The use of quadratic forms in geometric algebra

Eckhard M.S. Hitzer

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1 Definition of geometric algebra with quadratic form

Quadratic forms allow a basis-free definition[1] of Clifford geometric algebra.[1, 7]

Definition. An associative algebra over a field F with unity 1 is the Clifford geometric algebra $\mathcal{G}(\mathcal{Q})$ of a non-degenerate *quadratic form* $\mathcal{Q}$ on a linear space V over the field F, if $\mathcal{G}(\mathcal{Q})$ contains the linear space V itself and the field $F = F \cdot 1$ as distinct subspaces so that

- $\mathbf{x}^2 = \mathcal{Q}(\mathbf{x}) \;\; \forall \mathbf{x} \in V$
- V generates $\mathcal{G}(\mathcal{Q})$ as a algebra over the field F
- $\mathcal{G}(\mathcal{Q})$ is not generated by any proper subspace of V.

The third condition guarantees the universal property for odd dimensions n with signatures 1 mod 4 of the quadratic form $\mathcal{Q}$, and the dimension of $\mathcal{G}(\mathcal{Q})$ to be 2^n. In general the quadratic form $\mathcal{Q}$ is associated to the following symmetric bilinear form:

$$\begin{aligned}\mathbf{x} * \mathbf{y} &\equiv \frac{1}{2}[\mathcal{Q}(\mathbf{x}+\mathbf{y}) - \mathcal{Q}(\mathbf{x}) - \mathcal{Q}(\mathbf{y})] = \frac{1}{2}[(\mathbf{x}+\mathbf{y})^2 - \mathbf{x}^2 - \mathbf{y}^2] \\ &= \frac{1}{2}[\mathbf{x}\mathbf{y} + \mathbf{x}\mathbf{y}] \in \mathbb{R} \;\; \forall \mathbf{x}, \mathbf{y} \in V.\end{aligned}$$

As an example let us look at the real ($F = \mathbb{R}$) linear quadratic space $V = \mathbb{R}^{p,q}$, which generates the geometric algebra $\mathbb{R}_{p,q} = \mathcal{G}(\mathbb{R}^{p,q})$ of signature $p - q$. With the help of $\mathcal{Q}$ (or the scalar $*$ product[5]), we can introduce an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^{p,q}$.

The quadratic form applied to the orthonormal basis vectors gives explicitely

$$\mathbf{e}_i^2 = 1, \;\; \forall\, 1 \leq i \leq p, \qquad \mathbf{e}_i^2 = -1, \;\; \forall\, p < i \leq n.$$

[1] The following definition is not the only one using quadratic forms. C. Chevalley gave another definition in terms of a tensor algebra devided by an ideal. For the generation of this ideal a quadratic form becomes again essential ([2], chapter 11).

For distinct *orthonormal* basis vectors we must have

$$\mathbf{e}_i * \mathbf{e}_j = \frac{1}{2}[\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i] = 0 \;\; \forall\, 1 \le i < j \le n.$$

This yields

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \;\; \forall\, 1 \le i < j \le n,$$

i.e. the geometric product of orthogonal basis vectors is antisymmetric - exactly like the exterior product of Grassmann. New for Clifford's geometric product $\mathbf{xy}$ of vectors $\mathbf{x}$ and $\mathbf{y}$ is the symmetric scalar part $\mathbf{x} * \mathbf{y} \equiv \frac{1}{2}[\mathbf{xy} + \mathbf{xy}]$.

2 Examples of quadratic forms and associated geometric algebras

A number of examples in the chapter "Basic Axioms of Geometric Algebra". This were:

- The geometric algebra $\mathbb{R}_2 = \mathbb{R}_{2,0}$ of the Euclidean plane R^2 with $n = p = 2, q = 0$ and signature $p - q = 2 - 0 = 2$. The even subalgebra $\mathbb{R}_2^+$ of $\mathbb{R}_2$ was found to be isomorphic to the complex numbers.

- The geometric algebra $\mathbb{R}_3 = \mathbb{R}_{3,0}$ of the three-dimensional Euclidean space $\mathbb{R}^3$ with $n = p = 3, q = 0$ and signature 3. Its even subalgebra $\mathbb{R}_3^+$ was found to be isomorphic to Hamilton's famous quaternion algebra, which allows the most elegant spinorial description of rotations.

- The geometric algebra $\mathbb{R}_{3,1}$ of the four-dimensional ($n = 4$) Minkowski space $\mathbb{R}^{3,1}$, whose quadratic form is characterized by $p = 3, q = 1$ and signature $p - q = 2$. This particular geometric algebra is named Space Time Algebra (STA), because it is of great use for uniformly describing physics. Its even part $\mathbb{R}_{3,1}^+$ is isomorphic to the geometric algebra of Euclidean space. This isomorphism depends on the particular choice of the (time) vector with negative square and singles out a laboratory frame for measurements.

