Gram-Schmidt Orthogonalization in Geometric Algebra

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Let $\mathbb{R}_n$ be the geometric algebra of the real linear space $\mathbb{R}^n$. Let $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r\}$, $r \le n$ be a set of r linearly independent vectors.[2] Then the $r-$multivector $A_r = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \dots \wedge \mathbf{a}_r$ will necessarily be different from zero: $A_r \neq 0$, and vice versa, because the $r-$volume defined by A_r will be different from zero.

The linearly independent set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r\}, r \le n$ can be systematically orthogonalized. We construct the graded sequence of multivectors

$$A_0 = 1,\ \ A_1 = \mathbf{a}_1,\ \ A_2 = \mathbf{a}_1 \wedge \mathbf{a}_2, \dots,\ \ A_r = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \dots \wedge \mathbf{a}_r.$$

We can use $A_0, A_1, \dots, A_r$ in order to define a new set of vectors

$$\mathbf{c}_k = \widetilde{A}_{k-1} A_k = \widetilde{A}_{k-1} \rfloor A_k,\ \ k = 1, \dots, r,$$

where the tilde over A_{k-1} means to reverse the order of vector factors. E.g. $\widetilde{A}_2 = \mathbf{a}_2 \wedge \mathbf{a}_1 = -A_2$, $\widetilde{A}_3 = \mathbf{a}_3 \wedge \mathbf{a}_2 \wedge \mathbf{a}_1 = -A_3$, etc. The geometric product $\widetilde{A}_{k-1} A_k$ can be replaced by the left contraction, because by construction, the $(k-1)-$subspace defined by A_{k-1} is fully contained in the $k-$subspace defined by A_k. Let us remember the meaning of the left contraction: $\widetilde{A}_{k-1} \rfloor A_k$ results in an $k-(k-1) = 1$ dimensional subspace of the $k-$subspace defined by A_k, which is orthogonal to the $(k-1)-$subspace defined by A_{k-1}. Therefore the set of r vectors $\mathbf{c}_k$, $k = 1, \dots, r$ must be an orthogonal set, and span the $r-$subspace defined by A_r. The last property, can be easily verified by calculating the geometric product of all $\mathbf{c}_k$, $k = 1, \dots, r$:

$$\begin{aligned}
&\mathbf{c}_1 \mathbf{c}_2 \dots \mathbf{c}_r \\
&= 1 A_1 \widetilde{A}_1 A_2 \widetilde{A}_2 \dots A_{r-1} \widetilde{A}_{r-1} A_r \\
&= A_1 * \widetilde{A}_1 A_2 * \widetilde{A}_2 \dots A_{r-1} * \widetilde{A}_{r-1} A_r \\
&= |A_1|^2 |A_2|^2 \dots |A_{r-1}|^2 A_r,
\end{aligned}$$

where the symbol $(*)$ signifies the scalar product, i.e. the scalar part of the geometric product of two multivectors, and $|A|$ is the positive scalar magnitude of the multivector A defined by $|A|^2 = \widetilde{A} * A$. Obviously the product $\mathbf{c}_1 \mathbf{c}_2 \dots \mathbf{c}_r$ constitutes a factorization of A_r into a product of orthogonal vectors.

This result fully corresponds to the conventional Gram-Schmidt orthogonalization process in linear algebra.

References

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[3] L. Dorst, The Inner Products of Geometric Algebra, in L. Dorst et. al. (eds.), Applications of Geometric Algebra in Computer Science and Engineering, Birkhaeuser, Basel, 2002.