# Experimental evidence for a non-globally trace-preserving POVM

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**Abstract.** A well-known experiment from 1986 involving entangled pairs is examined. The data, which until now have not been modeled quantitatively, is shown *not* to be in agreement with the quantum measurement postulate using von Neumann projectors. On the other hand, the data agree with the postulate using a more general positive operator valued measure (POVM). The peculiarity of the POVM proposed here is that it is only conditionally a POVM; *i.e.* it is not complete (trace-preserving) on the entire Hilbert space but only on a subset, although the POVM elements are positive semidefinite observables on the entire space. The state vector of the aforementioned experiment is in the subset where completeness holds. An extension of the conditional POVM is then applied to a proposed experiment involving three-particle Greenberger-Horne-Zeilinger (GHZ) entangled states. As with the Aspect experiment, completeness holds for the conditional POVM upon application to the GHZ state. Violation of the Bell inequality in the GHZ experiment does not occur upon application of von-Neumann projectors; however the conditional POVM allows for Bell inequality violation.

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## 1. Review of the experiment

The experimental evidence examined here is from Aspect, Grangier and Roger [1], hereafter referred to as the *Aspect experiment*. In the experiment, systems of correlated photon pairs were produced from a single source. The photons, being correlated, are represented by a system of the following form:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\varphi_1, \theta_1\rangle + \mathrm{e}^{\mathrm{i}\alpha}|\varphi_2, \theta_2\rangle) \tag{1}$$

where $\varphi$ and $\theta$ represent the two photons, the subscripts 1 and 2 signify polarity and $\alpha$ is a relative phase factor. For each pair produced by the source, photon $\varphi$ passes through an observers (Alice) Mach-Zehnder (MZ) interferometer and photon $\theta$ serves as a registration gatekeeper, collected by a second observer (Bob). The purpose of the latter is to eliminate noise from singles passing into the interferometer. The experimental set-up is shown in figure 1, and Alices data from the experiment is shown in figure 2.

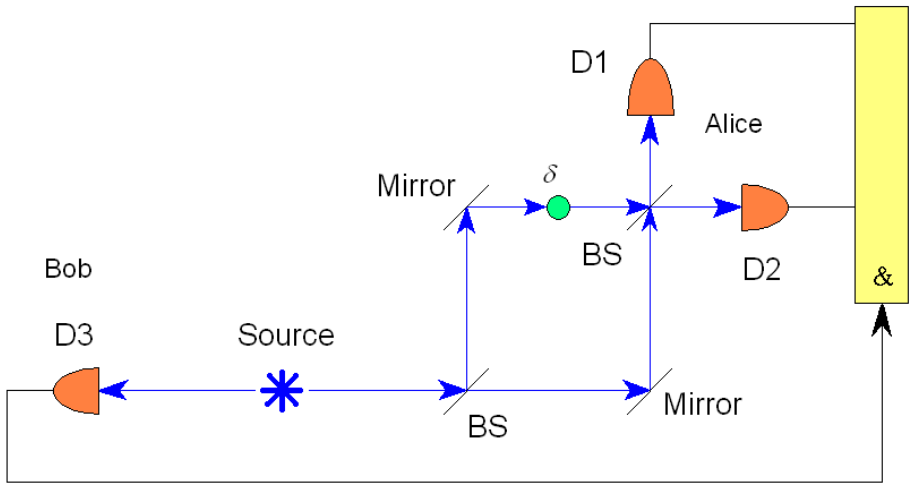


Fig 1. The experimental set-up of Grangier, Aspect and Roger [1]. A two-photon system is emitted by the source. Alices apparatus on the right is a MZ interferometer together with detectors D1 and D2. Bobs apparatus consists of a single detector D3 whose photon serves as a gatekeeper for Alices photon counting; *i.e.* data from D1 and D2 are collected only in coincidence with D3 registration.

In the original article, the authors did not provide a theoretical fit to the data in figure 2. Here, one is provided. First however, we will apply the quantum measurement postulate using von Neumann projectors, and show it does *not* match the experimental data.

