\partial_{\nu'}\left[(x'-x)^{\nu_1}\cdot\ldots\cdot(x'-x)^{\nu_n}\cdot\frac{1}{n!}\right] = \delta_{\nu}^{\nu_1}\cdot(x'-x)^{\nu_2}\cdot\ldots\cdot(x'-x)^{\nu_n}\cdot\frac{1}{(n-1)!}$$

which the functions (3.1) satisfy. But path integrals are nonsymmetric with respect to the indices $\nu_1, \ldots, \nu_n$, and by virtue of this the countable set of them contains all information about the oriented path between the points x and x'. The orientation is specified by the direction of integration.

The definition (3.2) does not impose any restrictions on the smoothness of the path and does not depend on the possible parametrization of the path $x^{\bar{\nu}}(t)$. Moreover, this definition can also be used when the path consists of a finite ordered set of disconnected continuous paths since the definition contains only the differential $dx^{\bar{\nu}}$. It is true that integrals of such a path are identical to the path integrals obtained by displacing all the paths in such a way that the beginning of a subsequent path coincides with the end of the preceding one. This shift is to be understood in the coordinate sense as a change in the coordinates of all points of the path by the same numbers, $dx^{\bar{\nu}}$ thereby remaining unchanged. In exactly the same way, the path integrals are not changed if at any point we add to the path an "appendix" traversed first in one direction and then immediately afterwards in the other. Thus, in this paper a path is an element of the group of paths introduced by Menskii (see, for example, the book [13]), and (3.2) defines a mapping of this group onto a group of sets of a countable number of multicomponent quantities called path integrals. An explicit expression for the group operation can be readily written down. If the path (x', x) consists of the parts $(x', \bar{x})$ and $(\bar{x}, x)$, then

$$(x', x)^{\nu_1\cdots\nu_n} = \sum_{m=0}^{n}(x', \bar{x})^{\bar{\mu}_1\cdots\bar{\mu}_m}\cdot\delta_{\bar{\mu}_1}^{\nu_1}\cdot\ldots\cdot\delta_{\bar{\mu}_m}^{\nu_m}\cdot(\bar{x}, x)^{\nu_{m+1}\cdots\nu_n}. \tag{3.3}$$

However, we defer the proof of this formula to Sec. 4. With this we conclude the discussion of the group properties of path integrals. We merely mention that the equivalence relation that, through displacement, combines disconnected (and "free") paths into elements of a group of paths depends on the coordinate system employed.

Multiple differentiation of path integrals with respect to the coordinates of the ends leads to derivatives that are nonsymmetric with respect to the indices, dependent on the order of differentiation:

$$\partial_{\mu_m'\cdots\mu_1'}(x', x)^{\nu_1\cdots\nu_n} = \delta_{\mu_1'}^{\nu_1}\cdots\delta_{\mu_m'}^{\nu_m}\cdot(x', x)^{\nu_{m+1}\cdots\nu_n}, \tag{3.4}$$

$$\partial_{\mu_m\cdots\mu_1}(x', x)^{\nu_1\cdots\nu_n} = (-1)^m\cdot(x', x)^{\nu_1\cdots\nu_{n-m}}\cdot\delta_{\mu_m}^{\nu_{n-m+1}}\cdot\ldots\cdot\delta_{\mu_1}^{\nu_n}. \tag{3.5}$$

Equation (3.4) is obtained directly from (3.2), and (3.5) is proved by induction. In both cases, we have in mind differentiation with respect to the coordinates of the ends of the path, the ends acquiring an infinitesimal displacement while all of the remainder of the path remains unchanged. It is in this sense that the derivative of a path-dependent function is understood in [5] and in the earlier studies [14,15].

4. Path-Dependent Functions. Noncovariant Expansion

The use of the path integrals (3.2) in (2.3) and (2.4) leads to the path-dependent functions

$$\Phi(x'\, x)=\sum_{n=0}^{\infty}\varphi_{,\nu_1\ldots\nu_n *}(x)\cdot(x',\, x)^{\nu_1\ldots\nu_n}=\sum_{n=0}^{\infty}\varphi_{*,\nu_1'\ldots\nu_n'}(x')\cdot(x,\, x')^{\nu_1'\ldots\nu_n'}. \tag{4.1}$$

Here, $\varphi_{,\nu_1\ldots\nu_n *}$ and $\varphi_{*,\nu_1'\ldots\nu_n'}$ are now nonsymmetric functions of the determining systems for which (2.5) and (2.6) remain valid by virtue of (3.4). Thus, the partial derivatives of a path-dependent function depend on the order of differentiation, like the derivatives of path integrals.