- The geometric algebra $\mathbb{R}_{4,1}$ of the five-dimensional ($n = 5$) quadratic linear space $\mathbb{R}^{4,1}$, whose quadratic form is characterized by $p = 4, q = 1$ with signature $p - q = 3$. $\mathbb{R}_{4,1}$ is very versatile as an algebraic model for three-dimensional Euclidean space, called homogeneous or conformal model of Euclidean space. Simple multivectors in $\mathbb{R}_{4,1}$ are in one-to-one correspondence with points, lines, planes, circles and spheres in Euclidean space. The operations of translation, rotation, join and intersection of these elements all become simple exception-free, monomial multivector product expressions.

3 Geometric algebras with degenerate quadratic forms

3.1 A new interpretation of the geommetric algebra of the Minkowski plane

Given a certain basis of n linearly independent vectors we have the freedom replace it by another set of n linearly independent vectors, performing a basis transformation. The expression for the quadratic form in the new basis depends on this basis transformation.

Let us look at the non-trivial example of the Minkowski plane $\mathbb{R}^{1,1}$ and its geometric algebra $\mathbb{R}_{1,1}$, with $n = 2$, $p = q = 1$, and signature $p-q = 1-1 = 0$. In the *orthonormal* basis $\{\mathbf{e}_0, \mathbf{e}_1\}$ the quadratic form relationship $\mathbf{x}^2 = \mathcal{Q}(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^{1,1}$ can be expressed as

$$\mathbf{e}_0^2 = -1, \mathbf{e}_1^2 = 1, \mathbf{e}_0\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_0 \equiv N,$$

and the orthonormality relation

$$\mathbf{e}_0 * \mathbf{e}_1 = 0,$$

where N is just another name for the bivector pseudoscalar I_2 of the geometric algebra $\mathbb{R}_{1,1}$ of the Minkowski plane.

A particular choice of a new basis, is the *null* basis $\{\mathbf{n}, \bar{\mathbf{n}}\}$ defined by the basis transformation[2]

$$\bar{\mathbf{n}} = \frac{1}{2}(\mathbf{e}_0 + \mathbf{e}_1), \;\; \mathbf{n} = \frac{1}{2}(\mathbf{e}_0 - \mathbf{e}_1).$$

Because the new basis is no longer orthonormal, the same quadratic form relationship $\mathbf{x}^2 = \mathcal{Q}(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^{1,1}$ looks now a little unfamiliar

$$\mathbf{n}^2 = \bar{\mathbf{n}}^2 = 0, \;\; \mathbf{n} \wedge \bar{\mathbf{n}} = \frac{1}{2}N, \text{ and } \;\; \mathbf{n} * \bar{\mathbf{n}} = \frac{1}{2}[\mathbf{n}\bar{\mathbf{n}} + \bar{\mathbf{n}}\mathbf{n}] = -\frac{1}{2}.$$

The last relationship $\mathbf{n} * \bar{\mathbf{n}} = \frac{1}{2}[\mathbf{n}\bar{\mathbf{n}} + \bar{\mathbf{n}}\mathbf{n}] = -\frac{1}{2}$ can be interpreted as a "duality" between the two one-dimensional spaces V^1 and V^{1*} spanned by the vectors $\mathbf{n}$ and $\bar{\mathbf{n}}$, respectively.[6] The duality condition being, that V^{1*} is the space of unique scalar-valued mappings, such that

$$\forall \mathbf{x} = x\mathbf{n} \in V^1, x \in \mathbb{R} \;\; \exists_1 \bar{\mathbf{x}} = \frac{1}{x}\bar{\mathbf{n}} \in V^{1*} : \mathbf{x} * \bar{\mathbf{x}} = -\frac{1}{2}.$$

The inverse basis transformation is given by

$$\mathbf{e}_0 = \bar{\mathbf{n}} + \mathbf{n}, \;\; \mathbf{e}_1 = \bar{\mathbf{n}} - \mathbf{n}.$$

[2] The factor $\frac{1}{2}$ in the definition of $\bar{\mathbf{n}}$ is the only major difference to the treatment in section 5 of the chapter on "Basic Axioms of Geometric Algebra".