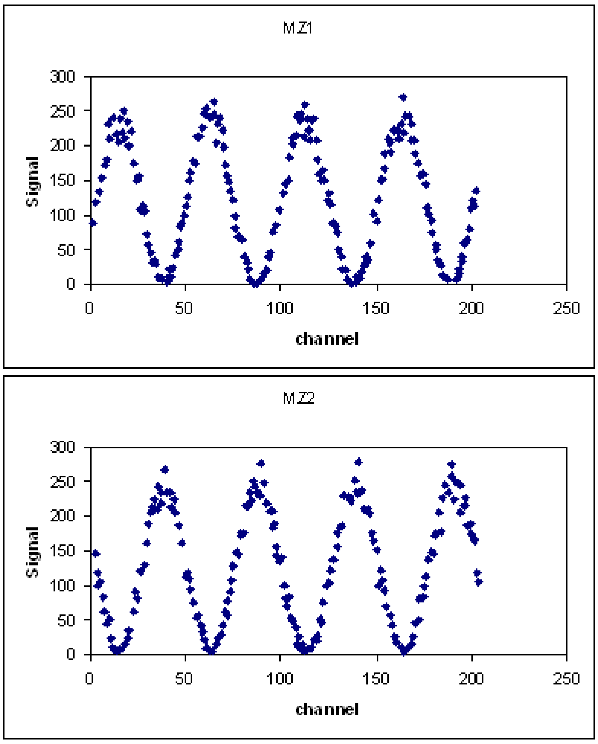


Fig 2. Data from the Aspect experiment [1]. Shown is signal intensity *vs.* relative phase $\delta$ between arms of MZ. These data match the calculations using the conditional POVM (13) and not the calculation using von Neumann projectors (3).

## 2. Data cannot be modeled by von Neumann Projectors $P_k$

The quantum measurement postulate (MP) for a discrete finite system is the following [2]:

MP *The probability of measuring eigenvalue $k$ of observable $A$ on a normalized system $|\psi\rangle = \sum_{ij} a_{ij}|\varphi_i^j\rangle \in E(A)$ = eigenspace of $A$ is given by*

$$p_A(k) = \sum_{j=1}^{J_k} |\langle\varphi_k^j|\psi\rangle|^2 \tag{2}$$

*where $J_k$ is the dimension of the $k^{th}$ eigensubspace.*

If the eigenspace is that of a cross product $A \otimes B$ of two observables $A$ and $B$ that do not have degenerate eigenvalues, then equation (2) can undergo a minor notational change in order to calculate the probability of Alice, whose observable is $A$, obtaining eigenvalue $k$, by treating the eigenvectors of Bobs $B$ eigenspace as degeneracies in $A$:

MP2 *The probability of Alice measuring eigenvalue $k$ of observable $A$ on a normalized non-degenerate two-particle system $|\psi\rangle = \sum_{ij} a_{ij}|\varphi_i, \theta_j\rangle \in E(A \otimes B)$ is given by:*

$$p_A(k) = \sum_{j=1}^{J} |\langle\varphi_k, \theta_j|\psi\rangle|^2. \tag{3}$$

*where $J$ is the dimension of Bobs $B$ eigenspace.*

Equation (3) can be rewritten in terms of von Neumann projectors $P_k$:

$$p_A(k) = \langle\psi|P_k|\psi\rangle \tag{4}$$

where

$$P_k = \sum_{j=1}^{J} |\varphi_k, \theta_j\rangle\langle\varphi_k, \theta_j|. \tag{5}$$

Next, let Alices eigenvalue $k = 1, 2$ in state (1) correspond to reflection or transmission upon interaction with the first half-silvered mirror in MZ respectively. These are in general not the same as those which correspond to detection in detectors D1 and D2. To obtain those, it is first necessary to transform the state (1) into a new basis using the rotation transform ("quantum gate"):

$$\begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \tag{6}$$

where $\delta$ is the rotation angle, proportional to the phase shift between arms in MZ. Applying transform (6) to state (1) results in:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\cos\delta|\varphi_{D1}, \theta_1\rangle - \sin\delta|\varphi_{D2}, \theta_1\rangle + \mathrm{e}^{\mathrm{i}\alpha}\sin\delta|\varphi_{D1}, \theta_2\rangle + \mathrm{e}^{\mathrm{i}\alpha}\cos\delta|\varphi_{D2}, \theta_2\rangle). \tag{7}$$