If the functions of the determining system are symmetric, $\varphi_{,\nu_1\ldots\nu_n *}=\varphi_{,(\nu_1\ldots\nu_n)*}$, then the function (4.1) does not depend on the path, like (2.3) and (2.4), because

$$(x',x)^{(\nu_1\ldots\nu_n)}=(x',x)^{\nu_1}\cdot\ldots\cdot(x',x)^{\nu_n}\cdot\frac{1}{n!}. \tag{4.2}$$

Thus, when the nonsymmetric functions of a determining system are made symmetric in a certain coordinate system, the path-dependent function is transformed into a path-independent two-point function equal to the values of the original functions for paths that are straight in the coordinate sense, i.e., have the equation $\bar{x} = x + kt$. Path-independent functions will be referred to by the abbreviation of stable functions.

If we consider a tensor path-dependent function, then the functions of the systems that determine it in the expansions

$$\Phi^{a'}{}_b(x',\, x)=\delta_a^{a'}\cdot\sum_{n=0}^{\infty}\varphi^a{}_{,\nu_1\ldots\nu_n * b}(x)\cdot(x',\, x)^{\nu_1\ldots\nu_n}= \tag{4.3}$$

$$=\delta_b^{b'}\cdot\sum_{n=0}^{\infty}\varphi^{a'}_{*b',\nu_1'\ldots\nu_n'}(x')\cdot(x,\, x')^{\nu_1'\ldots\nu_n'} \tag{4.4}$$

have special transformation properties under changes of the coordinates $x = x(\bar{x})$. They are not tensors, since the path integrals are nontensorial. In this paper, we shall not give the corresponding formulas. We note that the Kronecker delta with indices referring to different points of space is nontensorial. This symbol is widely used below. Naturally, the functions of the determining systems for the expansions at the left and right points of a two-point function are connected by certain relations. They can be obtained by multiple differentiation of the identity (4.3), (4.4) with allowance for (3.4) and (3.5) and with the subsequent equating $x' = x$. The results are given in the Appendix (Eq. (9.1)).

The function (4.3), which has the Latin indices a' and b, is said to be tensorial only for definiteness. In reality, the significance of these indices, i.e., the representation of the group of coordinate transformations, attached to the geometrical quantity to which they are appended can be arbitrary.

There exists an important class of path-dependent functions containing, in particular, the translator of parallel transport and the path-dependent functions of [6,7]. They satisfy equations of the type

$$\Theta^{\alpha'}{}_{\beta}=\Theta^{\alpha'}{}_{\bar{\gamma}}\cdot\Theta^{\bar{\gamma}}{}_{\beta} \tag{4.5}$$

whenever the path (x', x) consists of the paths $(x', \bar{x})$ and $(\bar{x}, x)$. We shall say that such functions are transitive. Differentiating Eq. (4.5) n times at the point x' with the subsequent equating $\bar{x} = x'$, we obtain (cf. the notation from (8.4))

$$\partial_{\nu_n'\ldots\nu_1'}\Theta^{a'}{}_b=\theta^{a'}{}_{,\nu_1'\ldots\nu_n' * c'}\cdot\Theta^{c'}{}_b. \tag{4.6}$$

Further, differentiating (4.6) at the point x' with subsequent equating $x' = x$, we obtain the recursion relation

$$\theta^a{}_{,\nu_1\ldots\nu_{n+1} * b}=\partial_{\nu_{n+1}}\theta^a{}_{,\nu_1\ldots\nu_n * b}+\theta^a{}_{,\nu_1\ldots\nu_n * c}\cdot\theta^c{}_{,\nu_{n+1} * b}, \tag{4.7}$$

which shows that transitive path-dependent functions are determined by the specification of only single-symbol functions of the determining system, in contrast to path-dependent functions of general type, for which all the functions of the determining systems must be specified independently. For the translator of parallel transport, single-symbol determining functions are the connection coefficients:

