On the real representations of the Poincare group

Leonardo Pedro

Centro de Fisica Teorica de Particulas, CFTP, Departamento de Fisica, Instituto Superior Tecnico, Universidade Tecnica de Lisboa

Abstract

DRAFT VERSION The formulation of quantum mechanics with a complex Hilbert space is equivalent to a formulation with a real Hilbert space and particular density matrix and observables. We study the real representations of the Poincare group, motivated by the fact that the localization of complex unitary representations of the Poincare group is incompatible with causality, Poincare covariance and energy positivity.

We review the map from the complex to the real irreducible representations—finite-dimensional or unitary—of a Lie group on a Hilbert space. Then we show that all the finite-dimensional real representations of the identity component of the Lorentz group are also representations of the parity, in contrast with many complex representations.

We show that any localizable unitary representation of the Poincare group, compatible with Poincare covariance, verifies: 1) it is self-conjugate (regardless it is real or complex); 2) it is a direct sum of irreducible representations which are massive or massless with discrete helicity. 3) it respects causality; 4) it is an irreducible representation of the Poincare group (including parity) if and only if it is: a)real and b)massive with spin 1/2 or massless with helicity 1/2. Finally, the energy positivity problem is discussed in a many-particles context.

A state [of a spin-0 elementary system] *which is localized at the origin in one coordinate system, is not localized in a moving coordinate system, even if the origins coincide at t=0. Hence our* [position] *operators have no simple covariant meaning under relativistic transformations.[...]*
For higher but finite [spin of a massless representation] *s, beginning with s=1 (i.e. Maxwell's equations) we found that no localized states in the above sense exist. This is an unsatisfactory, if not unexpected, feature of our work.*

—E.P.Wigner & T.D.Newton (1949)[1]

The concepts of mathematics are not chosen for their conceptual simplicity—even sequences of pairs of numbers [i.e. the real numbers] *are far from being the simplest concepts—but for their amenability to clever manipulations and to striking, brilliant arguments. Let us not forget that the Hilbert space of quantum mechanics is the complex Hilbert space, with a Hermitean scalar product. Surely to the unpreoccupied mind, complex numbers are far from natural or simple and they cannot be suggested by physical observations. Furthermore, the use of complex numbers is in this case not a calculational trick of applied mathematics but comes close to being a necessity in the formulation of the laws of quantum mechanics.*

—E.P.Wigner (1959)[2]

1. Introduction

1.1. Motivation

Henri Poincaré defined the Poincare group as the set of transformations that leave invariant the Maxwell equations for the classical electromagnetic field. The classical electromagnetic field transforms as a real representation of the Poincare group.

The complex representations of the Poincare group were systematically studied[3–8] and used in the definition of quantum fields[9]. These studies were very important in the evolution of the role of symmetry in the Quantum Theory[10].

The formulation of quantum mechanics with a complex Hilbert space is equivalent to a formulation with a real Hilbert space and particular density matrix and observables[11]. Quantum Theory on real Hilbert spaces was investigated before[12–16], the main conclusion was that the formulation of non-relativistic Quantum Mechanics with a real Hilbert space is necessarily equivalent to the complex formulation. We could not find in the literature a systematic study on the real representations of the Poincare group, as it seems to be common assumptions that if non-relativistic Quantum Mechanics is necessarily complex then the relativistic version must also be—it is hard to accept this specially because a relativistic Quantum Theory for a single-particle is inconsistent, as relativistic causality requires the existence of anti-particles[7]—or that the energy positivity implies complex Poincare representations—it is a long shot, as it happens for the relativistic causality, only in a many-particles description the energy positivity is well defined.

The reasons motivating this study are:

1) The real representations of the Poincare group play an important role in the classical electromagnetism and general relativity[17–19] and in Quantum Theory— e.g. the Higgs

boson, Majorana fermion or quantum electromagnetic fields transform as real representations under the action of the Poincare group.

2) The parity—included in the full Poincare group—and charge-parity transformations are not symmetries of the Electroweak interactions[20]. It is not clear why the charge-parity is an apparent symmetry of the Strong interactions[21] or how to explain the matter-antimatter asymmetry[22] through the charge-parity violation. Since the self-conjugate finite-dimensional representations of the identity component of the Lorentz group are also representations of the parity, this work may be useful in future studies of the parity and charge-parity violations.

3) The localization of complex irreducible unitary representations of the Poincare group is incompatible with causality, Poincare covariance and energy positivity[23–25], while the complex representation corresponding to the photon is not localizable[1, 26, 27]. The localization problems in the complex representations may come from the representation of the charge and matter-antimatter properties in relativistic Quantum Mechanics—which has always been problematic, remember the Dirac sea[28]—and so a study of the real representations, necessarily independent of the charge and matter-antimatter properties, may be useful.

1.2. Systems on real and complex Hilbert spaces

The position operator in Quantum Mechanics is mathematically expressed using a system of imprimitivity: a set of projection operators— associated with the coordinate space—on a Hilbert space; a group acting both on the Hilbert space and on the coordinate space in a consistent way[27, 29].

Many representations of a group—such as the finite-dimensional representations of semisimple Lie groups[30] or the unitary representations of separable locally compact groups[31]—are direct sums (or integrals) of irreducible representations, hence the study of these representations reduces to the study of the irreducible representations.

If the set of normal operators commuting with an irreducible real unitary representation of the Poincare group is isomorphic to the quaternions or to the complex numbers, then the most general position operator that the representation space admits is not complex linear, but real linear. Therefore, in this case, the real irreducible representations generalize the complex ones and these in turn generalize the quaternionic ones.

The study of irreducible representations on complex Hilbert spaces is in general easier than on real Hilbert spaces, because the field of complex numbers is the algebraic closure — where any polynomial equation has a root — of the field of real numbers. There is a well studied map, one-to-one or two-to-one and surjective up to equivalence, from the complex to the real linear finite-dimensional irreducible representations of a real Lie algebra[32, 33].

Section 2 reviews a similar map from the complex to the real irreducible representations—finite-dimensional or unitary—of a Lie group on a Hilbert space. Using Mackey's imprimitivity theorem, we extend the map to systems of imprimitivity. This section follows closely the reference[32], with the addition that we will also use the Schur's lemma for unitary representations on a complex Hilbert space[34].

Related studies can be found in the references [35, 36].

1.3. Finite-dimensional representations of the Lorentz group

The Poincare group, also called inhomogeneous Lorentz group, is the semi-direct product of the translations and Lorentz Lie groups[30]. Whether or not the Lorentz and Poincare groups include the parity and time reversal transformations depends on the context and authors. To be clear, we use the prefixes full/restricted when including/excluding parity and time reversal transformations. The Pin(3,1)/SL(2,C) groups are double covers of the full/restricted Lorentz group. The semi-direct product of the translations with the Pin(3,1)/SL(2,C) groups is called IPin(3,1)/ISL(2,C) Lie group — the letter (I) stands for inhomogeneous.

A projective representation of the Poincare group on a complex/real Hilbert space is an homomorphism, defined up to a complex phase/sign, from the group to the automorphisms of the Hilbert space. Since the IPin(3,1) group is a double cover of the full Poincare group, their projective representations are the same[37]. All finite-dimensional projective representations of a simply connected group, such as SL(2,C), are usual representations[7]. Both SL(2,C) and Pin(3,1) are semi-simple Lie groups, and so all its finite-dimensional representations are direct sums of irreducible representations[30]. Therefore, the study of the finite-dimensional projective representations of the restricted Lorentz group reduces to the study of the finite-dimensional irreducible representations of SL(2,C).

The Dirac spinor is an element of a 4 dimensional complex vector space, while the Majorana spinor is an element of a 4 dimensional real vector space[38–41]. The complex finite-dimensional irreducible representations of SL(2,C) can be written as linear combinations of tensor products of Dirac spinors.

In Section 2.3 we will review the Pin(3,1) and SL(2,C) semi-simple Lie groups and its relation with the Majorana, Dirac and Pauli matrices. We will obtain all the real finite-dimensional irreducible representations of SL(2,C) as linear combinations of tensor products of Majorana spinors, using the map from Section 2. Then we will check that all these real representations are also projective representations of the full Lorentz group, in contrast with the complex representations which are not all projective representations of the full Lorentz group.

1.4. Unitary representations of the Poincare group

According to Wigner's theorem, the most general transformations, leaving invariant the modulus of the internal product of a Hilbert space, are: unitary or anti-unitary operators, defined up to a complex phase, for a complex Hilbert; unitary, defined up to a signal, for a real Hilbert[27, 42]. This motivates the study of the (anti-)unitary projective representations of the full Poincare group.

All (anti-)unitary projective representations of ISL(2,C) are, up to isomorphisms, well defined unitary representations, because ISL(2,C) is simply connected[7]. Both ISL(2,C) and IPin(3,1) are separable locally compact groups and so all its (anti-)unitary projective representations are direct integrals of irreducible representations[31]. Therefore, the study of the (anti-)unitary projective representations of the restricted Poincare group reduces to the study of the unitary irreducible representations of ISL(2,C).

The spinor fields, space-time dependent spinors, are solutions of the free Dirac equation[43]. The real/complex Bargmann-Wigner fields[44, 45], space-time dependent linear combinations of tensor products of Majorana/Dirac spinors, are solutions of the free Dirac equation in each tensor index. The complex unitary irreducible projective representations of the Poincare group with discrete spin or helicity can be written as complex Bargmann-Wigner fields.

In Section 2.4, we will obtain all the real unitary irreducible projective representations of the Poincare group, with discrete spin or helicity, as real Bargmann-Wigner fields, using the map from Section 2. For each pair of complex representations with positive/negative energy, there is one real representation. We will define the Majorana-Fourier and Majorana-Hankel unitary transforms of the real Bargmann-Wigner fields, relating the coordinate space with the linear and angular momenta spaces. We show that any localizable unitary representation of the Poincare group, compatible with Poincare covariance, verifies: 1) it is self-conjugate (regardless it is real or complex); 2) it is a direct sum of irreducible representations which are massive or massless with discrete helicity. 3) it respects causality; 4) it is an irreducible representation of the Poincare group (including parity) if and only if it is: a)real and b)massive with spin 1/2 or massless with helicity 1/2.

The free Dirac equation is diagonal in the Newton-Wigner representation[1], related to the Dirac representation through a Foldy-Wouthuysen transformation[46, 47] of Dirac spinor fields. The Majorana-Fourier transform, when applied on Dirac spinor fields, is related with the Newton-Wigner representation and the Foldy-Wouthuysen transformation. In the context of Clifford Algebras, there are studies on the geometric square roots of -1 [16, 48] and on the generalizations of the Fourier transform[49], with applications to image processing[50].

1.5. Energy Positivity

In non-relativistic Quantum Mechanics the time is invariant under the Galilean transformations —excluding the time reversal transformation—and so the generator of translations in time is also invariant. Therefore, the positivity of the Energy and the localization in space of a state can be defined simultaneously. In relativistic Quantum Mechanics, the time is not invariant under Lorentz transformations, as a consequence the positivity of the Energy and the localization in space of a state cannot be defined simultaneously— the corresponding projection operators do not commute. The solution can be found in a many particles system.

In the canonical quantization description of a many particles system, the positivity of Energy is well defined by construction and the localization problem is handled by introducing anti-particles—causality implies the existence of anti-particles[7], a related approach led Dirac to predict the positron[28]. Yet, it should also be possible to build a description of a many particles system where the localization in space of a state is well defined by construction and the Energy positivity problem can be handled, as we can infer from the canonical quantization that both Energy positivity and localization are important and, in some way, complementary. Dirac himself was the first to consider an approach which do not assume the positivity of Energy by construction[51] and quantization in de Sitter space-time may be achieved in a related approach[52, 53].

The description of a many-particles system based on the localization will be discussed in the section 5.

2. Systems on real and complex Hilbert spaces

Definition (System)**.** A system (M, V) is defined by:
1) the (real or complex) Hilbert space V;
2) a set M of bounded endomorphisms on V.

The representation of a symmetry is an example of a system: a representation space plus a set of operators representing the action of the symmetry group in the representation space[54].