Note that state (7) has degeneracies: $|\varphi_{D1}, \theta_1\rangle, |\varphi_{D1}, \theta_2\rangle$ which span the eigensubspace of Alices eigenvalue 1, corresponding to registration by detector D1, and $|\varphi_{D2}, \theta_1\rangle, |\varphi_{D2}, \theta_2\rangle$ span that of eigenvalue 2, corresponding to detector D2 registration. These two eigenvalues Alice will measure. Applying state (7) to equation (4) results in the following probabilities for D1 and D2 registration, respectively:

$$p_A(1) = p_A(2) = \frac{1}{2}. \tag{8}$$

Note that these probabilities do *not* match the data in figure 2; they are flat lines vs. $\delta$. Thus the experimental data in the Aspect experiment *cannot* be accounted for using von Neumann projectors. In the next section, we will define a generalization of the von Neumann projectors called a "positive operator valued measure" (POVM) and construct a POVM which predicts the data. A peculiarity of the POVM constructed is that it is *not* a POVM for the entire space $E(A \otimes B)$, but only on a proper subset. Nevertheless, the state (7) is in that subset.

## 3. Construction of the conditional POVM $\tilde{P}_k$ which models the data

First, a definition [3]:

Definition. *A set $\{E_j\}$ of observable positive semidefinite operators $E_j$ over some space $X$ is said to be a POVM on $X$ if the sum of the operators over the entire set is the identity in that space; i.e.:*

$$\sum_j E_j = I_X. \tag{9}$$

This property is referred to as completeness. In particular, von Neumann projectors obey the completeness requirement (9), and so are a subcategory of POVM operators. Eigenvalue probabilities of a system $|\psi\rangle$ are calculated from POVM operators in the same manner as with projectors; *i.e.*:

$$p(k) = \langle\psi|E_k|\psi\rangle. \tag{10}$$

(*c.f.* equation (4).) Only with projectors $P_k$, there is the added restriction that they be idempotent:

$$P_k^2 = P_k. \tag{11}$$

Equation (11) does *not* hold for POVM operators in general. Note that a POVM is defined as a set. However, occasionally elements of that set are referred to as a "POVM" as well.

The conditional POVM operator on $E(A \otimes B)$ which will be used to match the Aspect data is defined as follows:

$$\tilde{P}_k = \sum_{j_1,j_2=1}^{J} |\varphi_k, \theta_{j_1}\rangle\langle\varphi_k, \theta_{j_2}|. \tag{12}$$

This operator was introduced previously by R. Srikanth [4] who applied it to an experiment involving entangled photons and a double slit [5]. The probability of obtaining eigenvalue $k$ using operator (12) is

$$\tilde{p}_A(k) = \frac{1}{N^2}\langle\psi|\tilde{P}_k|\psi\rangle \tag{13}$$

where $N = \sqrt{\sum_k \tilde{p}_A(k)}$ is a normalization constant which equals 1 if and only if completeness (9) holds for $\tilde{P}_k$. Let $U \subseteq E(A \otimes B)$ be the subset where completeness holds; *i.e.*

$$U_A = \{|\psi\rangle \in E(A \otimes B)| \sum_k \langle\psi|\tilde{P}_k|\psi\rangle = 1\}. \tag{14}$$

$U_A$ is nonempty, since *e.g.* $\sum_i a_{ii}|\varphi_i, \theta_i\rangle \in U_A$. The "$A$" subscript signifies that measurement is done on Alices end. It will be shown below that $\tilde{P}_k$ is a positive semidefinite observable on the entirety of $E(A \otimes B)$. Thus $\tilde{P}_k$ is a POVM on $U_A$. Since in general $U_A \neq E(A \otimes B)$, also shown below, $\tilde{P}_k$ is said to be a conditional POVM.

