Error Propagation of the Mean Value Error for Harney's Triangulation Method

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Abstract:

In a paper given by Harney (2006) the author derives the equation for calculating the distances to a source of radiation based on the inverse-square law of radiation and the relative coordinates of the source [1]. In this paper we derive the error propagation of the mean-value error for this method.

Introduction:

Given the system of equations as described in Harney (2006)

$$ar_1^2 = \left(x + x_{off}\right)^2 + y^2 + z^2 \tag{1}$$

$$br_1^2 = \left(x - x_{off}\right)^2 + y^2 + z^2 \tag{2}$$

$$cr_1^2 = x^2 + \left(y - y_{off}\right)^2 + z^2 \tag{3}$$

$$dr_1^2 = x^2 + y^2 + z^2 \tag{4}$$

We can simultaneously solve for x, y, z and r1 (distance from sensor 1 to the source) which eliminates the need for calibration of the sensors. Let us now solve the system with the offsets x_{off} and y_{off} be the same and equal to e. Solving the system using MATHEMATICA 7.2 we obtain the following solutions.

Solution set 1

$$x = \frac{(aq - bq)}{2(a + b - 2d)} \tag{5}$$

$$y = \frac{(aq + bq - 2cq)}{2(a + b - 2d)} \tag{6}$$

$$z = -\sqrt{\frac{aq^2}{(a + b - 2d)} + \frac{bq^2}{(a + b - 2d)} - \frac{(aq - bq)^2}{4(a + b - 2d)^2} - \frac{(aq + bq - 2cq)^2}{4(a + b - 2d)^2} - q^2} \tag{7}$$

$$r_1 = -\sqrt{\frac{2}{a + b - 2d}}\, q \tag{8}$$

Solution set 2

$$x = \frac{(aq - bq)}{2(a + b - 2d)} \tag{5}$$

$$y = \frac{(aq + bq - 2cq)}{2(a + b - 2d)} \tag{10}$$

$$z = -\sqrt{\frac{aq^2}{(a + b - 2d)} + \frac{bq^2}{(a + b - 2d)} - \frac{(aq - bq)^2}{4(a + b - 2d)^2} - \frac{(aq + bq - 2cq)^2}{4(a + b - 2d)^2} - q^2} \tag{11}$$

$$r_1 = \sqrt{\frac{2}{a+b-2d}}\, q \tag{12}$$

Solution set 3

$$x = \frac{(ae-be)}{2(a+b-2d)} \tag{5}$$

$$y = \frac{(ae+be-2ce)}{2(a+b-2d)} \tag{14}$$

$$z = \sqrt{\frac{ae^2}{(a+b-2d)} + \frac{be^2}{(a+b-2d)} - \frac{(ae-be)^2}{4(a+b-2d)^2} - \frac{(ae+be-2ce)^2}{4(a+b-2d)^2} - e^2} \tag{15}$$

$$r_1 = -\sqrt{\frac{2}{a+b-2d}}\, e \tag{16}$$

Solution set 4

$$x = \frac{(aq-bq)}{2(a+b-2d)} \tag{17}$$

$$y = \frac{(aq+bq-2cq)}{2(a+b-2d)} \tag{18}$$

$$z = \sqrt{\frac{aq^2}{(a+b-2d)} + \frac{bq^2}{(a+b-2d)} - \frac{(aq-bq)^2}{4(a+b-2d)^2} - \frac{(aq+bq-2cq)^2}{4(a+b-2d)^2} - q^2} \tag{19}$$

$$r_1 = \sqrt{\frac{2}{a+b-2d}}\, e \tag{20}$$

2 Solutions in the case that the offsets are different

Start by calling the x offset $x_{off} = q$ and $y_{off} = p$, the solving the system

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$$ar_1^2 = (x+q)^2 + y^2 + z^2 \tag{21}$$

$$br_1^2 = (x-q)^2 + y^2 + z^2 \tag{22}$$

$$cr_1^2 = x^2 + (y-p)^2 + z^2 \tag{23}$$

$$dr_1^2 = x^2 + y^2 + z^2 \tag{24}$$

we obtain the following sets of solutions:

Solution set 1

$$x = \frac{(aq-bq)}{2(a+b-2d)} \tag{25}$$

$$y = \frac{ap^2+bp^2-2dp^2-2cq^2+2dq^2}{2(a+b-2d)p} \tag{26}$$

$$z = -\sqrt{\frac{aq^2}{(a+b-2d)} + \frac{bq^2}{(a+b-2d)} - \frac{(aq-bq)^2}{4(a+b-2d)^2} - \frac{(ap^2+bp^2-2dp^2-2cq^2+2dq^2)}{4p^2(a+b-2d)^2} - q^2} \tag{27}$$

$$r_1 = +\sqrt{\frac{2}{1+b-2d}}\, q \tag{28}$$

Solution set 2

$$x=\frac{(aq-bq)}{2(a+b-2d)} \tag{29}$$

$$y=\frac{ap^2+bp^2-2dp^2-2cq^2+2dq^2}{2(a+b-2d)p} \tag{30}$$

$$z=-\sqrt{\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(ap^2+bp^2-2dp^2-2cq^2+2dq^2)}{4p^2(a+b-2d)^2}-q^2} \tag{31}$$

$$r_1=+\sqrt{\frac{2}{a+b-2d}}\,q \tag{32}$$

Similarly

Solution set 3

$$x=\frac{(aq-bq)}{2(a+b-2d)} \tag{33}$$

$$y=\frac{ap^2+bp^2-2dp^2-2cq^2+2dq^2}{2(a+b-2d)p} \tag{34}$$

$$z=\sqrt{\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(ap^2+bp^2-2dp^2-2cq^2+2dq^2)}{4p^2(a+b-2d)^2}-q^2} \tag{35}$$

$$r_1=-\sqrt{\frac{2}{a+b-2d}}\,q \tag{36}$$

Solution set 4

$$x=\frac{(aq-bq)}{2(a+b-2d)} \tag{37}$$

$$y=\frac{ap^2+bp^2-2dp^2-2cq^2+2dq^2}{2(a+b-2d)p} \tag{38}$$

$$z=\sqrt{\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(ap^2+bp^2-2dp^2-2cq^2+2dq^2)}{4p^2(a+b-2d)^2}-q^2} \tag{39}$$

$$r_1=+\sqrt{\frac{2}{a+b-2d}}\,q \tag{40}$$

3. Error Propagation of the Mean Value Error

In order to proceed with the calculation of the errors involved with the triangulation results above let us first define the error propagation of the mean value error to be (Harris and Stocker, 1998):

$$\left\langle \overline{\Delta f(x_0,y_0,z_0...)} \right\rangle=\sqrt{\left[\left(\frac{\partial f}{\partial x}\right)\overline{\Delta x}\right]^2_{x_0,y_0..}+\left[\left(\frac{\partial f}{\partial y}\right)\overline{\Delta y}\right]^2_{x_0,y_0..}+\left[\left(\frac{\partial f}{\partial z}\right)\overline{\Delta z}\right]^2_{x_0,y_0..}+....} \tag{41}$$

therefore using Eqs. (5) – (8) we obtain that:

$$\left\langle \overline{\Delta x(a_0,b_0,d_0,q_0)} \right\rangle=\sqrt{\begin{array}{l}\left[\left(\frac{\partial}{\partial a}\frac{(aq-bq)}{2(a+b-2d)}\right)\overline{\Delta a}\right]^2_{a_0,b_0,d_0,e_0..}+\left[\left(\frac{\partial}{\partial b}\frac{(aq-bq)}{2(a+b-2d)}\right)\overline{\Delta b}\right]^2_{a_0,b_0,d_0,e_0..}\\+\left[\left(\frac{\partial}{\partial d}\frac{(aq-bq)}{2(a+b-2d)}\right)\overline{\Delta d}\right]^2_{a_0,b_0,d_0,e_0.}+\left[\left(\frac{\partial}{\partial q}\frac{(aq-bq)}{2(a+b-2d)}\right)\overline{\Delta q}\right]^2_{a_0,b_0,d_0,e_0.}\end{array}}. \tag{42}$$

$$\left\langle \overline{\Delta y(a_0, b_0, c_0, d_0, q_0)} \right\rangle = \sqrt{\begin{aligned} &\left[\left(\frac{\partial}{\partial a}\frac{(aq+bq-2cq)}{2(a+b-2d)}\right)\overline{\Delta a}\right]^2_{a_0,b_0,c_0,d_0,e_0..} + \left[\left(\frac{\partial}{\partial b}\frac{(aq+bq-2cq)}{2(a+b-2d)}\right)\overline{\Delta b}\right]^2_{a_0,b_0,c_0 d_0,e_0..} \\ &+\left[\left(\frac{\partial}{\partial c}\frac{(aq+bq-2cq)}{2(a+b-2d)}\right)\overline{\Delta c}\right]^2_{a_0,b_0,c_0\ d_0,e_0.} + \left[\left(\frac{\partial}{\partial d}\frac{(aq+bq-2cq)}{2(a+b-2d)}\right)\overline{\Delta d}\right]^2_{a_0,b_0,c_0\ d_0,e_0.} \\ &+\left[\left(\frac{\partial}{\partial q}\frac{(aq+bq-2cq)}{2(a+b-2d)}\right)\overline{\Delta q}\right]^2_{a_0,b_0,c_0\ d_0,e_0.} \end{aligned}} . \tag{43}$$

