# Chisholm-Caianiello-Fubini Identities for $S = 1$ Barut-Muzinich-Williams Matrices*

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**Abstract**

The formulae of the relativistic products are found $S = 1$ Barut-Muzinich-Williams matrices. They are analogs of the well-known Chisholm-Caianiello-Fubini identities. The obtained results can be useful in the higher-order calculations of the high-energy processes with $S = 1$ particles in the framework of the $2(2S+1)$ Weinberg formalism, which recently attracted attention again.
PACS numbers: 02.90.+p, 11.90.+t, 12.20.Ds

*This the modernized version of a EF-UAZ FT-95-14 unpublished preprint of 1995 of the second author. The modifications are due to the Thesis of the first author.

The attractive Weinberg $2(2S+1)$ component formalism for description of higher spin particles [1] is based on the same principles as the Dirac formalism for spin-1/2, Ref. [2]. Further developments [3, 4, 5, 6, 7] showed that many interesting things can be found therein. For instance, the connections with the modified Bargmann-Wigner formalism (BWW) [7] or the connections with the so-called Bargmann-Wightman-Wigner formalism [8, 9, 5, 6]. On the basis of the analysis of the $(S,0)\oplus(0,S)$ representation space it was found there that the intrinsic parities of boson and its antiboson can be opposite, see also [10]. "If a neutrino is identified with the self/anti-self charge-conjugate representation space, then it may be coupled with the BWW bosons to generate physics beyond the present day gauge theories.", see the above-cited references. One more hint at the possible future application of these formalisms is the tentative experimental evidence for a tensor coupling in the $\pi^- \to e^- + \tilde{\nu}_e + \gamma$ decay, for instance [11]. There exist experimental opportunities to check the existence of the "unconventional" bosons and fermions, and different types of interactions as well, beyond the Standard Model, e. g., Ref. [12].

The principal equation in this formalism is that of the "$2s$"-order in the momentum operators. The analogs of the Dirac $\gamma$-matrices have also "$2s$" vectorial indices:

$$[\gamma_{\mu_1\mu_2\ldots\mu_{2s}}\partial^{\mu_1}\partial^{\mu_2}\ldots\partial^{\mu_{2s}} + m^{2s}]\Psi(x) = 0. \tag{1}$$

The covariant-defined $\Gamma$- matrices for any spin have been introduced by Barut, Muzinich and Williams [13], see also [14, 15]. For the case of spin $S=1$ they have the following form:[1]

$$\begin{aligned}
&\Gamma^{(1)} \equiv I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\\
&\Gamma^{(2)} \equiv \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},\\
&\Gamma^{(3)}_{\alpha\beta} \equiv \gamma_{\alpha\beta} = \begin{pmatrix} 0 & -\tilde{S}^\dagger_{\alpha\beta} \\ -\tilde{S}_{\alpha\beta} & 0 \end{pmatrix},\\
&\Gamma^{(4)}_{\alpha\beta} \equiv \gamma_{4,\alpha\beta} = i\gamma_5\gamma_{\alpha\beta},\\
&\Gamma^{(5)}_{\alpha\beta} \equiv \gamma_{5,\alpha\beta} = i\left[\gamma_{\alpha\lambda},\gamma_{\beta\lambda}\right]_-,\\
&\Gamma^{(6)}_{\alpha\beta,\mu\nu} \equiv \gamma_{6,\alpha\beta,\mu\nu} = \left[\gamma_{\alpha\mu},\gamma_{\beta\nu}\right]_+ + 2\delta_{\alpha\mu}\delta_{\beta\nu} - \left[\gamma_{\alpha\nu},\gamma_{\beta\mu}\right]_+ - 2\delta_{\alpha\nu}\delta_{\beta\mu} =\\
&= -\frac{1}{12}\left[\gamma_{5,\alpha\beta},\gamma_{5,\mu\nu}\right]_+ + 4\left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\right) - 4\epsilon_{\alpha\beta\mu\nu}\gamma_5\,,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
\tilde{S}_{44} &= -I, \quad \tilde{S}_{i4} = \tilde{S}_{4i} = iS_i,\\
\tilde{S}_{ij} &= S_{ij} - \delta_{ij} = S_iS_j + S_jS_i - \delta_{ij}\,.
\end{aligned} \tag{3}$$

[1]The Eiclidean metric is used.