Before continuing with the theory, we show that $\tilde{P}_k$ predicts the Aspect data: applying equation (13) to the state (7) ($N = 1$; therefore it is in $U_A$), one obtains the following probabilities for detectors D1 and D2, respectively:

$$\begin{aligned} \tilde{p}_A(1) &= \frac{1}{2}[1 + \sin 2\delta\ \cos\alpha] \\ \tilde{p}_A(2) &= \frac{1}{2}[1 - \sin 2\delta\ \cos\alpha]. \end{aligned} \tag{15}$$

Unlike the probabilities (8), the probabilities (15) match the normalized experimental data with proper selection of $\delta$; in particular when $\alpha = 0$.

It still needs to be shown that $\tilde{P}_k$ is positive semidefinite and an observable. Let $X$ be a finite Hilbert space and $|\psi\rangle \in X$. A positive semidefinite operator $A$ on $X$, [6] is one which is self-adjoint and where

$$\langle\psi|A|\psi\rangle \geq 0. \tag{16}$$

An observable is a self-adjoint operator that has only real eigenvalues. Let $A$ and $B$ be $I'$ and $J$-dimensional respectively, and

$$\begin{aligned} |\psi\rangle = {} & a_{11}|\varphi_1,\theta_1\rangle + a_{12}|\varphi_1,\theta_2\rangle + ... + a_{1J}|\varphi_1,\theta_J\rangle \\ & + a_{21}|\varphi_2,\theta_1\rangle + a_{22}|\varphi_2,\theta_2\rangle + ... + a_{2J}|\varphi_2,\theta_J\rangle \\ & + ... \\ & + a_{k1}|\varphi_k,\theta_1\rangle + a_{k2}|\varphi_k,\theta_2\rangle + ... + a_{kJ}|\varphi_k,\theta_J\rangle \\ & + ... \\ & + a_{I'1}|\varphi_{I'},\theta_1\rangle + a_{I'2}|\varphi_{I'},\theta_2\rangle + ... + a_{I'J}|\varphi_{I'},\theta_J\rangle \end{aligned} \tag{17}$$

be an arbitrary normalized element of $E(A \otimes B)$ (thus $\sum_{j_1,j_2} |a_{j_1 j_2}|^2 = 1$). The operator (12), expanded, looks like:

$$\begin{aligned} \tilde{P}_k = {} & |\varphi_k,\theta_1\rangle\langle\varphi_k,\theta_1| + |\varphi_k,\theta_1\rangle\langle\varphi_k,\theta_2| + ... + |\varphi_k,\theta_1\rangle\langle\varphi_k,\theta_J| \\ & + |\varphi_k,\theta_2\rangle\langle\varphi_k,\theta_1| + |\varphi_k,\theta_2\rangle\langle\varphi_k,\theta_2| + ... + |\varphi_k,\theta_2\rangle\langle\varphi_k,\theta_J| \\ & + ... \\ & + |\varphi_k,\theta_J\rangle\langle\varphi_k,\theta_1| + |\varphi_k,\theta_J\rangle\langle\varphi_k,\theta_2| + ... + |\varphi_k,\theta_J\rangle\langle\varphi_k,\theta_J|, \end{aligned} \tag{18}$$

which, due to its symmetry, is clearly self-adjoint. Applying the operator (18) to the state (17), one obtains:

$$\begin{aligned} \tilde{P}_k|\psi\rangle = {} & a_{k1}|\varphi_k,\theta_1\rangle + a_{k1}|\varphi_k,\theta_2\rangle + ... + a_{k1}|\varphi_k,\theta_J\rangle \\ & + a_{k2}|\varphi_k,\theta_1\rangle + a_{k2}|\varphi_k,\theta_2\rangle + ... + a_{k2}|\varphi_k,\theta_J\rangle \\ & + ... \\ & + a_{kJ}|\varphi_k,\theta_1\rangle + a_{kJ}|\varphi_k,\theta_2\rangle + ... + a_{kJ}|\varphi_k,\theta_J\rangle \\ & = (|\varphi_k,\theta_1\rangle + |\varphi_k,\theta_2\rangle + ... + |\varphi_k,\theta_J\rangle)(a_{k1} + a_{k2} + ... + a_{kJ}). \end{aligned} \tag{19}$$