3.2 Generalizing to the geometric mother algebra with p=q=n

We can now generalize the interpretation of the last subsection. We extend the one-dimensional space V^1 to an $n-$dimensional space V^n with an orthogonal null-vector basis $\{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_n\}$ and the *dual* space V^{1*} to the $n-$dimensional space V^{n*} with the orthogonal null-vector basis $\{\bar{\mathbf{n}}_1, \bar{\mathbf{n}}_2, \ldots, \bar{\mathbf{n}}_n\}$. The anaolgous duality conditions on the two basis are now that the basis vectors $\mathbf{n}_i$ are related to dual map vectors $\bar{\mathbf{n}}_j$ by

$$\mathbf{n}_i * \bar{\mathbf{n}}_j = -\frac{1}{2}\delta_{i,j} \;\; \forall\, i,j = 1,2,\ldots,n.$$

$\delta_{i,j}$ is the usual Kronecker delta symbol. Assuming that all vectors $\mathbf{n}_i$ and $\bar{\mathbf{n}}_j$ are linearly independent, the direct sum of the two vector spaces V^n and V^{n*} is a $2n-$dimensional vector space

$$\mathbb{R}^{n,n} = V^n \oplus V^{n*}.$$

The goemetric algebra of $\mathbb{R}^{n,n}$ is denoted by

$$\mathbb{R}_{n,n} = \mathcal{G}(\mathbb{R}^{n,n})$$

and called (geometric) *mother algebra*. The name stems from the fact that it is very useful for working with (multi)linear functions on $n-$dimensional vector spaces.[3, 6]

Continuing the analogy, we change the combined null-vector basis of the vector space $\mathbb{R}^{n,n}$ into an orthonormal basis, that clearly shows the quadratic form to which the geometric mother algebra is associated

$$\bar{\mathbf{e}}_i = \bar{\mathbf{n}}_i + \mathbf{n}_i, \;\; \mathbf{e}_i = \bar{\mathbf{n}}_i - \mathbf{n}_i, \;\; 1 \le i \le n.$$

Based on the null-vector properties of the vectors $\mathbf{n}_i$ and $\bar{\mathbf{n}}_j$ and on the scalar product condition $\mathbf{n}_i * \bar{\mathbf{n}}_j = -\frac{1}{2}\delta_{i,j} \;\forall\, i,j = 1,2,\ldots,n$, we can calculate the orthonormality relationships

$$\bar{\mathbf{e}}_i * \bar{\mathbf{e}}_j = -\delta_{i,j}, \;\; \mathbf{e}_i * \bar{\mathbf{e}}_j = 0, \;\; \mathbf{e}_i * \mathbf{e}_j = \delta_{i,j} \;\; \forall\, i,j = 1,2,\ldots,n.$$

The orthonormal set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is seen to span a real Euclidean vector space $\mathbb{R}^n$, and the orthonormal set of vectors $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \ldots, \bar{\mathbf{e}}_n\}$ is seen to span an anti-Euclidean vector space $\bar{\mathbb{R}}^n$. Hence the $2n-$dimensional vector space $\mathbb{R}^{n,n}$ can also be written as the direct sum of

$$\mathbb{R}^{n,n} = \mathbb{R}^n \oplus \bar{\mathbb{R}}^n.$$

Finally the simple homogeneous grade $(p+q)$ multivectors ($(p+q)$-blades)

$$\mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_p\,\bar{\mathbf{e}}_1\bar{\mathbf{e}}_2\ldots\bar{\mathbf{e}}_q$$

can be used to project out any desired subspace $\mathbb{R}^{p,q}$, because each $(p+q)$-blade is in one-to-one correspondence a subspace of $\mathbb{R}^{n,n}$([4], p. 19).

We therefore now understand in general, how any null-vector (sub)space with even dimension $2n'$ can be reinterpreted by a change of basis as a vector space $\mathbb{R}^{n',n'}$. In the same way that vector spaces and quadratic forms are invariant under a change of basis, the geometric algebra generated by a vector space and its quadratic form according to the general definition on page 1 is also invariant. This gives a strategy how to deal with degenerate geometric algebras, i.e. geometric algebras with quadratic forms that result in subspaces whose vectors square to zero.

A general notation used for vector spaces with arbitrary signatures is $\mathbb{R}^{p,q,r}$, where p, q, r mean the dimensions of the subspaces whose vectors have positive, negative or zero squares, respectively.[8] The third index r is often omitted, if $r = 0$. According to the main argument of this subsection, any degenerate $n-$dimensional vector space is a subspace of the comprehensive vector space $\mathbb{R}^{n,n}$ and hence any degenerate geometric algebra can also be embedded in a larger non-degenerage geometric algebra $\mathbb{R}_{n,n}$, called the mother algebra of $n-$space.

As an example, the Minkowski plane algebra $\mathbb{R}_{1,1,0}$ is equivalent to the degenerate algebra $\mathbb{R}_{0,0,2}$ and the geometric algebra of the conformal model $\mathbb{R}_{4,1,0}$ is equivalent to the degenerate algebra $\mathbb{R}_{3,0,2}$. These two examples are closely related, because as argued in this section, the underlying vector spaces are related by

$$\mathbb{R}^{1,1,0} = \mathbb{R}^{0,0,2}, \ \ \mathbb{R}^{4,1,0} = \mathbb{R}^{3,0,0} \oplus \mathbb{R}^{1,1,0}.$$

Both relationships correspond to simple changes of basis.

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