$S_i$ are the spin-1 matrices, and $\epsilon_{1234} = 1$. They have the simmetry properties [14]:

$$
\begin{aligned}
&\gamma_{\alpha\beta} = \gamma_{\beta\alpha}, \quad \sum_\alpha \gamma_{\alpha\alpha} = 0, \\
&\gamma_{4,\alpha\beta} = \gamma_{4,\beta\alpha}, \quad \sum_\alpha \gamma_{4,\alpha\alpha} = 0, \\
&\gamma_{5,\alpha\beta} = -\gamma_{5,\beta\alpha}, \\
&\gamma_{6,\alpha\beta,\mu\nu} = -\gamma_{6,\beta\alpha,\mu\nu}, \quad \gamma_{6,\alpha\beta,\mu\nu} = \gamma_{6,\mu\nu,\alpha\beta}, \\
&\gamma_{6,\alpha\beta,\mu\nu} + \gamma_{6,\alpha\mu,\nu\beta} + \gamma_{6,\alpha\nu,\mu\beta} = 0.
\end{aligned} \tag{4}
$$

The relativistic perturbation calculations of the processes including the $S = 1$ bosons will require the development technical methods analogous to those which have been elaborated for the fermion-fermion interaction, namely, reducing contracted products of the corresponding $\Gamma$ matrices [16, 17, 18, 19]. Our aim with this paper is to find the formulae of the relativistic scalar products like that $\gamma_{\mu\alpha} \cdots \gamma_{\beta\mu}$.

The following relations can be deduced by straightforward calculations:[2]

$$\gamma_{\mu\alpha}\gamma_{\beta\mu} = 3\delta_{\alpha\beta} - \frac{i}{2}\gamma_{5,\alpha\beta}, \tag{5}$$

$$\gamma_{\mu\alpha}\gamma_5\gamma_{\beta\mu} = -3\gamma_5\delta_{\alpha\beta} - \frac{i}{4}\epsilon_{\alpha\beta\sigma\tau}\gamma_{5,\sigma\tau}, \tag{6}$$

$$
\begin{aligned}
\gamma_{\mu\alpha}\gamma_{\sigma\tau}\gamma_{\beta\mu} &= 2\gamma_{\sigma\tau}\delta_{\alpha\beta} + \gamma_{\alpha\beta}\delta_{\sigma\tau} - \gamma_{\alpha\sigma}\delta_{\tau\beta} - \gamma_{\alpha\tau}\delta_{\sigma\beta} - \\
&- \gamma_{\beta\sigma}\delta_{\alpha\tau} - \gamma_{\beta\tau}\delta_{\alpha\sigma} - i\epsilon_{\alpha\beta\sigma\mu}\gamma_{4,\tau\mu} - i\epsilon_{\alpha\beta\tau\mu}\gamma_{4,\sigma\mu},
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\gamma_{\mu\alpha}\gamma_{4,\sigma\tau}\gamma_{\beta\mu} &= -2\gamma_{4,\sigma\tau}\delta_{\alpha\beta} - \gamma_{4,\alpha\beta}\delta_{\sigma\tau} + \gamma_{4,\alpha\sigma}\delta_{\tau\beta} + \gamma_{4,\alpha\tau}\delta_{\sigma\beta} + \\
&+ \gamma_{4,\beta\sigma}\delta_{\alpha\tau} + \gamma_{4,\beta\tau}\delta_{\alpha\sigma} - i\epsilon_{\alpha\beta\sigma\mu}\gamma_{\tau\mu} - i\epsilon_{\alpha\beta\tau\mu}\gamma_{\sigma\mu},
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\gamma_{\mu\alpha}\gamma_{5,\sigma\tau}\gamma_{\beta\mu} &= 2\gamma_{5,\sigma\tau}\delta_{\alpha\beta} + 2\gamma_{5,\alpha\sigma}\delta_{\beta\tau} + 2\gamma_{5,\tau\beta}\delta_{\alpha\sigma} - \\
&- 2\gamma_{5,\sigma\beta}\delta_{\alpha\tau} - 2\gamma_{5,\alpha\tau}\delta_{\sigma\beta} + 12i\left(\delta_{\alpha\sigma}\delta_{\tau\beta} - \delta_{\alpha\tau}\delta_{\sigma\beta}\right) + \\
&+ 12i\epsilon_{\alpha\sigma\tau\beta}\gamma_5,
\end{aligned} \tag{9}
$$

$$\gamma_{\mu\alpha}\gamma_{6,\sigma\tau,\rho\phi}\gamma_{\beta\mu} = 0. \tag{10}$$

The formulae for the $S = 1$ matrices which have been used above are presented in Appendix.