Thus

$$\begin{aligned}\langle\psi|\tilde{P}_k|\psi\rangle &= (a^*_{k1} + a^*_{k2} + ... + a^*_{kJ})(a_{k1} + a_{k2} + ... + a_{kJ}) \\ &= (a_{k1} + a_{k2} + ... + a_{kJ})^*(a_{k1} + a_{k2} + ... + a_{kJ}) \\ &= |a_{k1} + a_{k2} + ... + a_{kJ}|^2 \geq 0,\end{aligned} \tag{20}$$

which proves that $\tilde{P}_k$ is positive semi-definite. All that remains to be done to demonstrate that $\tilde{P}_k$ is an observable is to show that it has only real eigenvalues. This is done by contradiction: suppose $z \in \mathbb{C} - \mathbb{R}$ is an eigenvalue of $\tilde{P}_k$. Then:

$$\tilde{P}_k|\psi\rangle = z|\psi\rangle \tag{21}$$

where $|\psi\rangle$ is the associated eigenvector. Combining equations (20) and (21):

$$|(a_{k1} + a_{k2} + ... + a_{kJ})|^2 = \langle\psi|\tilde{P}_k|\psi\rangle = z\langle\psi|\psi\rangle = z \tag{22}$$

which implies that $z \in \mathbb{R}$; a contradiction. Thus $\tilde{P}_k$ is an observable. Note that observability and positive semidefiniteness hold throughout the entire space, it is only completeness which does not hold globally. We summarize:

Theorem A. *The operator $\tilde{P}_k$ is a POVM on the set $U_A$ defined by equation* (14),

and

Theorem B. *The operators $\tilde{P}_k$ predict the Aspect data shown in figure 2 whereas the von Neumann projectors $P_k$ do not. Further, the state which gives the Aspect data is in $U_A$.*

In the case $I' = J = 2$ (which is the case in the Aspect experiment), the condition for the set $U_A$ (14) can be written in a way which makes it easy to check whether $|\psi\rangle$ is in $U_A$ or not. From equation (22) applied to the normalized state $|\psi\rangle = \sum_{i,j=1}^{2} a_{ij}|\varphi_i, \theta_j\rangle$ we have that:

$$\begin{aligned}\langle\psi|\tilde{P}_1|\psi\rangle &= |a_{11} + a_{12}|^2 \\ \langle\psi|\tilde{P}_2|\psi\rangle &= |a_{21} + a_{22}|^2.\end{aligned} \tag{23}$$

Now if $\{\tilde{P}_1, \tilde{P}_2\}$ has the completeness property; *i.e.* $|\psi\rangle \in U_A$, then

$$\begin{aligned}1 = \langle\psi|\tilde{P}_1|\psi\rangle + \langle\psi|\tilde{P}_2|\psi\rangle &= |a_{11} + a_{12}|^2 + |a_{21} + a_{22}|^2 \\ &= 1 + a^*_{11}a_{12} + a_{11}a^*_{12} + a^*_{21}a_{22} + a_{21}a^*_{22}.\end{aligned} \tag{24}$$

Equation (24) holds if and only if

$$a^*_{11}a_{12} + a_{11}a^*_{12} + a^*_{21}a_{22} + a_{21}a^*_{22} = 2[\mathrm{re}(a_{11}a^*_{12}) + \mathrm{re}(a_{21}a^*_{22})] = 0. \tag{25}$$

Note that this is a condition on symmetry. In fact, it is easily shown that if a two-particle state is symmetric under entanglement swapping *i.e.* "envariant" [7], the state is in $U_A$. For the general case of a $I' \times J$ space, the condition can be written as

$$0 = \sum_{i=1}^{I'} \sum_{j=1}^{J-1} \sum_{k=j+1}^{J} \mathrm{re}(a^*_{ij}a_{ik}) \tag{26}$$

