Self/anti-self charge conjugate states in the helicity basis

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Abstract. We construct self/anti-self charge conjugate (Majorana-like) states for the $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz group, and their analogs for higher spins within the quantum field theory. The problem of the basis rotations and that of the selection of phases in the Dirac-like and Majorana-like field operators are considered. The discrete symmetries properties (P, C, T) are studied. Particular attention has been paid to the question of (anti)commutation of the Charge conjugation operator and the Parity in the helicity basis. Dynamical equations have also been presented. In the $(1/2,0)\oplus(0,1/2)$ representation they obey the Dirac-like equation with eight components, which has been first introduced by Markov. Thus, the Fock space for corresponding quantum fields is doubled (as shown by Ziino). The chirality and the helicity (two concepts which are frequently confused in the literature) for Dirac and Majorana states have been discussed.

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The *self/anti-self* charge-conjugate 4-spinors have been introduced in, e. g., Refs. [1, 2, 3, 4] in the coordinate representation and in the momentum representation. Later, these spinors have been studied in Refs. [5, 6, 7, 8, 9]. The authors found corresponding dynamical equations, gauge transformations and other specific features of them. On using $C=-e^{i\theta}\gamma^2\mathscr{K}$ [1] we define the *self/anti-self* charge-conjugate 4-spinors in the momentum space $C\lambda^{S,A}(\mathbf{p})=\pm\lambda^{S,A}(\mathbf{p})\,, C\rho^{S,A}(\mathbf{p})=\pm\rho^{S,A}(\mathbf{p})$. Such definitions of 4-spinors differ, of course, from the original Majorana definition in the x-representation $\nu(x)=\frac{1}{\sqrt{2}}(\Psi_{Dirac}(x)+\Psi^c_{Dirac}(x))$, $C\nu(x)=\nu(x)$ that represents the positive real $C-$ parity only. Nevertheless, both definitions are connected each other, that permits to call them “Majorana-like". In the spinorial basis with the appropriate normalization (“mass dimension") the explicit forms of the 4-spinors of the second kind $\lambda^{S,A}_{\uparrow\downarrow}(\mathbf{p})$ (and the corresponding expressions for $\rho^{S,A}_{\uparrow\downarrow}(\mathbf{p})$) have been given, e. g., in Ref. [6]:

$$\lambda^S_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} ip_l\\ i(p^-+m)\\ p^-+m\\ -p_r\end{pmatrix},\ \lambda^S_\downarrow(\mathbf{p})=\frac{1}{2\sqrt{E+m}}\begin{pmatrix}-i(p^++m)\\ -ip_r\\ -p_l\\ (p^++m)\end{pmatrix}, \tag{1}$$

$$\lambda^A_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} -ip_l\\ -i(p^-+m)\\ (p^-+m)\\ -p_r\end{pmatrix},\ \lambda^A_\downarrow(\mathbf{p})=\frac{1}{2\sqrt{E+m}}\begin{pmatrix}i(p^++m)\\ ip_r\\ -p_l\\ (p^++m)\end{pmatrix}, \tag{2}$$

and

$$\rho^S_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p^++m\\ p_r\\ ip_l\\ -i(p^++m)\end{pmatrix},\ \rho^S_\downarrow(\mathbf{p})=\frac{1}{2\sqrt{E+m}}\begin{pmatrix}p_l\\ (p^-+m)\\ i(p^-+m)\\ -ip_r\end{pmatrix}, \tag{3}$$

$$\rho^A_\uparrow(\mathbf{p}) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p^++m\\ p_r\\ -ip_l\\ i(p^++m)\end{pmatrix},\ \rho^A_\downarrow(\mathbf{p})=\frac{1}{2\sqrt{E+m}}\begin{pmatrix}p_l\\ (p^-+m)\\ -i(p^-+m)\\ ip_r\end{pmatrix}, \tag{4}$$

where $p_{r,l}=p_x\pm ip_y$ and $p^\pm=E\pm p_z$. As claimed in [4], $\lambda-$ and $\rho-$ 4-spinors are *not* the eigenspinors of the helicity (h). Moreover, $\lambda-$ and $\rho-$ are *not* the eigenspinors of the parity, as opposed to the Dirac case. The authors claimed

[1] K is the complex conjugation operation. γ^2 is the matrix of the set of γ matrices in the Weyl basis.

in the cited works that indices $\uparrow\downarrow$ should be referred to the chiral helicity quantum number introduced in the 60s, $\eta = -\gamma^5 h$, Ref. [10]. The normalizations of the spinors have also been given in the previous works.