$$\theta^{\alpha}{}_{.\,\nu*\beta} = -\Gamma^{\alpha}_{\beta\nu}, \tag{4.8}$$

as can be seen by comparing the definition

$$V^{\alpha'} = \delta_{\alpha}{}^{\alpha'} \cdot (\delta_{\beta}{}^{\alpha} - \Gamma_{\beta\nu}{}^{\alpha} \cdot dx^{\nu}) \cdot V^{\beta}$$

of parallel transport of a vector with the first terms in the expansion of the translator $\Theta^{\alpha'}{}_{\beta}$ in accordance with (4.3).

Thus, specification of the affine connection in space is equivalent (in accordance with (4.7)) to the specification of a transitive translator. Note however that the geometrical properties of space can be determined directly by a translator of general type, which simply specifies the operation of transport of the geometrical entity along any path, the information about the manner in which such transport must be made being so extensive that it cannot be included in the field of the coefficients $\Gamma_{\beta\nu}{}^{\alpha}$.

Determining the derivatives of path integrals by means of the expressions (3.4) and (3.5), we thereby determined the derivatives of the path-dependent function $\Phi(x', x)$ at the point x' or x. In their turn, these derivatives make it possible to write down an expansion of the function $\Phi(x', x)$ at an intermediate point $\bar{x}$ on the path (x', x). For example,

$$\Phi(x', x) = \sum_{n=0}^{\infty} \partial_{\bar{\nu}_n \ldots \bar{\nu}_1} \Phi(\bar{x}, x) \cdot (x', \bar{x})^{\bar{\nu}_1 \ldots \bar{\nu}_n}. \tag{4.9}$$

The validity of this formula is established by the identity of all the derivatives of the left- and right-hand sides at the point $\bar{x}$. Applying (4.9) to path integrals, $\Phi(x', x) = (x', x)^{\mu_1 \ldots \mu_m}$, and taking into account the relations (3.4), we obtain the expression (3.3).

5. Path Tensors. Covariant Expansion of Path-Dependent Functions

To obtain the tensor expansion of path-dependent functions, it is necessary to replace the noncovariant path integrals (3.2) by path tensors $X^{\nu_1 \ldots \nu_n}(x', x)$, for the construction of which the Kronecker delta in formula (3.2) is replaced by a certain translator,* say $X_{\nu'}{}^{\mu}$:

$$X^{\nu_1 \ldots \nu_n}(x', x) = \int_{x}^{x'} X_{\bar{\nu}}{}^{\nu_1} \cdot X^{\nu_2 \ldots \nu_n}(\bar{x}, x) \cdot dx^{\bar{\nu}}. \tag{5.1}$$

As the root letter for denoting path tensors and the translator which generates them we have chosen the Greek letter X, this being the one most similar to the Latin letter x associated with the components of a radius vector. Using the path tensors (5.1), we can specify a tensor expansion of the path-dependent function $\Phi^{\alpha'}{}_{b}$ by means of one further translator, denoted by $\Psi^{\alpha'}{}_{b}$, instead of the Kronecker delta of formula (4.3):

$$\Phi^{\alpha'}{}_{b} = \Psi^{\alpha'}{}_{a} \cdot \sum_{n=0}^{\infty} \varphi^{a}{}_{|\nu_1 \ldots \nu_n * b}(x) \cdot X^{\nu_1 \ldots \nu_n}(x', x). \tag{5.2}$$

The coefficients $\varphi^{a}{}_{|\nu_1 \ldots \nu_n * b}$ of this expansion are tensors by definition. This is indicated by the fact that the indices $\nu_1 \ldots \nu_n$ are separated by a vertical bar rather than a comma. We shall say that these coefficients, which are functions of the expansion point, form a tensorial determining system of the path-dependent function $\Phi^{\alpha'}{}_{b}$.