Consider for example, the "asymmetric" state $|\psi\rangle$ where $a_{11} = a_{12} = a_{21} = 1/(2\sqrt{2}) = a_{22}/\sqrt{5}$. Then

$$a_{11}^* a_{12} + a_{11} a_{12}^* + a_{21}^* a_{22} + a_{21} a_{22}^* = \frac{1+\sqrt{5}}{4} \neq 0; \tag{27}$$

which means that completeness does not hold for the $\tilde{P}_k$ in that case; *i.e.* $|\psi\rangle \notin U_A$. This raises an important question, but first, some notes: At this point, it should be evident, from the Aspect data, that the POVM $\tilde{P}_k$ is one which accounts for interference effects in two-particle systems, whereas the von Neumann operator $P_k$ does not. This is because for the latter, the sum appears *outside* of the norm in equation (3), whereas for the former, we have, from equation (13),

$$\tilde{p}_A(k) = \frac{1}{N^2} | \sum_{j=1}^{J} \langle \varphi_k, \theta_j | \psi \rangle |^2; \tag{28}$$

*i.e.* the sum appears *inside* the norm. The reason for this difference between equations (3) and (28) is similar to that of the measurement equations for single-particle systems; they are necessary for application when Alice can in principle know, and when she cannot know, which degenerate eigenvector gave rise to her measurement result, respectively. With this in mind, the question arises as to whether a state $|\psi\rangle \notin U_A$ can give rise to interference effects. This is important because it is generally accepted that incomplete operations are forbidden in quantum theory. If that is the case, then the POVM $\tilde{P}_k$ is inapplicable to such a state; therefore interference effects should *not* be observable with such a state.

As was mentioned earlier, a POVM does not in general, have the idempotent property. In particular, for $\tilde{P}_k$ we have that:

Theorem C. *$\tilde{P}_k^2 = J\tilde{P}_k$ where $J$ is the dimension of Bobs eigenspace E(B).*

*Proof.* From equation (18):

$$\begin{aligned} \tilde{P}_k^2 = [ & |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_J| \\ & + |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_J| \\ & + ... \\ & + |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_J|]\tilde{P}_k. \end{aligned} \tag{29}$$

Consider the first term in the first row of equation (29), $|\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_1|$. Multiplying it by $\tilde{P}_k$, one obtains an entire copy of the first row of $\tilde{P}_k$:

$$|\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_1|\tilde{P}_k = |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_J|. \tag{30}$$

Multiplying the second term in the first row of equation (29) by $\tilde{P}_k$, one obtains an entire copy of the second row of $\tilde{P}_k$:

$$|\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_2|\tilde{P}_k = |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_J|, \tag{31}$$

...and so on until the last term in the first row of equation (29) gives an entire copy of the $J$th row of $\tilde{P}_k$:

$$|\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_J|\tilde{P}_k = |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_J|. \qquad (32)$$

Thus it is evident from equations (30), (31) and (32), that multiplying the entire first row in equation (29) with $\tilde{P}_k$ gives an entire copy of $\tilde{P}_k$ itself:

$$[|\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_1\rangle\langle\varphi_k, \theta_J|]\tilde{P}_k = \tilde{P}_k. \qquad (33)$$

Similarly, multiplying the second row of equation (29) by $\tilde{P}_k$ gives a copy of $\tilde{P}_k$ also:

$$[|\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_2\rangle\langle\varphi_k, \theta_J|]\tilde{P}_k = \tilde{P}_k \qquad (34)$$

...and so on, down to the final $J$th row, which also gives a copy of $\tilde{P}_k$:

$$[|\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_1| + |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_2| + ... + |\varphi_k, \theta_J\rangle\langle\varphi_k, \theta_J|]\tilde{P}_k = \tilde{P}_k. \qquad (35)$$

From equations(29), (33), (34) and (35), it follows that $J$ copies of $\tilde{P}_k$ result from $\tilde{P}_k^2$. □

In the case of ensembles of several states $|\psi_i\rangle$ with frequency $p_i$, one works with density operators $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, and a more general form of equation (2) for calculating Alices probabilities, if Bob performs a projective measurement:

$$p_A(k) = \mathrm{tr}(P_k\rho), \qquad (36)$$

where "tr" is trace. If Bob destroys his which-way information, then equation (13) generalizes to:

$$\tilde{p}_A(k) = \frac{1}{N^2}[\mathrm{tr}(\tilde{P}_k\rho)] \qquad (37)$$

where $N^2 = \sum_k \mathrm{tr}(\tilde{P}_k\rho)$. If $N^2 \neq 1$ then the $\tilde{P}_k$ are said to *not* be trace-preserving, hence the title of this article. Trace-preservation is equivalent to completeness. Since $\tilde{P}_k$ does sometimes preserve the trace, it is referred to here as a *non-globally trace-preserving POVM*, to distinguish it from, as well as elevate its status from, ordinary sets of operators which exhibit completeness nowhere within the space.