We can introduce the quantum fields $\Psi(x)$, Ref. [23], now composed of the $\lambda-$ and $\rho-$ spinors. For instance, $\lambda^S(\mathbf{p})\exp(-ip\cdot x)$ and $\rho^A(\mathbf{p})\exp(-ip\cdot x)$ correspond to the "positive-energy solutions", and $\lambda^A(\mathbf{p})\exp(+ip\cdot x)$ and $\rho^S(\mathbf{p})\exp(+ip\cdot x)$, to "the negative-energy solutions". In this case, the dynamical coordinate-space equations are:

$$i\gamma^\mu\partial_\mu\lambda^S(x) - m\rho^A(x) = 0\,, \; i\gamma^\mu\partial_\mu\rho^A(x) - m\lambda^S(x) = 0\,, \tag{5}$$

$$i\gamma^\mu\partial_\mu\lambda^A(x) + m\rho^S(x) = 0\,, \; i\gamma^\mu\partial_\mu\rho^S(x) + m\lambda^A(x) = 0\,. \tag{6}$$

These are *not* the 4-component Dirac equations. Similar formulations have been presented by M. Markov [11], and by A. Barut and G. Ziino [3]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [12], who first presented the theory in the 2-dimensional representation of the inversion group (later called as the Bargmann-Wightman-Wigner-type quantum field theory). The Lagrangian contains all 4-spinors ($\lambda^{S,A}$ and $\rho^{S,A}$). The connection with the Dirac spinors has been found [6, 8].

It was shown [6] that the covariant derivative (and, hence, the interaction) can be introduced in this construct in the following way $\partial_\mu \rightarrow \nabla_\mu = \partial_\mu - ig\text{Ł}^5 A_\mu$, where $\text{Ł}^5 = \mathrm{diag}(\gamma^5 \quad -\gamma^5)$. Next, the Majorana-like field operator ($b^\dagger \equiv a^\dagger$)[2] admits additional phase (and, in general, normalization) transformations $\nu^{ML\,\prime}(x^\mu) = [c_0 + i(\boldsymbol{\tau}\cdot\mathbf{c})]\,\nu^{ML\,\dagger}(x^\mu)$, where c_α are arbitrary parameters, $\tau-$ matrices are the analogs of the Pauli matrices, which are defined over the field of the 2×2 matrices. Recently, the interest to these models raised again [9, 13]. We showed that the helicity, chiral helicity and chirality operators are connected by unitary transformations, see below. The first one is

$$\mathscr{U}_1 = \begin{pmatrix} 1 & p_l/(p+p_3) \\ -p_r/(p+p_3) & 1 \end{pmatrix}, \; U_1 = \begin{pmatrix} \mathscr{U}_1 & 0 \\ 0 & \mathscr{U}_1 \end{pmatrix}, \; U_1 \hat{h} U_1^{-1} = |\frac{\mathbf{n}}{2}| \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \tag{7}$$

with subsequent applications of the matrices

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; U_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

The concept of the doubling of the Fock space has been developed in Ziino works (cf. [12, 14]).

Several formalisms have been used for higher spin fields, e. g., [15, 16]. Ahluwalia *et al.* write [4]: "For spin-1 ... the requirement of self/anti-self charge conjugacy *cannot* be satisfied. That is, there does not exist a ζ [the phase factors between right- and left- 3-"spinors"] that can satisfy the spin-1 ... requirement $S^c_{[1]}\,\lambda(p^\mu) = \pm\lambda(p^\mu)\,, \quad S^c_{[1]}\,\rho(p^\mu) = \pm\rho(p^\mu)$ ". This is due to the fact that $C^2 = -1$ within this definition of the charge conjugation operator. "We find, however, that the requirement of self/anti-self conjugacy under charge conjugation can be replaced by the requirement of self/anti-self conjugacy under the operation of $\Gamma^5\,S^c_{[1]}$" [precisely, which was used by Weinberg in Ref. [16] due to the different choice of the equation for the negative-frequency 6-"bispinors"]. The covariant equations for the $\lambda-$ and $\rho-$ objects in the $(1,0)\oplus(0,1)$ representation have been obtained in Ref. [6]. For instance,[3]

$$\gamma_{\mu\nu}p^\mu p^\nu \lambda^S_{\uparrow\downarrow\rightarrow}(p^\mu) - m^2\rho^S_{\uparrow\downarrow\rightarrow}(p^\mu) = 0, \gamma_{\mu\nu}p^\mu p^\nu \lambda^A_{\uparrow\downarrow\rightarrow}(p^\mu) - m^2\rho^A_{\uparrow\downarrow\rightarrow}(p^\mu) = 0, \tag{9}$$

$$\gamma_{\mu\nu}p^\mu p^\nu \rho^S_{\uparrow\downarrow\rightarrow}(p^\mu) - m^2\lambda^S_{\uparrow\downarrow\rightarrow}(p^\mu) = 0, \gamma_{\mu\nu}p^\mu p^\nu \rho^A_{\uparrow\downarrow\rightarrow}(p^\mu) - m^2\lambda^A_{\uparrow\downarrow\rightarrow}(p^\mu) = 0\,. \tag{10}$$