[2]We have also used the Wolfram MATEMATICA programm to check them.

# Appendix

Here we present the set of algebraic relations for $S=1$ spin matrices, cf. [20, 21]. We imply a summation on the repeated indices.

$$
\begin{aligned}
& S_k S_i S_k = S_i, && (1)\\
& S_k S_i S_j S_k = 2\delta_{ij} - S_j S_i, && (2)\\
& S_k S_i S_j S_l S_k = S_l S_i S_j + S_j S_l S_i - S_j \delta_{il}, && (3)\\
& S_k S_i S_j S_l S_m S_k = \delta_{ij}\delta_{lm} + \delta_{im}\delta_{jl} - S_m S_l S_j S_i, && (4)
\end{aligned}
$$

and

$$
\begin{aligned}
& S_{ij} S_k = \delta_{ij} S_k + \frac{1}{2}\delta_{jk} S_i + \frac{1}{2}\delta_{ik} S_j + \frac{i}{2}\left(\epsilon_{ikl} S_{jl} + \epsilon_{jkl} S_{il}\right), && (5)\\
& S_{ik} S_{jl} + S_{jl} S_{ik} = 2\delta_{ik} S_{jl} + 2\delta_{jl} S_{ik} + \left(\epsilon_{ilm}\epsilon_{jkn} - \epsilon_{ijm}\epsilon_{kln}\right) S_{mn}, && (6)\\
& S_l S_{ij} S_m = 2\delta_{ij}\delta_{lm} - \delta_{im}\delta_{jl} - \delta_{jm}\delta_{il} - \delta_{lm} S_{ij} + \\
& \qquad + \delta_{im} S_l S_j + \delta_{jm} S_l S_i + \delta_{il} S_j S_m + \delta_{jl} S_i S_m, && (7)\\
& S_l S_{ij} S_m - S_m S_{ij} S_l = \delta_{il}(S_j S_m - S_m S_j) + \delta_{jl}(S_i S_m - S_m S_i) + \\
& \qquad + \delta_{im}(S_l S_j - S_j S_l) + \delta_{jm}(S_l S_i - S_i S_l), && (8)\\
& or \qquad\qquad = -i\epsilon_{ilm} S_j - i\epsilon_{jlm} S_i - 2\delta_{ij}(S_m S_l - S_l S_m), && (9)\\
& S_i S_j S_k + S_j S_k S_i + S_k S_i S_j = S_i \delta_{jk} + S_k \delta_{ij} + S_j \delta_{ik} + \\
& \qquad + \frac{i}{4}\left(\epsilon_{ijl} S_{lk} + \epsilon_{kil} S_{jl} + \epsilon_{jkl} S_{il}\right). && (10)
\end{aligned}
$$

This set supplies the known formulae for $S=1$ spin matrices, e. g. presented in [21]:

$$
\begin{aligned}
& S_i S_j S_k + S_k S_j S_i = \delta_{ij} S_k + \delta_{jk} S_i, && (11)\\
& \tilde{S}_{ik} S_j = -\frac{i}{2}\left[\delta_{ij}\tilde{S}_{4k} + \delta_{jk}\tilde{S}_{4i} + \epsilon_{jil}\tilde{S}_{lk} + \epsilon_{jkl}\tilde{S}_{il}\right] && (12)\\
& \Sigma_i^2 = S_3^2, \ \text{no } summation, && (13)\\
& \Sigma_i \Sigma_j + \Sigma_j \Sigma_i = 2\delta_{ij} S_3^2, && (14)\\
& \Sigma_i \Sigma_j - \Sigma_j \Sigma_i = 2i\epsilon_{ijk}\Sigma_k, && (15)
\end{aligned}
$$

where

$$
\Sigma_1 \equiv S_1^2 - S_2^2, \quad \Sigma_2 \equiv \tilde{S}_{12} = S_1 S_2 + S_2 S_1, \quad \Sigma_3 \equiv S_3. \qquad (16)
$$

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