*We shall give the name translator to a function that takes the value of the Kronecker delta when the path is contracted to a point:

$$X_{\nu'}{}^{\mu} \to \delta_{\nu}{}^{\mu}, \quad \Theta^{\alpha'}{}_{\beta} \to \delta_{\beta}{}^{\alpha} \quad \text{as} \quad (x', x) \to 0.$$

6. Two-Point Covariant Derivative

The standard covariant derivative of a function $\Phi^{a'}{}_{b}$ based on the translator $\Psi_{b}{}^{a'}$ can be expressed in the form

$$\overset{\Psi}{\nabla}_{\nu'}\Phi^{a'}{}_{b}=[\partial_{\nu'}(\Psi^{-1\bar{a}}{}_{c'}\cdot\Phi^{c'}{}_{b})]_{\bar{x}=x'} \tag{6.1}$$

(Ψ^{-1} denotes the inverse translator: $\Psi_{b'}{}^{-1a}\cdot\Psi_{c}{}^{b'}=\delta_{c}{}^{a}$, which, in fact, does not have any importance here). Differentiating in (6.1), we readily arrive at the usual expression for the standard covariant derivative:

$$\overset{\Psi}{\nabla}_{\nu'}\Phi^{a'}{}_{b}=\psi^{-1a'}{}_{\bullet c',\nu'}\cdot\Phi^{c'}{}_{b}+\partial_{\nu'}\Phi^{a'}{}_{b}$$

(we can denote $\psi^{-1a'}{}_{\bullet c',\nu'}=\Gamma^{a'}{}_{c'\nu'}$).

However, in this paper we shall use a different definition for the covariant derivative. It can be shown that the tensors $\varphi^{a}{}_{|\nu_1\ldots\nu_n\bullet b}$ in (5.2) can be given the meaning of coincidence limits of covariant derivatives if we introduce for the covariant derivatives the nonstandard definition

$$\underset{\bar{x}}{\overset{\Psi}{\nabla}}_{\nu'}\Phi^{a'}{}_{b}=\Psi^{\bar{a}a'}{}_{a}\cdot\partial_{\nu'}(\Psi^{-1\bar{a}}{}_{c'}\cdot\Phi^{c'}{}_{b}). \tag{6.2}$$

We shall call the construction (6.2) the two-point covariant derivative of the function $\Phi^{a'}{}_{b}$ at the point x' based on the translator Ψ with reference point $\bar{x}$. It is readily verified that such a derivative (in contrast to the standard derivative) vanishes when $\Phi = \Psi$ and when the reference point coincides with the point of the right argument, $\bar{x} = x$:

$$\underset{x}{\overset{\Psi}{\nabla}}_{\nu'}\Psi^{a'}{}_{b}=0.$$

In addition, for the same coincidence we have the following formula, which is analogous to the nontensorial formula (3.4):

$$\underset{x}{\overset{X}{\nabla}}_{\mu_m'\ldots\mu_1'}X^{\nu_1\ldots\nu_n}(x',x)=X_{\mu_1'}{}^{\nu_1}\cdot\ldots\cdot X_{\mu_m'}{}^{\nu_m}\cdot X^{\nu_{m+1}\ldots\nu_n}(x',x),$$

These properties ensure the validity of (5.2) if we set

$$\varphi^{a}{}_{|\nu_1\ldots\nu_n\bullet b}=\Big[\underset{x}{\overset{X\Psi}{\nabla}}_{\nu_n'\ldots\nu_2'}\underset{x}{\overset{\Psi}{\nabla}}_{\nu_1'}\Phi^{a'}{}_{b}\Big]_{x'=x}.$$

It is borne in mind that the derivatives here are based on the translator Ψ with respect to the index a' and on the translator X with respect to the indices $\nu_1',\ldots,\nu_{n-1}'$.

The definition (6.2) can be made standard if the reference point coincides with the differentiation point. However, the higher two-point derivatives cannot be made standard even when there is such coincidence, since in the case of multiple two-point differentiation the reference points of the foregoing differentiations are assumed to be constants.

For the indicated coincidence of the differentiation point with the reference point it is possible to have a simplified notation of the two-point derivative by means of the addition of subscripts, which are separated, not by a semicolon, as in the case of standard differentiation, but by a vertical bar:

$$\underset{x'}{\overset{\Psi}{\nabla}}_{\nu_n'\ldots\nu_1'}\Phi^{a'}{}_{b}\overset{\text{def}}{=}\Phi^{a'}{}_{|\nu_1'\ldots\nu_n'b}.$$

It is this that explains the appearance of the bar in the notation of the tensorial determining system in (5.2).