Next, R. da Rocha *et al.* write [13] that in the Dirac case $\{C,P\}_+ = 0$. On the other hand, it follows that $[C,P]_- = 0$ when acting on the Majorana-like states. In the previous works of the 50s-60s, see, e. g., Ref. [17], it is the latter case which has been attributed to the $Q=0$ eigenvalues (the truly neutral particles). You may compare these results with those of [4, 7, 18], where the same statements have been done on the quantum-field level even at the earlier time comparing with Ref. [13]. The acronym "ELKO" is the synonym for the self/anti-self charge conjugated states (the Majorana-like spinors). It is easy to find the correspondence between "the new notation", Refs. [19, 13] and the

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[2] The creation operators for a particle and an anti-particle are identical here.
[3] $\gamma_{\mu\nu}$ are the covariantly-defined matrices of this representation space.

previous one. Namely, $\lambda_{\uparrow}^{S,A} \to \lambda_{-,+}^{S,A}$, $\lambda_{\downarrow}^{S,A} \to \lambda_{+,-}^{S,A}$. However, the difference is also in the choice of the basis for the 2-spinors. As in Ref. [20], Ahluwalia, Grumiller and da Rocha have chosen the well-known helicity basis (cf. [21, 22]). I have shown that the helicity-basis 4-spinors satisfy the same Dirac equation, the parity matrix can be defined in the similar fashion as in the spinorial basis, but the helicity-basis 4-spinors are *not* the eigenspinors of the parity, Ref. [23].

Usually, the 2-spinors are parametrized in the following way, cf. [21, 22], in the helicity basis:[4]

$$\phi_L^+ = Ne^{i\theta_1}\begin{pmatrix}\cos(\vartheta/2)e^{-i\varphi/2}\\ \sin(\vartheta/2)e^{+i\varphi/2}\end{pmatrix},\ \phi_L^+ = Ne^{i\theta_2}\begin{pmatrix}\sin(\vartheta/2)e^{-i\varphi/2}\\ -\cos(\vartheta/2)e^{+i\varphi/2}\end{pmatrix}. \tag{11}$$

Next, their analogs for spin 1 are:

$$\phi_L^+ = N\,e^{i\theta_1}\begin{pmatrix}\frac{1}{2}(1+\cos\vartheta)e^{-i\varphi}\\ \sqrt{\frac{1}{2}}\sin\vartheta\\ \frac{1}{2}(1-\cos\vartheta)e^{+i\varphi}\end{pmatrix}\quad ,\ \phi_L^- = N\,e^{i\theta_2}\begin{pmatrix}-\frac{1}{2}(1-\cos\vartheta)e^{-i\varphi}\\ \sqrt{\frac{1}{2}}\sin\vartheta\\ -\frac{1}{2}(1+\cos\vartheta)e^{+i\varphi}\end{pmatrix}, \tag{12}$$

$$\phi_L^0 = N\,e^{i\theta_0}\begin{pmatrix}-\sqrt{\frac{1}{2}}\sin\vartheta\, e^{-i\varphi}\\ \cos\vartheta\\ \sqrt{\frac{1}{2}}\sin\vartheta\, e^{+i\varphi}\end{pmatrix}, \tag{13}$$

In this basis, the parity transformation for 2-spinors leads to the properties (provided that the overall phases do *not* transform):

$$\begin{aligned} R\phi_L^-(\mathbf{0}) &= -ie^{i(\theta_2-\theta_1)}\phi_L^+(\mathbf{0})\,, R\phi_L^+(\mathbf{0}) = -ie^{i(\theta_1-\theta_2)}\phi_L^-(\mathbf{0}), && (14)\\ R\Theta(\phi_L^-(\mathbf{0}))^* &= -ie^{-2i\theta_2}\phi_L^-(\mathbf{0})\,, R\Theta(\phi_L^+(\mathbf{0}))^* = +ie^{-2i\theta_1}\phi_L^+(\mathbf{0}), && (15)\end{aligned}$$