Definition (Complexification)**.** Consider a system (M, W) on a real Hilbert space. The system (M, W^c) is the complexification of the system (M, W), defined as $W^c \equiv \mathbb{C} \otimes W$, with the multiplication by scalars such that $a(bw) \equiv (ab)w$ for $a, b \in \mathbb{C}$ and $w \in W$. The internal product of W^c is defined—for $u_r, u_i, v_r, v_i \in W$ and $< v_r, u_r >$ the internal product of W—as:

$$< v_r + iv_i, u_r + iu_i >_c \equiv < v_r, u_r > + < v_i, u_i > +i < v_r, u_i > -i < v_i, u_r >$$

Definition (Realification)**.** Consider a system (M, V) on a complex Hilbert space. The system(M, V^r) is the realification of the system (M, V), defined as $V^r \equiv V$ is a real Hilbert space with the multiplication by scalars restricted to reals such that $a(v) \equiv (a + i0)v$ for $a \in \mathbb{R}$ and $v \in V$. The internal product of V^r is defined—for $u, v \in V$ and $< v, u >$ is the internal product of V—as:

$$< v, u >_r \equiv \frac{< v, u > + < u, v >}{2}$$

Note 2.1. *Let H_n, with $n \in \{1, 2\}$, be two Hilbert spaces with internal products $<, >$: $H_n \times H_n \to \mathbb{F}$,($\mathbb{F} = \mathbb{R}, \mathbb{C}$). A (anti-)linear operator $U : H_1 \to H_2$ is (anti-)unitary iff:*
1) it is surjective;
2) for all $x \in H_1$, $< U(x), U(x) >=< x, x >$.

Proposition 2.2. *Let H_n, with $n \in \{1, 2\}$, be two complex Hilbert spaces and H_n^r its complexification. The following two statements are equivalent:*
1) The operator $U : H_1 \to H_2$ is (anti-)unitary;
2) The operator $U^r : H_1^r \to H_2^r$ is (anti-)unitary, where $U^r(h) \equiv U(h)$, for $h \in H_1$.

Proof. Since $< h, h >=< h, h >_r$ and $U^r(h) = U(h)$, for $h \in H_1$, we get the result. □

Definition (Equivalence)**.** Consider the systems (M, V) and (N, W):
1) A normal endomorphism of (M, V) is a bounded endomorphism $S : V \to V$ commuting with $S^\dagger$ and m, for all $m \in M$; an anti-endomorphism in a complex Hilbert space is an anti-linear endomorphism;

2) An isometry of (M, V) is a unitary operator $S : V \to V$ commuting with m, for all $m \in M$;
3) The systems (M, V) and (N, W) are unitary equivalent iff there is a isometry $\alpha : V \to W$ such that $N = \{\alpha m \alpha^\dagger : m \in M\}$.

We use the trivial extension of the definition of irreducibility from representations to systems.

Definition (Irreducibility)**.** Consider the system (M, V) and let W be a linear subspace of V:
1) (M, W) is a (topological) subsystem of (M, V) iff W is closed and invariant under the system action, that is, for all $w \in W$:$(mw) \in W$, for all $m \in M$;
2) A system (M, V) is (topologically) irreducible iff their only sub-systems are the non-proper (M, V) or trivial $(M, \{0\})$ sub-systems, where $\{0\}$ is the null space.

Definition (Structures)**.** 1) Consider a system (M, V) on a complex Hilbert space. A C-conjugation operator of (M, V) is an anti-unitary involution of V commuting with m, for all $m \in M$;
2) Consider a system (M, W) on a real Hilbert space. A R-imaginary operator of (M, W), J, is an isometry of (M, W) verifying $J^2 = -1$.

2.1. The map from the complex to the real systems

Definition. Consider an irreducible system (M, V) on a complex Hilbert space:
1) The system is C-real iff there is a C-conjugation operator;
2) The system is C-pseudoreal iff there is no C-conjugation operator but there is an anti-unitary operator of (M, V);
3) The system is C-complex iff there is no anti-unitary operator of (M, V).

Definition 2.3. Consider the system (M, W) on a real Hilbert space and let (M, W^c) be its complexification: 1) (M, W) is R-real iff (M, W^c) is C-real irreducible;
2) (M, W) is R-pseudoreal iff (M, V) is C-pseudoreal irreducible, with $W^c = V \oplus \bar{V}$; 3) (M, W) is R-complex iff (M, V) is C-complex irreducible, with $W^c = V \oplus \bar{V}$.

Proposition 2.4. *Any irreducible real system is R-real or R-pseudoreal or R-complex.*

Proof. Consider an irreducible system (M, W) on a real Hilbert space. There is a C-conjugation operator of (M, W^c), θ, defined by $\theta(u + iv) \equiv (u - iv)$ for $u, v \in W$, verifying $(W^c)_\theta = W$.

Let (M, X^c) be a proper non-trivial subsystem of (M, W^c). Then θ is a C-conjugation operator of the subsystems (M, Y^c) and (M, Z^c), where $Y^c \equiv \{u + \theta v : u, v \in X^c\}$ and $Z^c \equiv \{u : u, \theta u \in X^c\}$. Therefore, $Y^c = \{u + iv : u, v \in Y\}$ and $Z^c = \{u + iv : u, v \in Z\}$, where $Y \equiv \{\frac{1+\theta}{2} u : u \in Y^c\}$ and $Z \equiv \{\frac{1+\theta}{2} u : u \in Z^c\}$, are invariant closed subspaces of W. If $Y = \{0\}$ then $Z = \{0\}$ and $Y^c = X^c = \{0\}$, in contradiction with X^c being non-trivial. If

$Z = W$ then $Y = W$ and $Z^c = X^c = W^c$, in contradiction with X^c being proper. Therefore $Z = \{0\}$ and $Y = W$, which implies $Z^c = \{0\}$ and $Y^c = W^c$.

So, (M, W) is equivalent to $(M, (X^c)^r)$, due to the existence of the bijective linear map $\alpha : (X^c)^r \to W$, $\alpha(u) = u + \theta u$, $\alpha^{-1}(u + \theta u) = u$, for $u \in (X^c)^r$. Suppose that there is a C-conjugation operator of (M, X^c), θ'. Then $(M, W_\pm)$ is a proper non-trivial subsystem of (M, W), where $W_\pm \equiv \{\frac{1\pm\theta'}{2} w : w \in W\}$, in contradiction with (M, W) being irreducible. □

Proposition 2.5. *Any real system which is R-real or R-pseudoreal or R-complex is irreducible.*

Proof. Consider an irreducible system on a complex Hilbert space (M, V). There is a R-imaginary operator J of the system (M, V^r), defined by $J(u) \equiv iu$, for $u \in V^r$.

Let (M, X^r) be a proper non-trivial subsystem of (M, V^r). Then J is an R-imaginary operator of (M, Y^r) and (M^r, Z^r), where $Y^r \equiv \{u + Jv : u, v \in X^r\}$ and $Z^r \equiv \{u : u, Ju \in X^r\}$. Then (M, Y) and (M, Z) are subsystems of (M, V), where the complex Hilbert spaces $Y \equiv Y^r$ and $Z \equiv Z^r$ have the scalar multiplication such that $(a + ib)(y) = ay + bJy$, for $a, b \in \mathbb{R}$ and $y \in Y$ or $y \in Z$. If $Y = \{0\}$, then $Z = X^r = \{0\}$ which is in contradiction with X^r being non-trivial. If $Z = V$, then $Y = V$ and $X^r = V^r$ which is in contradiction with X^r being non-trivial. So $Z = \{0\}$ and $Y = V$, which implies that $V = (X^r)^c$.

Then there is a C-conjugation operator of (M, V), θ, defined by $\theta(u + iv) \equiv u - iv$, for $u, v \in X^r$. We have $X^r = V_\theta$. Suppose there is a R-imaginary operator of (M, V_θ), J'. Then $(M, V_\pm)$, where $V_\pm \equiv \{\frac{1\pm iJ'}{2} v : v \in V\}$, are proper non-trivial subsystems of (M, V), in contradiction with (M, V) being irreducible.

Therefore, if (M, V) is C-real, then (M, V_θ) is R-real irreducible. If (M, V) is C-pseudoreal or C-complex, then (M, V_θ^r) is R-pseudoreal or R-complex, irreducible. □

2.2. Schur Systems

Definition 2.6 (Schur System)**.** A system (M, V), on a complex Hilbert space V, is a Schur system if the set of normal operators of (M, V) is isomorphic to $\mathbb{C}$.
Consider an irreducible system (M, W), on a real Hilbert space W and let (M, W^c) be its complexification: 1) (M, W) is Schur R-real iff (M, W^c) is Schur C-real;
2) (M, W) is Schur R-pseudoreal iff (M, V) is Schur C-pseudoreal, with $W^c = V \oplus \bar{V}$;
3) (M, W) is Schur R-complex iff (M, V) is Schur C-complex, with $W^c = V \oplus \bar{V}$.

Lemma 2.7. *Consider a Schur system (M, V) on a complex Hilbert space. An anti-isometry of (M, V), if it exists, is unique up to a complex phase.*

Proof. Let θ_1,θ_2 be two anti-isometries of (M, V). The product $(\theta_2\theta_1)$ is an isometry of (M, V); since (M, V) is irreducible, $(\theta_2\theta_1) = e^{i\phi}$; with $\phi \in \mathbb{R}$.

Therefore $\theta_2 = \alpha\theta_1\alpha^{-1}$; where $\alpha \equiv e^{i\frac{\phi}{2}}$ is a complex phase. □

Proposition 2.8. *Two R-real Schur systems are isometric iff their complexifications are isometric.*

Proof. Let (M, V) and (N, W) be C-real Schur systems, with θ_M and θ_N the respective C-conjugation operators. If there is an isometry $\alpha : V \to W$ such that $\alpha M = N\alpha$, then $\vartheta \equiv \alpha\theta_M\alpha^{-1}$ is an anti-isometry of (N, W). Since it is unique up to a phase, then $\theta_N = e^{i\phi}\vartheta$. Therefore $e^{i\frac{\phi}{2}}\alpha$ is an isometry between (M, V_θ) and (N, W_θ), where $V_{\theta_M} \equiv \{(1 + \theta_M)v : v \in V\}$. $\square$

Proposition 2.9. *Two C-complex or C-pseudoreal Schur systems are isometric or anti-isometric iff their realifications are isometric.*

Proof. Let (M, V) and (N, W) be R-complex or R-pseudoreal Schur systems, with J_M and J_N the respective R-imaginary operators. If there is an isometry $\alpha : V \to W$ such that $\alpha M = N\alpha$, then $K \equiv \alpha J_M\alpha^{-1}$ is a R-imaginary operator of (N, W). When considering (N, W_{J_N}) and (M, V_{J_M}), where $W_{J_N} \equiv \{(1 - iJ_N)w : w \in W\}$, we get that $(1 - J_N K)(1 - KJ_N) = r$ as an operator of W_{J_N}, where r is a non-negative null real scalar. If $c = 0$ then $K = -J_N$ and α defines an anti-isometry between (M, V_{J_M}) and (N, W_{J_N}). If $c \neq 0$ then $(1 - J_N K)\alpha c^{-\frac{1}{2}}$ is an isometry between (M, V_{J_M}) and (N, W_{J_N}). $\square$

Proposition 2.10. *The space of normal operators of a R-real Schur system is isomorphic to* $\mathbb{R}$.

Proof. Let (M, V) be a C-real Schur system, with θ the C-conjugation operator. If there is an endomorphism $\alpha : V \to V$ such that $\alpha M = M\alpha$, we know that $\alpha = re^{i\varphi}$. Then the endomorphism of V_θ is a real number. $\square$

Proposition 2.11. *The space of normal operators of a R-complex Schur system is isomorphic to* $\mathbb{C}$.