The two-point derivative does not depend on the position of the reference point and in any order is identical to the standard covariant derivative if it is based on a transitive translator. This can be readily verified and evidently explains why the two-point derivative has not hitherto been introduced into tensor analysis.

7. Loop Functions

We call a closed path a loop. Because of the coincidence of the beginning and the end, such a path is conveniently denoted by a single letter, say l. Accordingly, we denote loop integrals by $(l, x)^{\nu_1\ldots\nu_n}$. As before, the orientation of the loop is specified by the direction of the integration in accordance with (3.2). Single-symbol loop integrals $(l, x)^{\nu_1}$ and, by virtue of (4.2), the symmetric parts of all loop integrals are equal to zero: $(l, x)^{(\nu_1\ldots\nu_n)}=0$. The antisymmetric part of the two-symbol integral $(l, x)^{[\nu_1\nu_2]}$ of an infinitesimal loop is a bivector corresponding to the two-dimensional element with internal orientation bounded by this loop, the orientation of the element being opposite to that of the loop. This can be seen from the fact that $(l, x)^{[\nu_1\nu_2]}$ corresponds in accordance with (3.2) to the sum of triangular areas $dx^{\bar\nu}\cdot\delta_{\bar\nu}^{[\nu_1}\cdot(\bar x, x)^{\nu_2]}$. Therefore, if we denote by $df^{\nu_1\nu_2}$ the bivector corresponding to the two-dimensional element oriented in accordance with the loop we shall have $df^{\nu_1\nu_2}=-(l, x)^{[\nu_1\nu_2]}$.

As can be seen from (4.3), for the translator $\Phi^{a'}{}_b$ the expression

$$dA^a=\varphi^a{}_{,[\nu_1\nu_2]*b}\cdot(l, x)^{\nu_1\nu_2}\cdot A^b=-\varphi^a{}_{,[\nu_1\nu_2]*b}\cdot df^{\nu_1\nu_2}\cdot A^b,$$

written down for an infinitesimal loop, gives the change of a certain geometrical entity A^b when it is transported around the loop by means of $\Phi^{a'}{}_b$ in the direction of the loop orientation. Therefore, independently of the transitivity of the translator $\Phi^{a'}{}_b$ it is natural to call $\varphi^a{}_{,[\nu_1\nu_2]*b}$ the curvature tensor of the space whose geometrical properties are specified by the translator $\Phi^{a'}{}_b$:

$$\varphi^a{}_{,[\nu_1\nu_2]*b}={}^1\!/_2 R^a_{b\nu_1\nu_2}.$$

And it is only if the translator $\Phi^{a'}{}_b$ is transitive that formulas (4.7) lead to the ordinary expression for the curvature tensor in terms of single-symbol determining functions of the type (4.8).

8. Examples of Two-Point Functions

The simplest example of a path-dependent function is the work (1.1). Its determining system can be readily found by multiple differentiation*:

$$[\partial_{\nu_n'\ldots\nu_1'}W(x', x)]_{x'=x}\overset{\text{def}}{=}w_{,\nu_1\ldots\nu_n*}=\partial_{\nu_n\ldots\nu_2}F_{\nu_1}. \tag{8.1}$$

The function $W(x', x)$ obviously satisfies the relation $W(x', x) = W(x', \bar x) + W(\bar x, x)$ for any path (x', x) divided by the point $\bar x$ into two parts: $(x', \bar x)$ and $(\bar x, x)$. It can be called an additive path function. It is therefore not surprising that its entire determining system can be found from the single-symbol function $w_{,\nu_1*}=F_{\nu_1}$ in accordance with (8.1). The expansion of the work (1.1) with respect to path integrals takes the form

$$W(x', x)=\sum_{n=0}^{\infty}\partial_{\nu_n\ldots\nu_2}F_{\nu_1}\cdot(x', x)^{\nu_1\ldots\nu_n}.$$

We now consider the path-dependent function of [6]:

$$\Phi(x', x)=\exp\left\{ie\int_x^{x'}A_{\bar\mu}\cdot dx^{\bar\mu}\right\} \tag{8.2}$$

