Legendre's Algorithm and Pascal's Triangle

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Abstract

In this paper we show the relation that exists between Pascal's Triangle and Legendre's algorithm for calculating the exact amount of prime numbers that are less than a given number.

1 The Formula $P(n) = n - C(n) - 2$

To begin with, if n is an integer greater than 1, $P(n)$ the amount of prime numbers that are less than n and $C(n)$ the amount of composite numbers that are less than n, then we have

$$P(n) = n - C(n) - 2.$$

Therefore, if we know the value of $C(n)$ we can calculate the value of $P(n)$.

Let us suppose that n is in fact an integer greater than 4. The fact that $n > 4$ implies that there exists at least one prime number p such that $p < \sqrt{n}$. According to Legendre's algorithm, in order to calculate $C(n)$ we need to work with the prime numbers that are less than $\sqrt{n}$.

2 Calculating $C(n)$

Let us suppose that n is an integer greater than 4 and that there are exactly m prime numbers less than $\sqrt{n}$ (of course, we have $m \geq 1$). In order to calculate $C(n)$ we need to follow these steps:

1. We need to find all the prime numbers that are less than $\sqrt{n}$. These m prime numbers form a set which we will call *Set M*.

2. We need to calculate the amount of composite numbers divisible by 2 that are less than n. This amount equals $n/2-2$ or $\lfloor n/2 \rfloor - 1$ depending on whether $n/2$ is an integer or not respectively. If applicable (that is to say, if there exists more than one prime p such that $p < \sqrt{n}$), we need to repeat this with the rest of the prime numbers that are less than $\sqrt{n}$ (in general, if p is a prime number such that $p < \sqrt{n}$, then the amount of composite numbers divisible by p that are less than n equals $n/p - 2$ or $\lfloor n/p \rfloor - 1$ depending on whether n/p is an integer or not respectively). After having done so we will have obtained m results. By adding up these m results we will obtain a new result which we will denote by $T_1(n)$.

3. We need to find all possible combinations without repetition of 2 primes from Set M (the number of these combinations is equal to the binomial coefficient $\binom{m}{2}$). Then we need to express these combinations as multiplications (as an example, $2 * 3 = 6$ is the combination {2,3} expressed as a multiplication). By solving these multiplications we will obtain a series of products (precisely, $\binom{m}{2}$ products): r_1, r_2, r_3, etc. We need to calculate the amount of positive integers divisible by r_1 that are less than n. This amounts equals $n/r_1 - 1$ or $\lfloor n/r_1 \rfloor$ depending on whether n/r_1 is an integer or not respectively. Then we need to repeat this with the rest of the products (r_2, r_3, etc.). After having done so, we will have obtained $\binom{m}{2}$ results. By adding up these $\binom{m}{2}$ results we will obtain a new result which we will denote by $T_2(n)$.

4. In order to calculate $T_3(n)$ we need to use the same method that we used to calculate $T_2(n)$. The difference is that in this case we need to work with all possible combinations without repetition of 3 primes from Set M. In other words, we need to work with all possible 3-combinations of Set M.

5. We calculate $T_4(n)$. In this case we need to work with all possible 4-combinations of Set M.

6. Following the same rule we calculate $T_5(n)$, $T_6(n)$, $T_7(n)$, ..., $T_m(n)$.

7. Finally, if m is odd we have

$$C(n) = T_1(n) - T_2(n) + T_3(n) - T_4(n) + \cdots + T_m(n),$$

whereas if m is even we have

$$C(n) = T_1(n) - T_2(n) + T_3(n) - T_4(n) + \cdots + T_{m-1}(n) - T_m(n).$$

These two formulas can be combined into one by stating that

$$C(n) = -\left(\sum_{j=1}^{m}(-1)^j T_j(n)\right).$$

Remark 2.1. As we can see, $T_i(n)$ has a positive sign when i is odd and a negative sign when i is even.

Remark 2.2. In order to calculate the value of $T_i(n)$ we need to work with all possible i-combinations of the primes that are less than $\sqrt{n}$.

Remark 2.3. The symbol $\lfloor\rfloor$ represents the *floor function*.

Remark 2.4. Steps 3 to 6 can only be followed when applicable.

3 Relation Between Legendre's Algorithm and Pascal's Triangle

The relation that exists between Legendre's algorithm and Pascal's Triangle can be summarized in the following two statements:

1. If

   (a) the number n is an integer such that $n \geq 5$,

   (b) the number c is a composite number such that $c < n$, and

   (c) c is divisible by a maximum of x prime numbers that are less than $\sqrt{n}$,

   then c appears (is included) $\binom{x}{y}$ times in $T_y(n)$.

2. It is correct to say that if a is any positive odd integer, then

$$\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 1,$$

   whereas if b is any positive even integer, then

$$\binom{b}{1} - \binom{b}{2} + \binom{b}{3} - \binom{b}{4} + \cdots + \binom{b}{b-1} - \binom{b}{b} = 1.$$

Remark 3.1. This last statement describes one of the properties of Pascal's Triangle (see viXra:1303.0163). This property could also be expressed by stating that for every positive integer n we have

$$-\left(\sum_{j=1}^{n}(-1)^j\binom{n}{j}\right)=1.$$

Statements 1 and 2 imply that if a composite number $c < n$ is divisible by a maximum of x prime numbers that are less than $\sqrt{n}$, then the following occurs when we use Legendre's algorithm to calculate $C(n)$:

- The number c gets included $\binom{x}{1}-\binom{x}{2}+\binom{x}{3}-\binom{x}{4}+\cdots+\binom{x}{x}=1$ time in $C(n)$ if x is odd.
- The number c gets included $\binom{x}{1}-\binom{x}{2}+\binom{x}{3}-\binom{x}{4}+\cdots+\binom{x}{x-1}-\binom{x}{x}=1$ time in $C(n)$ if x is even.

In other words, the number of times every composite number $c < n$ gets included in $C(n)$ when we use Legendre's algorithm to calculate $C(n)$ is equal to

$$-\left(\sum_{j=1}^{x}(-1)^j\binom{x}{j}\right)=1.$$

This means that by using Legendre's algorithm the correct values of $C(n)$ and $P(n)$ are obtained.

Remark 3.2. In this paper we used the notation $P(n)$ instead of $\pi(n)$, since $P(n)$ denotes the amount of prime numbers that are less than n, whereas $\pi(n)$ denotes the amount of primes that are less than or equal to n. Similarly, we used $C(n)$ to denote the amount of composite numbers that are less than n, not the amount of composites that are less than or equal to n.

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