Proof. Let (M, V) be a R-complex Schur system, with J the R-imaginary operator. If there is a normal operator α of (M, V), then $KK^\dagger$ is a normal operator of the C-complex Schur system (M, V_J), where $K \equiv (\alpha + J\alpha J)$ and $V_J \equiv \{(1 - iJ)v : v \in V\}$. If $KK^\dagger = r > 0$, then $\frac{K}{\sqrt{r}}$ is unitary and V_J is equivalent to $\overline{V}_J$ which would imply that (M, V) is C-pseudoreal. Therefore $K = 0$ and hence α is a normal operator of (M, V_J), so $\alpha = re^{J\theta}$. $\square$

Proposition 2.12. *The space of normal operators of a R-pseudoreal Schur system is isomorphic to* $\mathbb{H}$ *(quaternions).*

Proof. Let (M, V) be a R-pseudoreal Schur system, with J the R-imaginary operator. If there is an endomorphism α of (M, V), then $SS^\dagger$ and $TT^\dagger$ are a self-adjoint endomorphisms of the C-complex Schur system (M, V_J), where $S \equiv (\alpha - J\alpha J)/2$, $T \equiv (\alpha + J\alpha J)/2$ and $V_J \equiv \{(1 - iJ)v : v \in V\}$. Let K be an unitary operator of (M, V) and anti-commuting with J, then $K^2 = e^{J\theta}$ and $Ke^{J\theta} = K(K^2) = (K^2)K = e^{J\theta}K$, therefore $K^2 = -1$. If $TT^\dagger = t > 0$, then $\frac{T}{\sqrt{t}}$ is unitary and anti-commutes with J, TK is a normal endomorphism of (M, V_J) and therefore $T = Kc + KJd$; if $TT^\dagger = 0$ then $c = d = 0$. If $SS^\dagger = s > 0$, then $\frac{S}{\sqrt{s}}$ is unitary and commutes with J, S is a normal endomorphism of (M, V_J) and therefore $S = a + Jb$; if $SS^\dagger = 0$ then $a = b = 0$.

Therefore $\alpha = S + T = a + Jb + Kc + KJd$, which is isomorphic to the quaternions. $\square$

2.3. Finite-dimensional representations

Lemma 2.13 (Schur's lemma for finite-dimensional representations[34])**.** *Consider an irreducible finite-dimensional representation (M_G, V) of a Lie group G on a complex Hilbert space V. If the representation (M_G, V) is irreducible then any endomorphism S of (M_G, V) is a complex scalar.*

Lemma 2.14. *Consider an irreducible complex finite-dimensional representation (M, V) on a complex Hilbert space. Then there is internal product such that: 1) The system is C-real iff there is an anti-linear involution of (M, V);*
2) The system is C-pseudoreal iff there is not an anti-linear bounded involution of (M, V), but there is an anti-isomorphism of (M, V);
3) The system is C-complex iff there is no anti-isomorphism of (M, V).

Proof. Let S be an anti-isomorphism of an irreducible representation (M, V). Then $S^2 = re^{i\varphi}$. But S^2 commutes with S which is anti-linear, so $S^2 = \pm r$. So, there is an internal product such that S is anti-unitary. □

Definition 2.15. A finite-dimensional system is completely reducible iff it can be expressed as a direct sum of irreducible systems.

Note 2.16 (Weyl theorem)**.** *All finite-dimensional representations of a semi-simple Lie group (such as SL(2,C)) are completely reducible.*

2.4. Unitary representations and Systems of Imprimitivity

Definition 2.17 (Normal System)**.** A System (M, V) is normal iff M is a set M of normal operators on V closed under Hermitian conjugation—for all $m \in M$ there is $n \in M$ such that $n = m^\dagger$.

A unitary representation or a System of Imprimitivity are examples of a normal System.

Note 2.18. *$W^\perp$ is the orthogonal complement of the subspace W of the Hilbert space V if: 1) $V = W \oplus W^\perp$, that is, all $v \in V$ can be expressed as $v = w + x$, where $w \in W$ and $x \in W^\perp$;*
2) if $w \in W$ and $x \in W^\perp$, then $x^\dagger w = 0$.

Lemma 2.19. *Consider a normal system (M, V). Then, for all subsystem (M, W) of (M_G, V), $(M_G, W^\perp)$ is also a subsystem of (M, V), where $W^\perp$ is the orthogonal complement of the subspace W.*

Proof. Let (M, W) be a subsystem of (M, V). $W^\perp$ is the orthogonal complement of W.
For all $x \in W^\perp$, $w \in W$ and $m \in M$, $< mx, w >=< x, m^\dagger w >$.
Since W is invariant and there is $n \in M$, such that $n = m^\dagger$, then $w' \equiv (m^\dagger w) \in W$.
Since $x \in W^\perp$ and $w' \in W$, then $< x, w' >= 0$.
This implies that if $x \in W^\perp$), also $(mx) \in W^\perp$, for all $m \in M$. □

Lemma 2.20. *Any Schur normal system on a complex Hilbert space is irreducible.*

Proof. Let (M, W) and $(M, W^{\perp})$ be sub-systems of the complex Schur system (M, V), where $W^{\perp}$ is the orthogonal complement of W.

There is a bounded endomorphism $P : V \to V$, such that, for $w, w' \in W$, $x, x' \in W^{\perp}$, $P(w + x) = w$. $P^2 = P$ and P is hermitian:

$$< w' + x', P(w + x) >=< w', w >=< P(w' + x'), w + x > \quad (1)$$

Let $w' \equiv mw \in W$ and $x' \equiv mx \in W^{\perp}$:

$$mP(w + x) = mw = w' \quad (2)$$
$$Pm(w + x) = P(w' + x') = w' \quad (3)$$

Which implies that P commutes with all $m \in M$, so $P \in \{0, 1\}$. If $P = 1$, then $W = V$, if $P = 0$, then W is the null space. $\square$

So a complex Schur normal system is irreducible, and hence, from Defns.2.3,2.6 and Prop.2.5, a real Schur normal system is also irreducible.

Lemma 2.21 (Schur's lemma for unitary representations[34]). *Consider an irreducible unitary representation (M, V) of a Lie group G on a complex Hilbert space V. If the representation (M, V) is irreducible then any normal operator N of (M, V) is a scalar.*

Definition 2.22. A unitary system is completely reducible iff it can be expressed as a direct integral of irreducible systems.

Note 2.23. *All unitary representations of a separable locally compact group (such as the Poincare group) are completely reducible.*

Definition 2.24. Consider a measurable space (X, M), where M is a σ-algebra of subsets of X. A projection-valued-measure, π, is a map from M to the set of self-adjoint projections on a Hilbert space H such that $\pi(X)$ is the identity operator on H and the function $<\psi, \pi(A)\psi>$, with $A \in M$ is a measure on M, for all $\psi \in H$.

Definition 2.25. Suppose now that X is a representation of G. Then, a system of imprimitivity is a pair (U, π), where π is a projection valued measure and U an unitary representation of G on the Hilbert space H, such that $U(g)\pi(A)U^{-1}(g) = \pi(gA)$.

Note 2.26 (Imprimitivity Theorem (thrm 6.12 [4, 27, 55, 56])). *Let G be a Lie group, H its closed subgroup. Let a pair (V, E) be a system of imprimitivity for G based on G/H on a separable complex Hilbert space. Then there exists a representation L of H such that (V, E) is equivalent to the canonical system of imprimitivity (V L , E L). For any two representations L, L' of the subgroup H the corresponding canonical systems of imprimitivity are equivalent if and only if L, L' are equivalent. The sets of normal operators commuting of C(VL, EL) and of C(L) are isomorphic.*

So we can define a map from the real to the complex systems of imprimitivity—analogous to the one for unitary representations.

3. Finite-dimensional representations of the Lorentz group

3.1. Majorana, Dirac and Pauli Matrices and Spinors

Definition 3.1. $\mathbb{F}^{m\times n}$ is the vector space of $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$.

In the next remark we state the Pauli's fundamental theorem of gamma matrices. The proof can be found in the reference[57].

Note 3.2 (Pauli's fundamental theorem)**.** *Let A^μ, B^μ, $\mu \in \{0,1,2,3\}$, be two sets of 4×4 complex matrices verifying:*

$$A^\mu A^\nu + A^\nu A^\mu = -2\eta^{\mu\nu} \tag{4}$$

$$B^\mu B^\nu + B^\nu B^\mu = -2\eta^{\mu\nu} \tag{5}$$

Where $\eta^{\mu\nu} \equiv diag(+1,-1,-1-1)$ is the Minkowski metric.

1) There is an invertible complex matrix S such that $B^\mu = SA^\mu S^{-1}$, for all $\mu \in \{0,1,2,3\}$. S is unique up to a non-null scalar.

2) If A^μ and B^μ are all unitary, then S is unitary.

Proposition 3.3. *Let α^μ, β^μ, $\mu \in \{0,1,2,3\}$, be two sets of 4×4 real matrices verifying:*

$$\alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu = -2\eta^{\mu\nu} \tag{6}$$

$$\beta^\mu\beta^\nu + \beta^\nu\beta^\mu = -2\eta^{\mu\nu} \tag{7}$$

Then there is a real matrix S, with $|detS| = 1$, such that $\beta^\mu = S\alpha^\mu S^{-1}$, for all $\mu \in \{0,1,2,3\}$. S is unique up to a signal.

Proof. From remark 3.2, we know that there is an invertible matrix T', unique up to a non-null scalar, such that $\beta^\mu = T'\alpha^\mu T'^{-1}$. Then $T \equiv T'/|det(T')|$ has $|detT| = 1$ and it is unique up to a complex phase.

Conjugating the previous equation, we get $\beta^\mu = T^*\alpha^\mu T^{*-1}$. Then $T^* = e^{i2\theta}T$ for some real number θ. Therefore $S \equiv e^{i\theta}T$ is a real matrix, with $|detS| = 1$, unique up to a signal. $\square$

Definition 3.4. The Majorana matrices, $i\gamma^\mu$, $\mu \in \{0,1,2,3\}$, are 4×4 complex unitary matrices verifying:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2\eta^{\mu\nu} \tag{8}$$

The Dirac matrices are $\gamma^\mu \equiv -i(i\gamma^\mu)$.

In the Majorana bases, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned}
&i\gamma^1 = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
&i\gamma^0 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{9}$$

In reference [58] it is proved that the set of five anti-commuting 4×4 real matrices is unique up to isomorphisms. So, for instance, with 4×4 real matrices it is not possible to obtain the euclidean signature for the metric.

Definition 3.5. The Dirac spinor is a 4×1 complex column matrix, $\mathbb{C}^{4\times 1}$.

The space of Dirac spinors is a 4 dimensional complex vector space.

Lemma 3.6. *The charge conjugation operator Θ, is an anti-linear involution commuting with the Majorana matrices $i\gamma^\mu$. It is unique up to a complex phase.*

Proof. In the Majorana bases, the complex conjugation is a charge conjugation operator. Let Θ and Θ' be two charge conjugation operators operators. Then, $\Theta\Theta'$ is a complex invertible matrix commuting with $i\gamma^\mu$, therefore, from Pauli's fundamental theorem, $\Theta\Theta' = c$, where c is a non-null complex scalar. Therefore $\Theta' = c^*\Theta$ and from $\Theta'\Theta' = 1$, we get that $c^*c = 1$. $\square$

Definition 3.7. Let Θ be a charge conjugation operator.

The set of Majorana spinors, *Pinor*, is the set of Dirac spinors verifying the Majorana condition (defined up to a complex phase):

$$Pinor \equiv \{u \in \mathbb{C}^{4\times 1} : \Theta u = u\} \tag{10}$$

The set of Majorana spinors is a 4 dimensional real vector space. Note that the linear combinations of Majorana spinors with complex scalars do not verify the Majorana condition.

There are 16 linear independent products of Majorana matrices. These form a basis of the real vector space of endomorphisms of Majorana spinors, $End(Pinor)$. In the Majorana bases, $End(Pinor)$ is the vector space of 4×4 real matrices.

Definition 3.8. The Pauli matrices σ^k, $k \in \{1, 2, 3\}$ are 2×2 hermitian, unitary, anti-commuting, complex matrices. The Pauli spinor is a 2×1 complex column matrix. The space of Pauli spinors is denoted by *Pauli*.

The space of Pauli spinors, *Pauli*, is a 2 dimensional complex vector space and a 4 dimensional real vector space. The realification of the space of Pauli spinors is isomorphic to the space of Majorana spinors.

3.2. On the Lorentz, SL(2,C) and Pin(3,1) groups

Note 3.9. *The Lorentz group, $O(1,3) \equiv \{\lambda \in \mathbb{R}^{4\times 4} : \lambda^T \eta \lambda = \eta\}$, is the set of real matrices that leave the metric, $\eta = diag(1, -1, -1, -1)$, invariant.*

The proper orthochronous Lorentz subgroup is defined by $SO^+(1,3) \equiv \{\lambda \in O(1,3) : det(\lambda) = 1, {\lambda^0}_0 > 0\}$. It is a normal subgroup. The discrete Lorentz subgroup of parity and time-reversal is $\Delta \equiv \{1, \eta, -\eta, -1\}$.