(we have denoted it specifically, indicating explicitly the path ends x and x'). It has (apart from the notation) the same single-symbol determining function

$$[\partial_{\nu_1'}\Phi(x', x)]_{x'=x}\overset{\text{def}}{=}\varphi_{,\nu_1*}=ieA_{\nu_1}, \tag{8.3}$$

and the complete determining system can also be found from this single-symbol function (8.3). However, the path-dependent function (8.2) is, in contrast to (1.1), transitive: $\Phi(x', x)=\Phi(x', \bar x)\cdot\Phi(\bar x, x)$. Therefore, its determining system can be found from a recursion relation of the type (4.7):

$$\varphi_{,\nu_1\ldots\nu_n*}=\partial_{\nu_n}\varphi_{,\nu_1\ldots\nu_{n-1}*}+\varphi_{,\nu_1\ldots\nu_{n-1}*}\cdot\varphi_{,\nu_n*}.$$

We now turn to the translator of parallel transport

$$\Theta^{\alpha'}{}_\beta=\delta_\alpha{}^{\alpha'}\cdot\sum_{n=0}^{\infty}\theta^\alpha{}_{,\nu_1\ldots\nu_n*\beta}\cdot(x', x)^{\nu_1\ldots\nu_n}. \tag{8.4}$$

If we symmetrize the functions of its determining system in a normal coordinate system with pole at the point x, we obtain the stable translator of parallel transport along geodesics that Synge denotes by $g_j^{i'}$ ([3], p. 60 of the Russian translation). We have denoted such a translator by $\Theta^{s\alpha'}{}_\beta$:

$$\theta^{s\alpha}{}_{,\nu_1\ldots\nu_n*\beta}\overset{x}{=}\theta^\alpha{}_{,(\nu_1\ldots\nu_n)*b}.$$

*As an exception, we use Latin letters to denote a path-dependent function and the system determining it.

The letter x above the equals sign means that the equation holds only in the normal coordinate system with pole at the point x (this is a certain modernization of Schouten's notation $\overset{*}{=}$ [1]).

We now introduce the dual translator $\Theta_{\alpha'}{}^{\beta}: \Theta_{\alpha'}{}^{\beta}\cdot\Theta^{\gamma'}{}_{\beta}=\delta_{\alpha'}{}^{\gamma'}$. Integration of this translator along a certain path in accordance with (5.1) leads to a path-dependent path vector

$$\Theta^{\nu}(x', x)=\int_{x}^{x'}\Theta_{\bar{\nu}}{}^{\nu}\cdot dx^{\bar{\nu}}. \tag{8.5}$$

Substituting in the integral in (8.5) an expansion of the type (4.3),

$$\Theta_{\bar{\nu}}{}^{\mu}=\delta_{\bar{\nu}}{}^{\nu}\cdot\sum_{n=0}^{\infty}\theta_{\nu,\nu_1\ldots\nu_n\bullet}{}^{\mu}\cdot(\bar{x}, x)^{\nu_1\ldots\nu_n},$$

and integrating, we find a nontensorial expansion of the path vector $\Theta^{\mu}(x', x)$:

$$\Theta^{\mu}(x', x)=\sum_{n=0}^{\infty}\theta_{,\nu_1\ldots\nu_n\bullet}{}^{\mu}\cdot(x', x)^{\nu_1\ldots\nu_n},$$

where $\theta_{,\nu_1\ldots\nu_n}{}^{\mu}=\theta_{\nu_1,2\ldots\nu_n\bullet}{}^{\mu}$. If we symmetrize with respect to the indices $\nu_1,\ldots,\nu_n$ the last expression in a certain coordinate system, we arrive at a definite stable path vector. We make this stabilization in a normal coordinate system with pole at the point x and denote the stable path vector then obtained by $\Theta^{e\mu}(x', x)$:

$$\Theta^{e\mu}(x', x)=\sum_{n=0}^{\infty}\theta^{e}{}_{,\nu_1\ldots\nu_n\bullet}{}^{\mu}\cdot(x', x)^{\nu_1\ldots\nu_n},\quad \theta^{e}{}_{,\nu_1\ldots n_n\bullet}{}^{\mu}\overset{x}{=}\theta_{(\nu_1,\nu_2\ldots\nu_n)\bullet}{}^{\mu}. \tag{8.6}$$