$$\left\langle \overline{\Delta z(a_0, b_0, c_0, d_0, q_0)} \right\rangle = \left[\begin{aligned} &\left[\left(\frac{\partial}{\partial a}\left(\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(aq+bq-2cq)^2}{4(a+b-2d)^2}-q^2\right)^{1/2}\right)\overline{\Delta a}\right]^2_{a_0,b_0,c_0,d_0,e_0..} \\ &+\left[\left(\frac{\partial}{\partial b}\left(\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(aq+bq-2cq)^2}{4(a+b-2d)^2}-q^2\right)^{1/2}\right)\overline{\Delta b}\right]^2_{a_0,b_0,c_0 d_0,e_0..} \\ &+\left[\left(\frac{\partial}{\partial c}\left(\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(aq+bq-2cq)^2}{4(a+b-2d)^2}-q^2\right)^{1/2}\right)\overline{\Delta c}\right]^2_{a_0,b_0,c_0\ d_0,e_0.} \\ &+\left[\left(\frac{\partial}{\partial d}\left(\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(aq+bq-2cq)^2}{4(a+b-2d)^2}-q^2\right)^{1/2}\right)\overline{\Delta d}\right]^2_{a_0,b_0,c_0\ d_0,e_0.} \\ &+\left[\left(\frac{\partial}{\partial e}\left(\frac{aq^2}{(a+b-2d)}+\frac{bq^2}{(a+b-2d)}-\frac{(aq-bq)^2}{4(a+b-2d)^2}-\frac{(aq+bq-2cq)^2}{4(a+b-2d)^2}-q^2\right)^{1/2}\right)\overline{\Delta q}\right]^2_{a_0,b_0,c_0\ d_0,e_0.} . \end{aligned}\right]^{1/2} \tag{44}$$

$$\left\langle \overline{\Delta r_1(a_0, b_0, d_0, e_0)} \right\rangle = \sqrt{\begin{aligned} &\left[\left(\frac{\partial}{\partial a}\sqrt{\frac{2}{a+b-2d}}\ q\right)\overline{\Delta a}\right]^2_{a_0,b_0,d_0,e_0..} + \left[\left(\frac{\partial}{\partial b}\sqrt{\frac{2}{a+b-2d}}\ q\right)\overline{\Delta b}\right]^2_{a_0,b_0,d_0,e_0..} \\ &+\left[\left(\frac{\partial}{\partial d}\sqrt{\frac{2}{a+b-2d}}\ q\right)\overline{\Delta d}\right]^2_{a_0,b_0,d_0,e_0.} + \left[\left(\frac{\partial}{\partial q}\sqrt{\frac{2}{a+b-2d}}\ q\right)\overline{\Delta q}\right]^2_{a_0,b_0,d_0,e_0.} \end{aligned}} . \tag{45}$$

where

$$\overline{\Delta x_i} = \sigma_n = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2} \tag{46}$$

in our case

$$\overline{\Delta a} = \sigma_{n_a} = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(a_i - \bar{a})^2} \tag{47}$$

$$\overline{\Delta b} = \sigma_{n_b} = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(b_i - \bar{b})^2} \tag{48}$$

$$\overline{\Delta c} = \sigma_{n_c} = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(c_i - \bar{c})^2} \tag{49}$$

$$\overline{\Delta d} = \sigma_{n_d} = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(d_i - \bar{d})^2} \tag{50}$$

$$\overline{\Delta q} = \sigma_{n_e} = \sqrt{\frac{1}{(n-1)} \sum_{i=1}^{n} (q_i - \overline{q})^2} \tag{51}$$

$$\overline{\Delta r_1} = \sigma_{n_{r_1}} = \sqrt{\frac{1}{(n-1)} \sum_{i=1}^{n} (r_{1i} - \overline{r})^2} \tag{52}$$