The Lorentz group is the semi-direct product of the previous subgroups, $O(1,3) = \Delta \ltimes SO^+(1,3)$.

Definition 3.10. The set Maj is the 4 dimensional real space of the linear combinations of the Majorana matrices, $i\gamma^\mu$:

$$Maj \equiv \{a_\mu i\gamma^\mu : a_\mu \in \mathbb{R},\ \mu \in \{0,1,2,3\}\} \tag{11}$$

Definition 3.11. $Pin(3,1)$ [37] is the group of endomorphisms of Majorana spinors that leave the space Maj invariant, that is:

$$Pin(3,1) \equiv \Big\{S \in End(Pinor) :\ |detS| = 1,\ S^{-1}(i\gamma^\mu)S \in Maj,\ \mu \in \{0,1,2,3\}\Big\} \tag{12}$$

Proposition 3.12. *The map $\Lambda : Pin(3,1) \to O(1,3)$ defined by:*

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu \equiv S^{-1}(i\gamma^\mu)S \tag{13}$$

is two-to-one and surjective. It defines a group homomorphism.

Proof. 1) Let $S \in Pin(3,1)$. Since the Majorana matrices are a basis of the real vector space Maj, there is an unique real matrix $\Lambda(S)$ such that:

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu = S^{-1}(i\gamma^\mu)S \tag{14}$$

Therefore, Λ is a map with domain $Pin(3,1)$. Now we can check that $\Lambda(S) \in O(1,3)$:

$$(\Lambda(S))^\mu{}_\alpha \eta^{\alpha\beta}(\Lambda(S))^\nu{}_\beta = -\frac{1}{2}(\Lambda(S))^\mu{}_\alpha\{i\gamma^\alpha, i\gamma^\beta\}(\Lambda(S))^\nu{}_\beta = \tag{15}$$

$$= -\frac{1}{2}S\{i\gamma^\mu, i\gamma^\nu\}S^{-1} = S\eta^{\mu\nu}S^{-1} = \eta^{\mu\nu} \tag{16}$$

We have proved that Λ is a map from $Pin(3,1)$ to $O(1,3)$.

2) Since any $\lambda \in O(1,3)$ conserve the metric η, the matrices $\alpha^\mu \equiv \lambda^\mu{}_\nu i\gamma^\nu$ verify:

$$\{\alpha^\mu, \alpha^\nu\} = -2\lambda^\mu{}_\alpha \eta^{\alpha\beta}\lambda^\nu{}_\beta = -2\eta^{\mu\nu} \tag{17}$$

In a basis where the Majorana matrices are real, from Proposition 3.3 there is a real invertible matrix S_λ, with $|detS_\Lambda| = 1$, such that $\lambda^\mu{}_\nu i\gamma^\nu = S_\lambda^{-1}(i\gamma^\mu)S_\lambda$. The matrix S_Λ is unique up to a sign. So, $\pm S_\lambda \in Pin(3,1)$ and we proved that the map $\Lambda : Pin(3,1) \to O(1,3)$ is two-to-one and surjective.

3) The map defines a group homomorphism because:

$$\Lambda^\mu{}_\nu(S_1)\Lambda^\nu{}_\rho(S_2)i\gamma^\rho = \Lambda^\mu{}_\nu S_2^{-1} i\gamma^\nu S_2 \tag{18}$$

$$= S_2^{-1}S_1^{-1} i\gamma^\mu S_1 S_2 = \Lambda^\mu{}_\rho(S_1S_2)i\gamma^\rho \tag{19}$$

□

Note 3.13. *The group $SL(2,\mathbb{C}) = \{e^{\theta^j i\sigma^j + b^j\sigma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$ is simply connected. Its projective representations are equivalent to its ordinary representations[7].*

There is a two-to-one, surjective map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$, defined by:

$$\Upsilon^\mu{}_\nu(T)\sigma^\nu \equiv T^\dagger\sigma^\mu T \tag{20}$$

Where $T \in SL(2,\mathbb{C})$, $\sigma^0 = 1$ and σ^j, $j \in \{1,2,3\}$ are the Pauli matrices.

Lemma 3.14. *Consider that $\{M_+, M_-, i\gamma^5 M_+, i\gamma^5 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces Pinor and Pauli, respectively, verifying:*

$$\gamma^0\gamma^3 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm \tag{21}$$

The isomorphism $\Sigma : Pauli \to Pinor$ is defined by:

$$\Sigma(P_+) = M_+,\ \Sigma(iP_+) = i\gamma^5 M_+ \tag{22}$$
$$\Sigma(P_-) = M_-,\ \Sigma(iP_-) = i\gamma^5 M_- \tag{23}$$

The group $Spin^+(3,1) \equiv \{\Sigma \circ A \circ \Sigma^{-1} : A \in SL(2,\mathbb{C})\}$ is a subgroup of $Pin(1,3)$. For all $S \in Spin^+(1,3)$, $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$.

Proof. From remark 3.13, $Spin^+(3,1) = \{e^{\theta^j i\gamma^5\gamma^0\gamma^j + b^j\gamma^0\gamma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$. Then, for all $T \in SL(2,C)$:

$$-i\gamma^0 \Sigma \circ T^\dagger \circ \Sigma^{-1} i\gamma^0 = \Sigma \circ T^{-1} \circ \Sigma^{-1} \tag{24}$$

Now, the map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$ is given by:

$$\Upsilon^\mu{}_\nu(T) i\gamma^\nu = (\Sigma \circ T^{-1} \circ \Sigma^{-1}) i\gamma^\mu (\Sigma \circ T \circ \Sigma^{-1}) \tag{25}$$

Then, all $S \in Spin^+(3,1)$ leaves the space Maj invariant:

$$S^{-1} i\gamma^\mu S = \Upsilon^\mu{}_\nu(\Sigma^{-1} \circ S \circ \Sigma) i\gamma^\nu \in Maj \tag{26}$$

Since all the products of Majorana matrices, except the identity, are traceless, then $det(S) = 1$. So, $Spin^+(3,1)$ is a subgroup of $Pin(1,3)$ and $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$. □

Definition 3.15. The discrete Pin subgroup $\Omega \subset Pin(3,1)$ is:

$$\Omega \equiv \{\pm 1, \pm i\gamma^0, \pm\gamma^0\gamma^5, \pm i\gamma^5\} \tag{27}$$

The previous lemma and the fact that Λ is continuous, implies that $Spin^+(1,3)$ is a double cover of $SO^+(3,1)$. We can check that for all $\omega \in \Omega$, $\Lambda(\pm\omega) \in \Delta$. That is, the discrete Pin subgroup is the double cover of the discrete Lorentz subgroup. Therefore, $Pin(3,1) = \Omega \ltimes Spin^+(1,3)$

Since there is a two-to-one continuous surjective group homomorphism, $Pin(3,1)$ is a double cover of $O(1,3)$, $Spin^+(3,1)$ is a double cover of $SO^+(1,3)$ and $Spin^+(1,3) \cap SU(4)$ is a double cover of $SO(3)$. We can check that $Spin^+(1,3) \cap SU(4)$ is equivalent to $SU(2)$.

3.3. Finite-dimensional representations of SL(2,C)

Note 3.16. *Since SL(2,C) is a semisimple Lie group, all its finite-dimensional (real or complex) representations are direct sums of irreducible representations.*

Note 3.17. *The finite-dimensional complex irreducible representations of SL(2,C) are labeled by (m,n), where $2m, 2n$ are natural numbers. Up to equivalence, the representation space $V_{(m,n)}$ is the tensor product of the complex vector spaces V_m^+ and V_n^-, where $V_m^\pm$ is a symmetric tensor with $2m$ Dirac spinor indexes, such that ${\gamma^5}_k v = \pm v$, where $v \in V_m^\pm$ and ${\gamma^5}_k$ is the Dirac matrix γ^5 acting on the k-th index of v.*

The group homomorphism consists in applying the same matrix of $Spin^+(1,3)$, correspondent to the $SL(2,C)$ group element we are representing, to each index of v. $V_{(0,0)}$ is equivalent to $\mathbb{C}$ and the image of the group homomorphism is the identity.

These are also projective representations of the time reversal transformation, but, for $m \neq n$, not of the parity transformation, that is, under the parity transformation, $(V_m^+ \otimes V_n^-) \to (V_m^- \otimes V_n^+)$ and under the time reversal transformation $(V_m^+ \otimes V_n^-) \to (V_m^+ \otimes V_n^-)$.

Lemma 3.18. *The finite-dimensional real irreducible representations of SL(2,C) are labeled by (m,n), where $2m, 2n$ are natural numbers and $m \geq n$. Up to equivalence, the representation space $W_{(m,n)}$ is defined for $m \neq n$ as:*

$$W_{(m,n)} \equiv \{\frac{1 + (i\gamma^5)_1 \otimes (i\gamma^5)_1}{2} w : w \in W_m \otimes W_n\}$$
$$W_{(m,m)} \equiv \{\frac{1 + (i\gamma^5)_1 \otimes (i\gamma^5)_1}{2} w : w \in (W_m)^2\}$$

where W_m is a symmetric tensor with m Majorana spinor indexes, such that $(i\gamma^5)_1(i\gamma^5)_k w = -w$, where $w \in W_m$; $(i\gamma^5)_k$ is the Majorana matrix $i\gamma^5$ acting on the k-th index of w; $(W_m)^2$ is the space of the linear combinations of the symmetrized tensor products $(u \otimes v + v \otimes u)$, for $u, v \in W_m$.

The group homomorphism consists in applying the same matrix of $Spin^+(1,3)$, correspondent to the $SL(2,C)$ group element we are representing, to each index of the tensor. In the $(0,0)$ case, $W_{(0,0)}$ is equivalent to $\mathbb{R}$ and the image of the group homomorphism is the identity.

These are also projective representations of the full Lorentz group, that is, under the parity or time reversal transformations, $(W_{m,n} \to W_{m,n})$.

Proof. For $m \neq n$ the complex irreducible representations of SL(2,C) are C-complex. The complexification of $W_{(m,n)}$ verifies $W^c_{(m,n)} = (V_m^+ \otimes V_n^-) \oplus (V_m^- \otimes V_n^+)$.

For $m = n$ the complex irreducible representations of SL(2,C) are C-real. In a Majorana basis, the C-conjugation operator of $V_{(m,m)}$, θ, is defined as $\theta(u \otimes v) \equiv v^* \otimes u^*$, where $u \in V_m^+$ and $v \in V_m^-$. We can check that there is a bijection $\alpha : W_{(m,m)} \to (V_{(m,m)})_\theta$, defined by $\alpha(w) \equiv \frac{1 - i(i\gamma^5)_1 \otimes 1}{2} w$; $\alpha^{-1}(v) \equiv v + v^*$, for $w \in W_{(m,m)}$, $v \in (V_{(m,m)})_\theta$.

Using the map from Section 2, we can check that the representations $W_{(m,n)}$, with $m \geq n$, are the unique finite-dimensional real irreducible representations of SL(2,C), up to isomorphisms.

We can check that $W^c_{(m,n)}$ is equivalent to $W^c_{(n,m)}$, therefore, invariant under the parity or time reversal transformations. □

As examples of real irreducible representations of $SL(2,C)$ we have for $(1/2,0)$ the Majorana spinor, for $(1/2,1/2)$ the linear combinations of the matrices $\{1, \gamma^0\vec{\gamma}\}$, for $(1,0)$ the linear combinations of the matrices $\{i\vec{\gamma}, \vec{\gamma}\gamma^5\}$. The group homomorphism is defined as $M(S)(u) \equiv Su$ and $M(S)(A) \equiv SAS^\dagger$, for $S \in Spin^+(1,3)$, $u \in Pinor$, $A \in \{1, \vec{\gamma}\gamma^0\}$ or $A \in \{i\vec{\gamma}, \vec{\gamma}\gamma^5\}$.