$\Theta = -i\sigma_y$ is the Wigner matrix. In the $(1,0)\oplus(0,1)$ representation the situation is similar (see the formulas (31) in Ref. [22]). The analogs of (14,15) are:

$$\begin{aligned} R\phi_L^-(\mathbf{0}) &= +e^{i(\theta_2-\theta_1)}\phi_L^+(\mathbf{0})\,, R\phi_L^+(\mathbf{0}) = +e^{i(\theta_1-\theta_2)}\phi_L^-(\mathbf{0}), && (16)\\ R\Theta(\phi_L^-(\mathbf{0}))^* &= -e^{-2i\theta_2}\phi_L^-(\mathbf{0})\,, R\Theta(\phi_L^+(\mathbf{0}))^* = -e^{-2i\theta_1}\phi_L^+(\mathbf{0}), && (17)\\ R\phi_L^0(\mathbf{0}) &= -\phi_L^0(\mathbf{0})\,, R\Theta(\phi_L^0(\mathbf{0}))^* = +e^{-2i\theta_0}\phi_L^0(\mathbf{0}). && (18)\end{aligned}$$

This opposes to the spinorial basis, where, for instance: $R\phi_L^{\pm}(\mathbf{0}) = \phi_L^{\pm}(\mathbf{0})$.

Correspondingly, if we choose the Parity operator as $P = i\gamma^0 R$, cf. [1, 19], we have in the $(1/2,0)\oplus(0,1/2)$ representation:

$$\begin{aligned} P\lambda_{\uparrow}^S(\mathbf{0}) &= -i\cos(2\theta_1)\lambda_{\uparrow}^A(\mathbf{0}) - \sin(2\theta_1)\lambda_{\uparrow}^S(\mathbf{0})\,, && (19)\\ P\lambda_{\downarrow}^S(\mathbf{0}) &= i\cos(2\theta_2)\lambda_{\downarrow}^A(\mathbf{0}) + \sin(2\theta_2)\lambda_{\downarrow}^S(\mathbf{0})\,, && (20)\end{aligned}$$

and, in the $(1,0)\oplus(0,1)$ representation:

$$\begin{aligned} P\lambda_{\uparrow}^S(\mathbf{0}) &= -i\cos(2\theta_1)\lambda_{\uparrow}^S(\mathbf{0}) - \sin(2\theta_1)\lambda_{\uparrow}^A(\mathbf{0})\,, && (21)\\ P\lambda_{\rightarrow}^S(\mathbf{0}) &= i\cos(2\theta_0)\lambda_{\rightarrow}^A(\mathbf{0}) + \sin(2\theta_0)\lambda_{\rightarrow}^S(\mathbf{0}) && (22)\\ P\lambda_{\downarrow}^S(\mathbf{0}) &= -i\cos(2\theta_2)\lambda_{\downarrow}^S(\mathbf{0}) - \sin(2\theta_2)\lambda_{\downarrow}^A(\mathbf{0})\,. && (23)\end{aligned}$$

Analogous equations can be derived for $\lambda_{\uparrow\downarrow}^A$.

Further calculations are straightforward in both $(1/2,0)\oplus(0,1/2)$ and $(1,0)\oplus(0,1)$ representations. And, indeed, they can lead to $[C,P]_- = 0$ when acting on the "ELKO" states, provided that we take into account the phase factor in the definition of the $P-$ operator.

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[4] Nevertheless, this parametrization immediately implies some controversies in the properties of the overall and relative phases with respect to the space inversion.

We presented a review of the formalism in the momentum-space Majorana-like particles in the $(S,0)\oplus(0,S)$ representation of the Lorentz Group. The $\lambda-$ and $\rho-$ 4-spinors satisfy the 8- component analogue of the Dirac equation. Apart, they have different gauge transformations comparing with the usual Dirac 4-spinors. Their helicity, chirality and chiral helicity properties have been investigated in detail. At the same time, we showed that the Majorana-like 4-spinors can be obtained by the rotation of the spin-parity basis. Meanwhile, several authors have claimed that the physical results would be different on using calculations with these Majorana-like spinors. However, several statements made by other researchers concerning with chirality, helicity, chiral helicity, and C, P (anti)commutation should not be considered to be true until the time when experiments confirm them. Usually, it is considered that the rotations (unitary transformations) have *no* any physical consequences on the level of the Lorentz-covariant theories. Next, we discussed the $[C,P]_{\pm}=0$ dilemma for neutral and charged particles on using the analysis of the basis rotations and phases.

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