Differentiation of any stable path vector leads to a so-called exponential translator [16]. Differentiation of the vector (8.6) leads to the exponential translator

$$\Theta^{e}{}_{\nu'}{}^{\mu}=\partial_{\nu'}\Theta^{e\mu}(x', x), \tag{8.7}$$

which is remarkable in that in the chosen coordinate system it takes a δ-function form: $\Theta^{e}{}_{\nu'}{}^{\mu}\overset{x}{=}\delta_{\nu'}{}^{\mu}$, since for its determining system $\theta^{e}{}_{\nu,\nu_1\ldots\nu_n\bullet}{}^{\mu}$ we have the equations

$$\theta^{e}{}_{\nu\bullet}{}^{\mu}\overset{x}{=}\delta_{\nu}{}^{\mu};\quad \theta^{e}{}_{\nu,\nu_1\ldots\nu_n\bullet}{}^{\mu}\overset{x}{=}0.$$

This follows from (4.7) and the fact that $\partial_{(\nu_n\ldots\nu_2}\Gamma_{\nu\nu_1)}{}^{\mu}\overset{x}{=}0$ ([1], p.158). Simultaneously, $\Theta^{e\mu}(x', x)$ in the normal coordinate system is equal to the coordinate difference: $\Theta^{e\mu}(x', x)\overset{x}{=}(x'-x)^{\mu}$.

The vector constructed in a metric space by differentiating Synge's world function $\Omega(x', x)$ [3] has the same property. Therefore, in a metric space $\Theta^{e\mu}(x', x)=-g^{\mu\nu}\cdot\partial_{\nu}\Omega(x', x)$. The vector $\Theta^{e\mu}(x', x)$, denoted by $\sigma_{;}{}^{\mu}$, is widely used (see, for example, [17]).

We note finally a remarkable property of the two-point derivative (6.2) based on the exponential translator (8.7) (it is not identical to the standard covariant derivative because the exponential translator is nontransitive). In a normal coordinate system, this derivative is identical to the partial derivative for $x' = x$:

$$\overset{\Theta^{e}}{\underset{x'}{\nabla}}{}_{\nu_n'\ldots\nu_1'}\overset{x}{=}\partial_{\nu_n'\ldots\nu_1'}.$$

And therefore this derivative is precisely the covariant operation that leads to the so-called "extension" of tensors introduced in [18].

9. Appendix

The relations connecting the functions $\varphi^{a}{}_{,\nu_1\ldots\nu_n\bullet b}$ and $\varphi^{a}{}_{\bullet b,\nu_1\ldots\nu_n}$ in (4.3) and (4.4) can be conveniently represented in the recursive form

$$\varphi^{a}{}_{,\nu_1\bullet b}=\partial_{\nu_1}\varphi^{a}{}_{\bullet b}-\varphi^{a}{}_{\bullet b,\nu_1};\quad \varphi^{a}{}_{,\nu_1\ldots\nu_n\bullet b}={}_{\nu_n}\{\varphi^{a}{}_{,\nu_1\ldots\nu_{n-1}\bullet b}\}. \tag{9.1}$$

Here, the curly brackets with index ν_n preceding them denote the operation in which the index ν_n is added to all factors of the expression within the brackets, in turn, as in differentiation. At the same time, the differentiation symbol ∂_ν is regarded as a factor, and the terms that do not contain a differentiation must be equipped on the left with the symbol ∂ without indices (equivalent to unity). It is important that the addition of the index ν_n to the terms $\varphi^a{}_{*b,\,\nu_{n-1}\cdots\nu_1}$ is made between the comma and the index ν_{n-1} with simultaneous change of sign. For example,

$$\varphi^a{}_{,\,\nu_1\nu_2*b} = {}_{\nu_2}\{\partial_{\nu_1}\varphi^a{}_{*b} - \partial\varphi^a{}_{*b,\,\nu_1}\} = \partial_{\nu_2\nu_1}\varphi^a{}_{*b} - \partial_{\nu_1}\varphi^a{}_{*b,\,\nu_2} - \partial_{\nu_2}\varphi^a{}_{*b,\,\nu_1} + \varphi^a{}_{*b,\,\nu_2\nu_1}.$$

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