We can check that the domain of M can be extended to $Pin(1,3)$, leaving the considered vector spaces invariant. For $m = n$, we can define the "pseudo-representation" $W'_{(m,m)} \equiv \{((i\gamma^5)_1 \otimes 1)w : w \in W_{(m,m)}\}$ which is equivalent to $W_{(m,m)}$ as an $SL(2,C)$ representation, but under parity transforms with the opposite sign. As an example, the "pseudo-representation" $(1/2,1/2)$ is defined as the linear combinations of the matrices $\{i\gamma^5, i\gamma^5\vec{\gamma}\gamma^0\}$.

4. Unitary representations of the Poincare group

4.1. Bargmann-Wigner fields

Definition 4.1. Consider that $\{M_+, M_-, i\gamma^0 M_+, i\gamma^0 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces *Pinor* and *Pauli*, respectively, verifying:

$$\gamma^3\gamma^5 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm$$

Let H be a real Hilbert space. For all $h \in H$, the bijective linear map $\Theta_H : Pauli \otimes_{\mathbb{R}} H \to Pinor \otimes_{\mathbb{R}} H$ is defined by:

$$\Theta_H(h \otimes_{\mathbb{R}} P_+) = h \otimes_{\mathbb{R}} M_+,\ \Theta_H(h \otimes_{\mathbb{R}} iP_+) = h \otimes_{\mathbb{R}} i\gamma^0 M_+$$
$$\Theta_H(h \otimes_{\mathbb{R}} P_-) = h \otimes_{\mathbb{R}} M_-,\ \Theta_H(h \otimes_{\mathbb{R}} iP_-) = h \otimes_{\mathbb{R}} i\gamma^0 M_-$$

Definition 4.2. Let H_n, with $n \in \{1,2\}$, be two real Hilbert spaces and $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ be an operator. The operator $U^\Theta : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is defined as $U^\Theta \equiv \Theta_{H_2} \circ U \circ \Theta^{-1}_{H_1}$.

The space of Majorana spinors is isomorphic to the realification of the space of Pauli spinors.

Definition 4.3. The real Hilbert space $Pinor(\mathbb{X}) \equiv Pinor \otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in *Pinor*.

Definition 4.4. The complex Hilbert space $Pauli(\mathbb{X}) \equiv Pauli \otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in *Pauli*.

Note 4.5. *The Fourier Transform $\mathcal{F}_P : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{R}^3)$ is an unitary operator defined by:*

$$\mathcal{F}_P\{\psi\}(\vec{p}) \equiv \int d^n\vec{x} \frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^n}} \psi(\vec{x}),\ \psi \in Pauli(\mathbb{R}^3)$$

Where the domain of the integral is $\mathbb{R}^3$.

Note 4.6. *The inverse Fourier transform verifies:*

$$\begin{aligned}-\vec{\partial}^2\ \mathcal{F}_P^{-1}\{\psi\}(\vec{x}) &= (\mathcal{F}_P^{-1}\circ R)\{\psi\}(\vec{x})\\ i\vec{\partial}_k\ \mathcal{F}_P^{-1}\{\psi\}(\vec{x}) &= (\mathcal{F}_P^{-1}\circ R'_k)\{\psi\}(\vec{x})\end{aligned}$$

Where $\psi \in Pauli(\mathbb{R}^3)$ *and* $R, R'_k : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{R}^3)$, *with* $k \in \{1,2,3\}$, *are linear maps defined by:*

$$\begin{aligned}R\{\psi\}(\vec{p}) &\equiv (\vec{p})^2\psi(\vec{p})\\ R'_k\{\psi\}(\vec{p}) &\equiv \vec{p}_k\ \psi(\vec{p})\end{aligned}$$

Definition 4.7. Let $\vec{x} \in \mathbb{R}^3$. The spherical coordinates parametrization is:

$$\vec{x} = r(\sin(\theta)\sin(\varphi)\vec{e_1} + \sin(\theta)\sin(\varphi)\vec{e_2} + \cos(\theta)\vec{e_3})$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a fixed orthonormal basis of $\mathbb{R}^3$ and $r \in [0, +\infty[$, $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$.

Definition 4.8. Let

$$\mathbb{S}^3 \equiv \{(p, l, \mu) : p \in \mathbb{R}_{\geq 0}; l, \mu \in \mathbb{Z}; l \geq 0; -l \leq \mu \leq l\}$$

The Hilbert space $L^2(\mathbb{S}^3)$ is the real Hilbert space of real Lebesgue square integrable functions of $\mathbb{S}^3$. The internal product is:

$$< f, g >= \sum_{l=0}^{+\infty}\sum_{\mu=-l}^{l-1}\int_0^{+\infty} dp f(p,l,\mu)g(p,l,\mu),\ \ f, g \in L^2(\mathbb{S}^3)$$

Definition 4.9. The Spherical transform $\mathcal{H}_P : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_P\{\psi\}(p,l,\mu) \equiv \int r^2 dr d(\cos\theta) d\varphi \frac{2p}{\sqrt{2\pi}} j_l(pr) Y_{l\mu}(\theta,\varphi)\psi(r,\theta,\varphi),\ \ \psi \in Pauli(\mathbb{R}^3)$$

The domain of the integral is $\mathbb{R}^3$. The spherical Bessel function of the first kind j_l [59], the spherical harmonics $Y_{l\mu}$[60] and the associated Legendre functions of the first kind $P_{l\mu}$ are:

$$\begin{aligned}j_l(r) \equiv& r^l\Big(-\frac{1}{r}\frac{d}{dr}\Big)^l\frac{\sin r}{r}\\ Y_{l\mu}(\theta,\varphi) \equiv& \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^\mu(\cos\theta)e^{i\mu\varphi}\\ P_l^\mu(\xi) \equiv& \frac{(-1)^\mu}{2^l l!}(1-\xi^2)^{\mu/2}\frac{\mathrm{d}^{l+\mu}}{\mathrm{d}\xi^{l+\mu}}(\xi^2-1)^l\end{aligned}$$

Note 4.10. *Due to the properties of spherical harmonics and Bessel functions, the Spherical transform is an unitary operator. The inverse Spherical transform verifies:*

$$-\vec{\partial}^2\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1}\circ R)\{\psi\}(\vec{x})$$
$$(-x^1 i\partial_2 + x^2 i\partial_1)\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1}\circ R')\{\psi\}(\vec{x})$$

Where $\psi \in Pauli(\mathbb{S}^3)$ *and* $R, R' : Pauli(\mathbb{S}^3) \to Pauli(\mathbb{S}^3)$ *are linear maps defined by:*

$$R\{\psi\}(p,l,\mu) \equiv p^2\psi(p,l,\mu)$$
$$R'\{\psi\}(p,l,\mu) \equiv \mu\ \psi(p,l,\mu)$$

Definition 4.11. The real vector space $Pinor_j$, with $2j$ a positive integer, is the space of linear combinations of the tensor products of $2j$ Majorana spinors, symmetric on the spinor indexes. The real vector space $Pinor_0$ is the space of linear combinations of the tensor products of 2 Majorana spinors, anti-symmetric on the spinor indexes.

Definition 4.12. The real Hilbert space $Pinor_j(\mathbb{X}) \equiv Pinor_j \otimes L^2(\mathbb{X})$ is the space of square integrable functions with domain $\mathbb{X}$ and image in $Pinor_j$.

Definition 4.13. The Hilbert space $Pinor_{j,n}$, with $(j-\nu)$ an integer and $-j \leq n \leq j$ is defined as:

$$Pinor_{j,n} \equiv \{\Psi \in Pinor_j : \sum_{k=1}^{k=2j} (\gamma^0)_1 \Big(\gamma^0\gamma^3\gamma^5\Big)_k \Psi = 2n\Psi\}$$

Where $\Big(\gamma^3\gamma^5\Big)_k$ is the matrix $\gamma^3\gamma^5$ acting on the Majorana index k.

Definition 4.14. The Spherical transform $\mathcal{H}'_P : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}'_P\{\psi\}(p,l,J,\nu) \equiv \sum_{\mu=-l}^{l}\sum_{n=-j}^{j} < l\mu jn|J\nu > \Big(\mathcal{H}_P^{\Theta}\Big)_1\{\psi\}(p,l,\mu,n),\ \psi \in Pinor_j(\mathbb{R}^3)$$

$< l\mu jn|J\nu >$ are the Clebsh-Gordon coefficients and $\psi(p,l,\mu,n) \in Pinor_{j,n}$ such that $\psi(p,l,\mu) = \sum_{n=-j}^{j}\psi(p,l,\mu,n)$. $(j-n)$, $(J-\nu)$ and $(J-j)$ are integers, with $-J \leq \nu \leq J$ and $|j-l| \leq J \leq j+l$. $\Big(\mathcal{H}_P^{\Theta}\Big)_1$ is the realification of the transform $\mathcal{H}_P$, with the imaginary number replaced by the matrix $i\gamma^0$ acting on the first Majorana index of ψ.

Proposition 4.15. *Consider a unitary operator* $U : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{X})$ *such that* $U \circ H^2 = E^2 \circ U$*, where*

$$iH\{\Psi\}(\vec{x}) \equiv \Big(\gamma^0\vec{\not{\partial}} + i\gamma^0 m\Big)_k \Psi(\vec{x})$$

the Majorana matrices act on some Majorana index k; $E^2\{\Phi\}(X) \equiv E^2(X)\Phi(X)$ with $E(X) \geq m \geq 0$ a real number.

Then the operator $U' : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{X})$ is unitary, where U' is defined by:

$$U' \equiv \frac{E + UH\gamma^0U^\dagger}{\sqrt{E+m}\sqrt{2E}}$$

Proof. Note that since $E^2 = U^\dagger H^2 U$, $E = \sqrt{E^2}$ commutes with $UH\gamma^0U^\dagger$. We have that

$$(U')^\dagger(U') = \frac{E + U\gamma^0 HU^\dagger}{\sqrt{E+m}\sqrt{2E}}\frac{E + UH\gamma^0U^\dagger}{\sqrt{E+m}\sqrt{2E}} = 1$$

We also have that $(U')(U')^\dagger = 1$. Therefore, U' is unitary. □

Definition 4.16. The Fourier-Majorana transform $\mathcal{F}_M : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{R}^3)$ is an unitary operator defined by:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \int d^3\vec{x}\Big(\frac{e^{-i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\Big)_1 \prod_{k=1}^{2j}\Big(\frac{E_p + H(\vec{x})\gamma^0}{\sqrt{E_p+m}\sqrt{2E_p}}\Big)_k \Psi(\vec{x}),\ \ \Psi \in Pinor_j(\mathbb{R}^3)$$

The matrices with the index k apply on the corresponding spinor index of Ψ.

Definition 4.17. The Hankel-Majorana transform $\mathcal{H}_M : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{S}^3)$ is an unitary operator defined by:

$$\mathcal{H}_M\{\Psi\}(p,l,J,\nu) \equiv \sum_{\mu=-l}^{l}\sum_{n=-j}^{j} < l\mu jn|J\nu > \int d^3\vec{x}$$
$$\Big(\frac{2p}{\sqrt{2\pi}} j_l(pr)Y_{l\mu}(\theta,\varphi)\Big)_1 \prod_{k=1}^{2j}\Big(\frac{E_p + H(\vec{x})\gamma^0}{\sqrt{E_p+m}\sqrt{2E_p}}\Big)_k \Psi(\vec{x},n)$$

The matrices with the index k apply on the corresponding spinor index of $\Psi \in Pinor_j(\mathbb{R}^3)$. $< l\mu jn|J\nu >$ are the Clebsh-Gordon coefficients and $\Psi(\vec{x},n) \in Pinor_{j,n}$ such that $\Psi(\vec{x}) = \sum_{n=-j}^{j}\Psi(\vec{x},n)$.