Therefore the final expression for the errors becomes

$$\left\langle \overline{\Delta f(a_0, b_0, c_0, d_0.q_0..)} \right\rangle = \left[\begin{array}{l} \dfrac{(a+b-2c)^2}{2(a+b-2d)^2} + \dfrac{\left(2(a+b-2d)q^2 + 4(b-d)^2 q^2 + 4(c-d)^2 q^2 - \dfrac{2\left(b^2 - bc + 2c^2 - a(b+c) - 2d(b+c) + 4d^2\right)}{a^2 + b^2 - 2c(a+b) + 2c^2 - 4d(a+b) + 8d^2} \right) \Delta a^2}{2(a+b-2d)^4} \\ + \dfrac{\left(2(a+b-2d)q^2 + 4(a-d)^2 q^2 + 4(c-d)^2 q^2 - \dfrac{2\left(b^2 - bc + 2c^2 - 2cd + 4d^2 - a(b+c+2d)\right)^2 q^2}{\left(a^2 + b^2 - 2c(a+b) + 2c^2 - 4d(a+b) + 8d^2\right)} \right) \Delta b^2}{2(a+b-2d)^4} \\ + \dfrac{\left(4(a+b-2d)^2 q^2 - \dfrac{2(a+b-2c)^2 (a+b-2d)^2 q^2}{\left(a^2 + b^2 - 2c(a+b) + 2c^2 - 4d(a+b) + 8d^2\right)} \right) \Delta c^2}{2(a+b-2d)^4} \\ + \dfrac{\left(4(a-b)^2 q^2 + 4(a+b-2c)^2 q^2 + 8(a+b-2d)q^2 - \dfrac{32\left(c^2 - a(b+c-d) + b(d-c)\right)^2 q^2}{\left(a^2 + b^2 - 2c(a+b) + 2c^2 - 4d(a+b) + 8d^2\right)} \right) \Delta d^2}{2(a+b-2d)^4} \\ \dfrac{\left(a^2 + b^2 - 4b(2+c+2d) + 4\left(c^2 + 4d(1+d)\right) + 2a\left(b - 2\left(b - 2(2+c+2d)\right)\right)\right) \Delta q^2}{2(a+b-2d)^2} \end{array} \right]^{1/2} \tag{53}$$

4. Error Propagation of the Mean Value Error $x_{off} \neq y_{0ff}$ case

Similarly in the case where $x_{off} = q$ and $y_{0ff} = p$ we obtain the following expression

$$
\left\langle\overline{\Delta f(a_0,b_0,c_0,d_0.q_0.p_0.)}\right\rangle=\left[\begin{aligned}
&\frac{q^2\left(a+b+2(b-d)^2-2d+\frac{2(c-d)^2q^2}{p^2}-\frac{2q^2\left(\left(a(b+c)-b^2-2d(c+d)+b(c+2d)\right)p^2-2q^2(c-d)^2\right)}{p^2\left((a+b-2d)^2p^4+\left(a^2+b^2-4b(c+d)+8d(c+d)-2a(b+2(c+d))\right)p^2q^2+4q^4(c-d)^2\right)}\right)\Delta a^2}{2(a+b-2d)^4}\\
&+\frac{q^2\left(a+b+2(a-d)^2-2b+\frac{2(c-d)^2q^2}{p^2}-\frac{2q^2\left(\left(a^2-bc+2d(c+d)-a(b+c+2d)\right)p^2+2(c-d)^2q^2\right)}{p^2\left((a+b-2d)^2p^4+\left(a^2+b^2-4b(c+d)+8d(c+d)-2a(b+2(c+d))\right)p^2q^2+4q^4(c-d)^2\right)}\right)\Delta b^2}{2(a+b-2d)^4}\\
&+\frac{\left(a^2+(b-4d)^2-2a(b+4d)\right)q^6\Delta c^2}{(a+b-2d)^2\left((a+b-2d)^2p^4+\left(a^2+b^2-4b(c+d)+8d(c+d)-2a(b+2(c+d))\right)p^2q^2+4q^4(c-d)^2\right)}\\
&+\frac{q^2\left((a-b)^2+2(a+b-2d)+\frac{(a+b-2c)^2q^2}{p^2}-\frac{4q^2\left((2ab+ca+bc-d(a+b+2c))p^2+(a-b-2c)(c-d)q^2\right)^2\Delta d^2}{p^2\left((a+b-2d)^2p^4+\left(a^2+b^2-4b(c+d)+8d((c+d)-2)-2a(b+2(c+d))\right)p^2q^2+4(c-d)^2q^4\right)}\right)}{2(a+b-2d)^4}\\
&\left(\left(\frac{1}{2}+\frac{(c-d)q^2}{(a+b-2d)p^2}\right)^2-\frac{\left((a+b-2d)^2p^4-4(c-d)^2q^4\right)^2}{4(a+b-2d)^2p^4\left((a+b-2d)^2p^4+\left(a^2+b^2-4b(c+d)+8d(c+d)-2a(b+2(c+d))\right)p^2q^2+4(c-d)^2q^4\right)}\right)\Delta p^2\\
&+\frac{\left(4(a-b)^2+32\,(a+b-2d)+\frac{64(c-d)^2q^2}{p^2}-\frac{\left(2\left(a^2+b^2-4b(c+d)+8d(c+d)-2a(b+2(c+d))\right)p^2q+16(c-d)^2q^3\right)^2}{p^2\left((a+b-2d)^2p^4+\left(a^2+b^2-4b(c+d)+8d(c+d)-2a(b+2(c+d))\right)p^2q^2+4(c-d)^2q^4\right)}\right)}{16(a+b-2d)^2}
\end{aligned}\right]^{1/2}
\tag{54}
$$

References

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