The inverse Fourier-Majorana transform verifies:

$$(iH(\vec{x}))_k\ \mathcal{F}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_M^{-1} \circ R)\{\psi\}(\vec{x})$$
$$\vec{\partial}_l\ \mathcal{F}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{F}_M^{-1} \circ R')\{\psi\}(\vec{x})$$

Where $\psi \in Pinor_j(\mathbb{R}^3)$ and $R, R' : Pinor_j(\mathbb{R}^3) \to Pinor_j(\mathbb{R}^3)$ are linear maps defined by:

$$R\{\psi\}(\vec{p}) \equiv (i\gamma^0)_k E_p \psi(\vec{p})$$
$$R'\{\psi\}(\vec{p}) \equiv (i\gamma^0)_1 \vec{p}_l\ \psi(\vec{p})$$

The inverse Hankel-Majorana transform verifies:

$$(iH(\vec{x}))_k\ \mathcal{H}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_M^{-1}\circ R)\{\psi\}(\vec{x})$$
$$(-x^1\partial_2 + x^2\partial_1 + \sum_{k=1}^{2j}(i\gamma^0\gamma^3\gamma^5)_k)\ \mathcal{H}_M^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_M^{-1}\circ R')\{\psi\}(\vec{x})$$

Where $\psi \in Pinor_j(\mathbb{S}^3)$ and $R, R' : Pinor_j(\mathbb{S}^3) \to Pinor_j(\mathbb{S}^3)$ are linear maps defined by:

$$R\{\psi\}(p,l,J,\nu) \equiv (i\gamma^0)_k E_p \psi(p,l,J,\nu)$$
$$R'\{\psi\}(p,l,J,\nu) \equiv (i\gamma^0)_1 \nu\ \psi(p,l,J,\nu)$$

Definition 4.18. The space of (real) Bargmann-Wigner fields $BW_j(\mathbb{R}^3)$ is defined as:

$$BW_j \equiv \{\Psi \in Pinor_j(\mathbb{R}^3) : \left(e^{iH(\vec{x})t}\right)_k \Psi = \left(e^{iH(\vec{x})t}\right)_1 \Psi; 1 \leq k \leq 2j; t \in \mathbb{R}\}$$

Note that if the equality $e^{-iH_1 t}\Psi = e^{-iH_2 t}\Psi$ holds for all differentiable $\Psi \in H$ then for the continuous linear extension the equality holds for all $\Psi \in H$, by the bounded linear transform theorem.

Definition 4.19. The complex Hilbert space $Dirac_j(\mathbb{X}) \equiv Pinor_j(\mathbb{X}) \otimes \mathbb{C}$ is the complexification of $Pinor_j(\mathbb{X})$. The space of complex Bargmann-Wigner fields is the complexification of the space of real Bargmann-Wigner fields.

4.2. Real unitary representations of the Poincare group

Definition 4.20. The $IPin(3,1)$ group is defined as the semi-direct product $Pin(3,1) \ltimes \mathbb{R}^4$, with the group's product defined as $(A,a)(B,b) = (AB, a + \Lambda(A)b)$, for $A, B \in Pin(3,1)$ and $a, b \in \mathbb{R}^4$ and $\Lambda(A)$ is the Lorentz transformation corresponding to A.

The $ISL(2,C)$ group is isomorphic to the subgroup of $IPin(3,1)$, obtained when $Pin(3,1)$ is restricted to $Spin^+(1,3)$. The full/restricted Poincare group is the representation of the $IPin(3,1)/ISL(2,C)$ group on Lorentz vectors, defined as $\{(\Lambda(A), a) : A \in Pin(3,1), a \in \mathbb{R}^4\}$.

Definition 4.21. Given a Lorentz vector l, the little group G_l is the subgroup of $SL(2,C)$ such that for all $g \in G_l$, $g\not{l} = \not{l}g$.

Proposition 4.22. *Given a Lorentz vector l, consider a set of matrices $\alpha_k \in SL(2,C)$ verifying $\alpha_k \not{l} = \not{k}\alpha_k$. Let $H_k \equiv \{\alpha^{-1}_{\Lambda_S(k)} S \alpha_k : S \in SL(2,C)\}$. Then $H_k = G_l$.*

Proof. We can check that $H_k \subset G_l$. For any $s \in G_l$, there is $S = \alpha_{\Lambda_S(k)} s \alpha_k^{-1}$ such that $s \in H_k$. □

For $i\not{l} = i\gamma^0$, we can set $\alpha_p = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2m}}$ and $G_l = SU(2)$. For $i\not{l} = (i\gamma^0 + i\gamma^3)$, we can set $\alpha_p = B_v R_p$, where the boost velocity is $v = \frac{E_p^2 - 1}{E_p^2 + 1}$ along $\vec{p}$ and $R_p = e^{-\gamma^2\gamma^1\theta/2} e^{-\gamma^1\gamma^3\phi/2}$ is a rotation from the z axis to the axis $\frac{\vec{\not{p}}}{E_p} = (\sin\phi\cos\theta\gamma_1 + \sin\phi\sin\theta\gamma_2 + \cos\phi\gamma_3)$; $G_l = SE(2)$

$$SE(2) = \{(1 + i\gamma^5(\gamma^1 a + \gamma^2 b)(\gamma^0 + \gamma^3))e^{i\gamma^0\gamma^3\gamma^5\theta} : a, b, \theta \in \mathbb{R}\}. \tag{28}$$

Note 4.23. *The complex irreducible projective representations of the Poincare group with finite mass split into positive and negative energy representations, which are complex conjugate of each other. They are labeled by one number j, with $2j$ being a natural number. The positive energy representation spaces V_j are, up to isomorphisms, written as a symmetric tensor product of Dirac spinor fields defined on the 3-momentum space, verifying $(\gamma^0)_k\Psi_j(\vec{p}) = \Psi_j(\vec{p})$. The matrices with the index k apply in the corresponding spinor index of Ψ_j.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as a complex scalar defined on the 3-momentum space.

The representation map is given by:

$$\begin{aligned} L_S\{\Psi\}(\vec{p}) &= \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \prod_{k=1}^{2j} (\alpha^{-1}_{\Lambda(p)} S \alpha_p)_k \Psi(\vec{\Lambda}^{-1}(p)) \\ T_a\{\Psi\}(\vec{p}) &= e^{-ip\cdot a}\Psi(\vec{p}) \end{aligned}$$

Where $\alpha_p = \frac{\not{p}\gamma^0+m}{\sqrt{E_p+m}\sqrt{2m}}$.

Proposition 4.24. *The real irreducible projective representations of the Poincare group with finite mass are labeled by one number j, with $2j$ being a natural number. The representation spaces W_j are, up to isomorphisms, written as a symmetric tensor product of Majorana spinor fields defined on the 3-momentum space, verifying $(i\gamma^0)_k\Psi_j(\vec{p}) = (i\gamma^0)_1\Psi_j(\vec{p})$. The matrices with the index k apply in the corresponding spinor index of Ψ_j.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as a real scalar defined on the 3-momentum space, times the identity matrix of a Majorana spinor space.

The representation map is given by:

$$\begin{aligned} L_S\{\Psi\}(\vec{p}) &= \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \prod_{k=1}^{2j} (\alpha^{-1}_{\Lambda(p)} S \alpha_p)_k \Psi(\vec{\Lambda}^{-1}(p)) \\ T_a\{\Psi\}(\vec{p}) &= e^{-i\gamma^0 p\cdot a}\Psi(\vec{p}) \end{aligned}$$

Note 4.25. *The complex irreducible projective representations of the Poincare group with null mass and discrete helicity split into positive and negative energy representations, which are complex conjugate of each other. They are labeled by one number j, with $2j$ being an integer number. The positive energy representation spaces V_j are, up to isomorphisms, written as a symmetric tensor product of Dirac spinor fields defined on the 3-momentum space, verifying $(\gamma^0)_k\Psi_j(\vec{p}) = \Psi_j(\vec{p})$ and $(\gamma^3\gamma^5)_k\Psi_j(\vec{p}) = \pm\Psi_j(\vec{p})$, with the plus sign if j is positive and the minus sign if j is negative.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as a scalar defined on the 3-momentum space.

The representation map is given by:

$$L_S\{\Psi\}(\vec{p}) = \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \prod_{k=1}^{2j} (e^{i\gamma^0\gamma^3\gamma^5\theta})_k \Psi(\vec{\Lambda}^{-1}(p))$$
$$T_a\{\Psi\}(\vec{p}) = e^{-ip\cdot a}\Psi(\vec{p})$$

Where θ is the angle of the rotation of the little group $SE(2)$.

Note 4.26. *The real irreducible projective representations of the Poincare group with null mass and discrete helicity are labeled by one number j, with $2j$ being an integer number. The positive energy representation spaces V_j are, up to isomorphisms, written as a symmetric tensor product of Majorana spinor fields defined on the 3-momentum space, verifying $(i\gamma^0)_k\Psi_j(\vec{p}) = (i\gamma^0)_1\Psi_j(\vec{p})$ and $(\gamma^3\gamma^5)_k\Psi_j(\vec{p}) = \pm\Psi_j(\vec{p})$, with the plus sign if j is positive and the minus sign if j is negative.*

The representation space V_0 is, up to isomorphisms, written in a Majorana basis as the realification of the complex functions defined on the 3-momentum space, with the operator correspondent to the imaginary unit given by the matrix $i\gamma^0$ of a Majorana spinor space.

The representation map is given by:

$$L_S\{\Psi\}(\vec{p}) = \sqrt{\frac{(\Lambda^{-1})^0(p)}{E_p}} \prod_{k=1}^{2j} (e^{i\gamma^0\gamma^3\gamma^5\theta})_k \Psi(\vec{\Lambda}^{-1}(p))$$
$$T_a\{\Psi\}(\vec{p}) = e^{-i\gamma^0 p\cdot a}\Psi(\vec{p})$$

Where θ is the angle of the rotation of the little group $SE(2)$.

4.3. Localization

The concept of a measure is essential in physics.

Definition 4.27 (Measure)**.** A measure on a set X, is a function which assigns a non-negative real number —the *size*—to some subsets of X, such that:

1) the subsets which are assigned a size are called *measurable* sets, the complement of a measurable set and the countable union of measurable sets are measurable sets;

2) the size of the countable union of disjoint measurable sets is the sum of their sizes.

Definition 4.28. Consider a measurable space (X, M), where M is a σ-algebra of subsets of X. A projection-valued-measure, π, is a map from M to the set of self-adjoint projections on a Hilbert space H such that $\pi(X)$ is the identity operator on H and the function $<\psi, \pi(A)\psi>$, with $A \in M$ is a measure on M, for all $\psi \in H$.

Definition 4.29. Suppose now that X is a representation of G. Then, a system of imprimitivity is a pair (U, π), where π is a projection valued measure and U an unitary representation of G on the Hilbert space H, such that $U(g)\pi(A)U^{-1}(g) = \pi(gA)$.

Note 4.30 (Theorem 6.12 of [27])**.** *There is a one-to-one correspondence between the complex system of imprimitivity (U,P), based on $\mathbb{R}^3$, and the representations of $SU(2)$. The system (U,P) is equivalent to the system induced by the representation of $SU(2)$.*

Definition 4.31. A covariant system of imprimitivity is a system of imprimitivity (U,P), where U is a representation of the Poincare group and P is a projection-valued measure based on $\mathbb{R}^3$, such that for the Euclidean group $U(g)\pi(A)U^{-1}(g) = \pi(gA)$ and for the Lorentz group, for a state at time null at point $\vec{x} = 0$, $L\{\Psi\}(0) = S\Psi(0)$.

Definition 4.32. A localizable real unitary representation of the Poincare group, compatible with Poincare covariance, consists of a system of imprimitivity on R^3 for which at time null and $\vec{x} = 0$, the Lorentz transformations do not act on the space coordinates.

So, the localization of a state in $x = 0$ is a property invariant under relativistic transformations.

Proposition 4.33. *Any localizable unitary representation of the Poincare group, compatible with Poincare covariance, verifies: 1) it is self-conjugate (regardless it is real or complex); 2) it is a direct sum of irreducible representations which are massive or massless with discrete helicity.*

Proof. Since the system is a unitary Poincare representation, it is a direct sum of irreducible unitary Poincare representations and so there must be an unitary transformation U, such that:

$$\Psi(x + a) = (Ue^{-JP\cdot a}U^{-1})\{\Psi\}(x) \tag{29}$$

$$S\Psi(\Lambda(x)) = (ULU^{-1})\{\Psi\}(x) \tag{30}$$

Where J is the operator corresponding to the imaginary unit after the realification of the Poincare representation, so L commutes with J.

The system of imprimitivity is a representation of $SU(2)$, hence the operator $i\gamma^0$ is well defined. If we make a Fourier transformation, then we get that:

$$(Ue^{J\vec{P}\cdot\vec{a}}U^{-1})\{\Psi\}(\vec{p}) = e^{i\gamma^0\vec{p}\cdot\vec{a}}\Psi(\vec{p}) \tag{31}$$

Note that this equation is valid for all $\vec{p}$. The system is a direct sum of irreducible unitary Poincare representations. Then, for $m^2 < 0$ only the subspace $\vec{p}^2 \geq |m^2|$ is valid. For $p = 0$ only the subspace $\vec{p} = 0$ is valid. Since the other types of irreducible representations verify $p \neq 0$ and $m^2 \geq 0$, the complementary subspaces $\vec{p}^2 < |m^2|$ or $\vec{p} \neq 0$ cannot be representation spaces and hence the representations with $m^2 < 0$ and $p = 0$ cannot be subspaces of a localizable representation.

So we are left with $p \neq 0$ and $m^2 \geq 0$.

Given an irreducible subspace with $p \neq 0$ and $m^2 \geq 0$, M, we consider the subspace N of the representation $M \oplus M_0$ verifying $e^{iHt}\Psi = e^{iH_0t}\Psi$, where M_0 is a spin-0 representation and e^{iH_0t} is the translation in time acting on M_0. Then, $e^{iH_0(\vec{\partial})t}U = Ue^{iH_0(J\vec{P})}$. Multiplying U by $\alpha_p\sqrt{m/E_p}$ we can check that $J\Psi = i\gamma^0\Psi$ and so N is equivalent to M.

Now we define the unitary transformation $\Lambda\{\Psi\}(p) = \sqrt{\frac{E_p}{\Lambda^0(p)}}\Psi(\Lambda^{-1}(p))$. Then, we can check that $S \equiv L\Lambda^{-1}$ and it does not depend on $\vec{p}$. If we redefine $U\{\Psi\}(\vec{p}) = \alpha_p\sqrt{\frac{1}{\Lambda^0(p)}}U'\{\Psi\}(\vec{p})$, then we get that $\Lambda S\alpha_p U'\{\Psi\}(\vec{p}) = \alpha_p\Lambda Q_p U'\{\Psi\}(\vec{p})$ and so U' commutes with the Poincare representation. Since the Poincare representation is R-complex, then $U' = e^{i\gamma^0\theta}$.

So, we are left with the condition that both Q_p and $S \equiv L\Lambda^{-1}$ (which does not depend on $\vec{p}$) are irreducible. In the case of massive representations or massless representations with discrete helicity this restricts the representations of S to be of type (j,0). As for the infinite spin, the boost in the z direction for a momenta in the z direction multiplies the modulus of the translations of $SE(2)$ by E_p, which is in contradiction with the fact that $S \equiv L\Lambda^{-1}$ does not depend on $\vec{p}$.

When we go back to coordinate space, the projector on the $i\gamma^0$s can be written as an equality of the time translations which is not part of the commuting ring of the SU(2) representation and hence it does not commute with the system of imprimitivity on R^3. So, the localizable Poincare representation may not be irreducible as a Poincare representation. As for the symmetry upon exchanges of spinor indexes, it can be maintained in coordinate space only on the indexes where the Hamiltonian does not act.

A real localizable Poincare representation is an R-real system (if it is irreducible as a system including the system of imprimitivity and the Poincare representation), because the operators $i\gamma^0$s in coordinate space, although commute with the system of imprimitivity on $\mathbb{R}^3$ (including parity), do not commute with the Boosts. □

Excluding parity, a massive localizable representation is R-real, but a massless representation is R-complex—$i\gamma^5$ is the R-imaginary operator.

Corollary. *A localizable Poincare representation is an irreducible representation of the Poincare group (including parity) if and only if it is: a)real and b)massive with spin 1/2 or massless with helicity 1/2.*

Notice that the condition of irreducibility of the representation admits localized solutions—the derivative of a bump function is a bump function, so we can find bump functions in the representation space—but it does not admit a position operator—the subspace of bump functions is not closed. Hence, we can say that a particular spin 1 state is in an arbitrarily small region of space, but the measurement of the position of an arbitrary spin 1 state might make it no longer a spin 1 state.

Going to complex systems, we can check that in the massive case, the condition of irreducibility does not admit localized solutions—given a localized solution Ψ in a region of space, then the result of the application of the projection operator to Ψ is not localized in a region of space. As for the massless representation, the condition of positive energy does not admit localized solutions either—for the same region as above—, but the condition for a chiral irreducible representation does admit localized solutions. The parity operator for such a chiral irreducible representation is anti-linear.

The localizable Poincare representation is Poincare covariant because for time $x^0 = 0$ at point $\vec{x} = 0$, we have for the Lorentz group $L\{\Psi\}(0) = S\Psi(0)$. The localizable Poincare representation is compatible with causality because the propagator $\Delta(x) = 0$ for $x^2 < 0$ (space-like x), where the propagator is defined for spin or helicity 1/2 as:

$$\Delta(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}} e^{-i\gamma^0 p\cdot x} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}} \tag{32}$$

And verifies:

$$\Psi(x) = \int d^3\vec{y}\Delta(x-y)\Psi(y) \tag{33}$$

To show it we just need to do a Lorentz transformation such that $x^0 = 0$ and then show that $\Delta((0,\vec{x})) = 0$ for $\vec{x} \neq 0$.

5. Energy Positivity

5.1. Vectors

The role played by the unitary representations of the Poincare group in Quantum Theory corresponds to the role played by the unitary representations of the rotation group in Newton's mechanics. Newton's mechanics is built using the 3-D vectors as its basic unit. Some features of the non-commutative operators are already present in Newton's physics: is a vector along the direction z? The answer can be yes, no, or part of it. And the result matters, for instance, the inner product of the vector Force and the vector of displacement is Work. The vector Force can come from someone pushing a block, from the gravitational or electric field or from friction. The Newton's vector is an abstraction to represent different physical quantities, all it matters are their properties as a unitary representation of the rotation group. In the same way, a vector in Quantum Theory is a unitary representation of the Poincare group, it may be used to represent different physical objects.

5.2. Density matrix and real Hilbert space

As a consequence of Schur's lemma—related with the Frobenious theorem—, the set of normal operators commuting with an irreducible real unitary representation of a Lie group is isomorphic to the reals, to the complex numbers or to the quaternions—the irreduciblity of a group representation on a Hilbert space is intuitively the minimization of the degrees of freedom of the Hilbert space. This fact turns the study of the Hilbert spaces over the reals, the complex or the quaternions interesting for Quantum Theory. However, once we consider the density matrix in Quantum Mechanics, it is a simple exercise to show that the complex and quaternion Hilbert spaces are special cases of the real Hilbert space.

In short, the complex Hilbert space case is achieved once we postulate that there is a unitary operator J, with $J^2 = -1$, which commutes with the density matrix and all the observables. The quaternionic Hilbert space corresponds to the case where both the unitary operators J and K commute with the density matrix and all the observables, with $J^2 = K^2 = -1$ and $JK = -KJ$. Note that a complex Hilbert space is an Hilbert space over a division algebra over the real numbers, hence it has an extra layer of mathematical structure, which is dispensable because of the already existing density matrix in Quantum Mechanics.

Of course, if the postulate corresponding to the complex Hilbert space is correct, there are practical advantages in using the complex notation. However, we should be aware that using the complex notation is a practical choice, not one of fundamental nature in the formalism of Quantum Mechanics. We cannot claim that the fact that the operator J exists is a deductible consequence of the formalism of Quantum Mechanics with a complex Hilbert space. It would be the same as claiming that we can derive from Newton's formalism that the space is 3 dimensional, instead of assuming that we use 3 dimensional vectors in Newton mechanics because we postulate that the space has 3 dimensions.

5.3. Localization

The Hilbert space of non-relativistic Quantum Mechanics may be necessarily isomorphic to a complex one[61–64], but there is no proper coordinate space—compatible with covariance under relativistic transformations—in the complex Hilbert space of an irreducible unitary representation of the Poincare group [1, 23–25, 65]—the group of symmetries of the space-time. The complete definition of a Newton's 3-D vector includes the vector and the point in space where the vector is applied, but in a Quantum Mechanics' vector the application points are to be deduced from the vector itself; so it is of fundamental importance to be able to recover the coordinate space from the Hilbert space of a representation of the symmetries of space-time, otherwise the vector is of little meaning.

Choosing real representations is, in practice, choosing real Majorana spinors instead of complex scalars as the basic elements of relativistic Quantum Theory. For instance, we will see that the state of a spin-0 elementary system is a tensor field of real Majorana spinors, which only in momenta space (not in coordinate space) can be considered a complex scalar field.

5.4. Many particles

In classical mechanics, the energy of a free body of mass m is $E_p = \frac{\vec{p}^2}{2m}$. Since it is proportional to the square of the momentum, it does not make sense to talk about a negative energy. However, if we consider a box in which we can insert and remove free bodies such that in both the initial and final states the box is empty, the insertion of a body with momentum $\vec{p}$ and negative energy $E_p = -\frac{\vec{p}^2}{2m}$ to the system is equivalent to the removal of a body with momentum $-\vec{p}$ positive energy $E_p = \frac{\vec{p}^2}{2m}$, because the equations of motion are invariant under time reversal. But time reversal transforms the act of adding a body on the act of removing a body.

So, how can we say that a body was added to the system and not that the movie of the removal of a body is playing backwards? The solution is to identify a feature on the system that is also affected by time reversal and we use it as a reference. For instance, if there is one body that—we know, or we define it as if—it was added to the system, then the addition of that body will appear a removal if we are watching the movie backwards. The product of the energies of two bodies is invariant under the Galilean transformations. Note that we can only remove a body which was previously added to the box, as well as only add a body which will later be removed, to keep the box empty in both the initial and final states.

Hence, the value of any quantity which is non-invariant under the space-time symmetries— including the sign of the Energy— by itself does not mean much without something to compare to, such that we can compute an invariant quantity.

In non-relativistic Quantum Mechanics, the translations in time are given by the operator $e^{i\frac{\vec{\partial}^2}{2m}t}$—where t is time—acting on a Hilbert space of *positive* energy solutions because there is the imaginary unit—which is invariant under Lorentz transformations and anti-commutes with the time reversal transformations— that we use as our reference.

In relativistic Quantum Mechanics, the translations in time are given by the operator $e^{(\gamma^0\vec{\gamma}\cdot\vec{\partial}+i\gamma^0 m)t}$, which is real—in the Majorana basis—and does *not* leave invariant a Hilbert space of *positive* Energy solutions. In other words, if we want a coordinate space which is relativistic covariant, the imaginary unit cannot be used as our reference for the sign of the energy. We cannot say that by considering real Hilbert spaces we are creating a new problem about Energy positivity. as if we insist on a covariant coordinate space, the problem about the Energy positivity does not vanish in complex Hilbert spaces. Remember that ever since the Dirac sea (which led to the prediction of the positron) the problem about Energy positivity was always solved in a many particle description.

In a system of particles, we can compare the energy of one particle with the energy of another particle we know it is positive, like we would do in classical mechanics. If our reference particle is massive and has momentum q, then the Poincare invariant condition $p \cdot q > 0$ will be respected by a massive or massless particle with momentum p if and only if p^0 has the same sign as q^0. Instead of the momenta we can use the translations generators to define the condition for energy positivity.

6. Conclusion

The complex irreducible representations are not a generalization of the real irreducible representations, in the same way that the complex numbers are a generalization of the real numbers. There is a map, one-to-one or two-to-one and surjective up to equivalence, from the complex to the real irreducible representations of a Lie group on a Hilbert space.

We obtained all the real unitary irreducible projective representations of the Poincare group, with discrete spin, as real Bargmann-Wigner fields. For each pair of complex representations with positive/negative energy, there is one real representation. The Majorana-Fourier and Majorana-Hankel unitary transforms of the real Bargmann-Wigner fields relate the coordinate space with the linear and angular momenta spaces. The localizable unitary representations of the Poincare group (compatible with Poincare covariance and causality) are direct sums of irreducible representations with discrete spin and helicity.

We might be interested in the position as an observable. Now the question is, given an irreducible representation of the Poincare group, should the position be invariant under a $U(1)$ symmetry? Unfortunately, everyone known to the author that studied this problem assumed that it should. But the answer a priori is no it should not, because the $U(1)$ is related with the gauge symmetry which is a local symmetry and it would be useful to have a well defined notion of localization before we start considering local symmetries. For the spin one-half in a real Hilbert space, the localization problems[1, 23–27] only appear if we require that all the observables are invariant under a U(1) symmetry—usually associated with the charge—, this is related with the result in quantum field theory that causality requires the existence of anti-particles[7].

[1] T. D. Newton and E. P. Wigner, "Localized states for elementary systems," *Rev. Mod. Phys.* **21** (Jul, 1949) 400–406.

[2] E. P. Wigner, "The unreasonable effectiveness of mathematics in the natural sciences. richard courant lecture in mathematical sciences delivered at new york university, may 11, 1959," *Communications on Pure and Applied Mathematics* **13** no. 1, (1960) 1–14. `http://dx.doi.org/10.1002/cpa.3160130102`.

[3] E. P. Wigner, "On Unitary Representations of the Inhomogeneous Lorentz Group," *Annals Math.* **40** (1939) 149–204.

[4] G. Mackey, *Unitary group representations in physics, probability, and number theory*. Advanced book classics. Addison-Wesley Pub. Co., 1989.

[5] Y. Ohnuki, *Unitary Representations of the Poincaré Group and Relativistic Wave Equations*. World Scientific Publishing Company Incorporated, 1988.

[6] N. Straumann, "Unitary Representations of the inhomogeneous Lorentz Group and their Significance in Quantum Physics," *ArXiv e-prints* (Sept., 2008) , `arXiv:0809.4942 [math-ph]`.

[7] S. Weinberg, *The Quantum Theory of Fields: Modern Applications*. The Quantum Theory of Fields. Cambridge University Press, 1995.

[8] A. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*. Princeton Landmarks in mathematics and physics. Princenton University Press, 2001.

[9] S. Weinberg, "Feynman Rules for Any Spin," *Physical Review* **133** (Mar., 1964) 1318–1332.

[10] D. Gross, "Symmetry in physics: Wigner's legacy," *Phys.Today* **48N12** (1995) 46–50.

[11] J. Myrheim, "Quantum mechanics on a real hilbert space," *ArXiv e-prints* (1999) , `arXiv:quant-ph/9905037 [quant-ph]`.

[12] E. Stueckelberg, "Quantum theory in real hilbert-space," *Helvetica Physica Acta* **33** (1960) 727–752.

[13] E. Stueckelberg and M. Guenin, "Quantum theory in real hilbert-space. ii, addenda and errats," *Helvetica Physica Acta* **34** (1961) 621–628.

[14] E. Stueckelberg, M. Guenin, and C. Piron, "Quantum theory in real hilbert-space. iii, fields of the first kind (linear field operators)," *Helvetica Physica Acta* **34** (1961) 621–628.

[15] L. Accardi and A. Fedullo, "On the statistical meaning of complex numbers in quantum mechanics," *Lettere al Nuovo Cimento* **34** no. 7, (1982) 161–172.

[16] D. Hestenes, "Real Spinor Fields," *J. Math. Phys.* **8** no. 4, (1967) 798–808.

[17] N. J. Poplawski, "Spacetime and fields," *ArXiv e-prints* (Nov., 2009) , `arXiv:0911.0334 [gr-qc]`.

[18] G. Grignani and G. Nardelli, "Gravity and the poincar group," *Phys. Rev. D* **45** (Apr, 1992) 2719–2731.

[19] F. W. Hehl, Y. N. Obukhov, and D. Puetzfeld, "On Poincaré gauge theory of gravity, its equations of motion, and Gravity Probe B," *Phys.Lett.* **A377** (2013) 1775–1781, `arXiv:1304.2769 [gr-qc]`.

[20] G. Branco, L. Lavoura, and J. Silva, *CP Violation*. International series of monographs on physics. Clarendon Press, 1999.

[21] R. D. Peccei, "The Strong CP Problem and Axions," in *Axions*, M. Kuster, G. Raffelt, and B. Beltrán, eds., vol. 741 of *Lecture Notes in Physics, Berlin Springer Verlag*, pp. 3–540. 2008. `arXiv:hep-ph/0607268`.

[22] L. Canetti, M. Drewes, and M. Shaposhnikov, "Matter and antimatter in the universe," *New Journal of Physics* **14** no. 9, (Sept., 2012) 095012, `arXiv:1204.4186 [hep-ph]`.

[23] J. F. Perez and I. F. Wilde, "Localization and causality in relativistic quantum mechanics," *Phys. Rev. D* **16** (Jul, 1977) 315–317.

[24] G. C. Hegerfeldt, "Instantaneous spreading and Einstein causality in quantum theory," *Annalen der Physik* **510** (Dec., 1998) 716–725, `quant-ph/9809030`.

[25] M. Plaschke and J. Yngvason, "Massless, string localized quantum fields for any helicity," *Journal of Mathematical Physics* **53** no. 4, (Apr., 2012) 042301, `arXiv:1111.5164 [math-ph]`. section 2.4 on the self-adjoint fields.

[26] A. Wightman, "On the localizability of quantum mechanical systems," *Rev.Mod.Phys.* **34** (1962) 845–872.

[27] V. Varadarajan, *Geometry of Quantum Theory*. Springer Science+Business Media LLC, 2007.

[28] Alvarez-Gaume, Luis and Vazquez-Mozo, Miguel A., "Introductory lectures on quantum field theory," `arXiv:hep-th/0510040 [hep-th]`.

[29] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*. Graduate studies in mathematics. American Mathematical Society, 2009.

[30] B. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003.

[31] R. W. Henrichs, "On decomposition theory for unitary representations of locally compact groups," *Journal of Functional Analysis* **31** no. 1, (1979) 101 – 114.

[32] A. Oniščik and E. S. I. I. for Mathematical Physics, *Lectures On Real Semisimple Lie Algebras And Their Representations*. ESI Lectures in Mathematics and Physics. European Mathematical Society Publishing, 2004.

[33] N. Iwahori, "On real irreducible representations of Lie algebras.," *Nagoya Math. J.* **14** (1959) 59–83.

[34] D. Ramakrishnan and R. Valenza, *Fourier Analysis on Number Fields*. Graduate Texts in Mathematics. Springer, 1999.

[35] T. Bröcker and T. Dieck, *Representations of Compact Lie Groups*. Graduate Texts in Mathematics. Springer, 1985.

[36] R. Plymen and P. Robinson, *Spinors in Hilbert Space*. Cambridge Tracts in Mathematics. Cambridge University Press, 1994.

[37] M. Berg, C. De Witt-Morette, S. Gwo, and E. Kramer, "The Pin Groups in Physics," *Reviews in Mathematical Physics* **13** (2001) 953–1034, `math-ph/0012006`.

[38] I. Todorov, "Clifford Algebras and Spinors," *ArXiv e-prints* (June, 2011) , `arXiv:1106.3197 [math-ph]`.

[39] A. Aste, "A direct road to Majorana fields," *Symmetry* **2** (2010) 1776–1809, `arXiv:0806.1690 [hep-th]`. See section 5 on the Majorana spinor irrep of SL(2,C).

[40] P. B. Pal, "Dirac, Majorana, and Weyl fermions," *American Journal of Physics* **79** (May, 2011) 485–498, `arXiv:1006.1718 [hep-ph]`.

[41] H. K. Dreiner, H. E. Haber, and S. P. Martin, "Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry," *Phys.Rept.* **494** (2010) 1–196, `arXiv:0812.1594 [hep-ph]`.

[42] L. Molnár, "An algebraic approach to wigner's unitary-antiunitary theorem," *Journal of the Australian Mathematical Society (Series A)* **65** (12, 1998) 354–369.

[43] P. A. M. Dirac, "The quantum theory of the electron," *Proc. R. Soc. Lond. A* **117** no. 778, (1928) 610–624.

[44] V. Bargmann and E. P. Wigner, "Group theoretical discussion of relativistic wave equations," *Proceedings of the National Academy of Sciences* **34** no. 5, (1948) 211–223, `http://www.pnas.org/content/34/5/211.full.pdf+html`.

[45] S.-Z. Huang, T.-N. Ruan, N. Wu, and Z.-P. Zheng, "Wavefunctions for particles with arbitrary spin," *Commun.Theor.Phys.* **37** (2002) 63–74.

[46] J. P. Costella and B. H. McKellar, "The Foldy-Wouthuysen transformation," *Am.J.Phys.* **63** (1995) 1119, `arXiv:hep-ph/9503416 [hep-ph]`.

[47] L. L. Foldy and S. A. Wouthuysen, "On the dirac theory of spin 1/2 particles and its non-relativistic

limit," *Phys. Rev.* **78** (Apr, 1950) 29–36.

[48] E. Hitzer, J. Helmstetter, and R. Abłamowicz, "Square roots of -1 in real clifford algebras," in *Quaternion and Clifford Fourier Transforms and Wavelets*, E. Hitzer and S. J. Sangwine, eds., Trends in Mathematics, pp. 123–153. Springer Basel, 2013.

[49] H. De Bie, "Clifford algebras, fourier transforms, and quantum mechanics," *Mathematical Methods in the Applied Sciences* **35** no. 18, (2012) 2198–2228.

[50] T. Batard, M. Berthier, and C. Saint-Jean, "Clifford-fourier transform for color image processing," in *Geometric Algebra Computing*, E. Bayro-Corrochano and G. Scheuermann, eds., pp. 135–162. Springer London, 2010.

[51] P. A. M. Dirac, "Bakerian Lecture. The Physical Interpretation of Quantum Mechanics," *Royal Society of London Proceedings Series A* **180** (Mar., 1942) 1–40.

[52] S. De Bièvre and J. Renaud, "Massless gupta-bleuler vacuum on the (1+1)-dimensional de sitter space-time," *Phys. Rev. D* **57** (May, 1998) 6230–6241.

[53] B. Forghan, M. Takook, and A. Zarei, "Krein Regularization of QED," *Annals Phys.* **327** (2012) 2388–2401, `arXiv:1206.2796 [hep-ph]`.

[54] A. Kirillov and E. Hewitt, *Elements of the Theory of Representations.* Grundlehren Der Mathematischen Wissenschaften. Springer London, Limited, 2011.

[55] H.-D. Doebner, P. Šťovíček, and J. Tolar, "Quantization of Kinematics on Configuration Manifolds," *Reviews in Mathematical Physics* **13** (2001) 799–845, `math-ph/0104013`.

[56] G. Cassinelli, E. De Vito, and A. Levrero, "Square-integrable imprimitivity systems," *Journal of Mathematical Physics* **41** no. 7, (2000) 4833–4859.

[57] R. H. Good, "Properties of the Dirac Matrices," *Reviews of Modern Physics* **27** (Apr., 1955) 187–211.

[58] L. O'Raifeartaigh, "The dirac matrices and the signature of the metric tensor," *Helvetica Physica Acta* **34** (1961) 675–698.

[59] G. S. Adkins, "Three-dimensional Fourier transforms, integrals of spherical Bessel functions, and novel delta function identities," *ArXiv e-prints* (Feb., 2013) , `arXiv:1302.1830 [math-ph]`.

[60] R. Szmytkowski, "Recurrence and differential relations for spherical spinors," *Journal of Mathematical Chemistry* **42** no. 3, (2007) 397–413.

[61] C. N. Yang, "Square root of minus one, complex phases and Erwin Schrödinger," in *Schrödinger*, C. W. Kilmister, ed. Cambridge University Press, 1987.

[62] S. Adler, *Quaternionic Quantum Mechanics and Quantum Fields.* Oxford University Press, USA, 1995.

[63] A. Trautman, "On Complex Structures in Physics," in *On Einstein's Path: essays in honor of Engelbert Schucking*, A. Harvey, ed., p. 487. 1999. `math-ph/9809022`.

[64] G. W. Gibbons, "The emergent nature of time and the complex numbers in quantum cosmology," *ArXiv e-prints* (Nov., 2011) , `arXiv:1111.0457 [gr-qc]`.

[65] S. Farkas, Z. Kurucz, and M. Weiner, "Poincar covariance of relativistic quantum position," *International Journal of Theoretical Physics* **41** no. 1, (2002) 79–88.