A proposed Greenberger-Horne-Zeilinger (GHZ) experiment is analyzed in the next section, using an extension of $\tilde{P}_k$ to three particle systems. As in the Aspect experiment, measurements are done in a complete basis, whether or not Bob destroys which-way information.

## 4. Operators $\tilde{P}_k$ model GHZ experiment

GHZ states involve entanglements of more than two particles. Here we will consider a 3-photon state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|A_1\rangle|B_1\rangle|C_1\rangle + |A_2\rangle|B_2\rangle|C_2\rangle] \qquad (38)$$

where $A$, $B$ and $C$ represent entangled photons, and these can each occur in one of two states, 1 or 2, hence the subscripts. In this gedanken experiment the three photons are emitted by a common source along coplanar trajectories. Photons $A$, $B$ and $C$ each encounter a beam splitter, which splits its beam within a plane, at an angle $\alpha$, $\beta$ and $\gamma$ respectively, with respect to the normal vector $\mathbf{n}$ to the original plane. Alice receives photons $B$ and $C$ and Bob receives photon $A$. The set-up has some similarities to that of Mermins gedanken experiment [8] and is sketched in figure 3. The basic idea of the experiment is as follows:

> *Alice measures correlation between her two photons A and B vs. relative angle between her two polarizers, and Bob has the choice of making a projective measurement on his photon C or destroying its "which-way" information.*

"Relative angle" is defined below. To the authors knowledge, this experiment has not yet been performed.

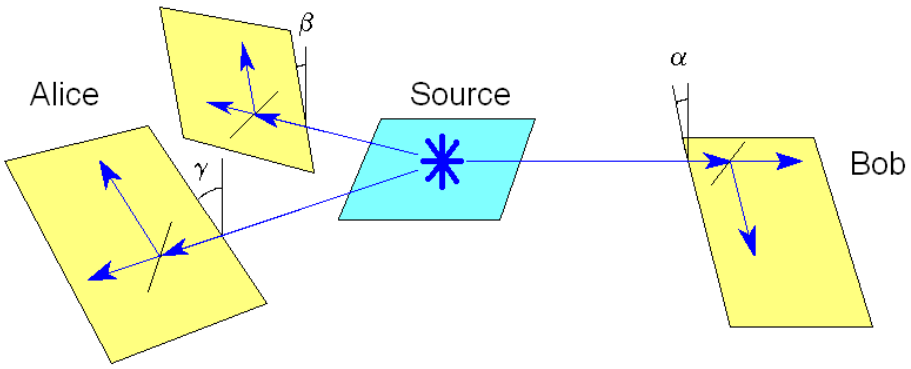


Fig 3. A sketch of the GHZ experimental set up. Three photons are emitted by a source and travel in the same blue plane. Two go to Alice, and one to Bob. Each photon reaches a beam splitter and afterwards may take one of two paths within a yellow plane. The yellow planes have relative angles $\alpha$, $\beta$ or $\gamma$ with respect to the normal of the blue plane. In the experiment, Alice measures correlation between her two photons with respect to relative angle $\beta - \gamma$, and Bob has a choice of measuring eigenvalue information from his photon or destroying it.

State (38) can be transformed into the basis of the beam splitters using the rotation transform (6):

$$\begin{aligned}
|\psi\rangle = \frac{1}{\sqrt{2}}[&(\cos\alpha\cos\beta\cos\gamma + \sin\alpha\sin\beta\sin\gamma)|A_1\rangle|B_1\rangle|C_1\rangle \\
&+ (\sin\alpha\sin\beta\cos\gamma - \cos\alpha\cos\beta\sin\gamma)|A_1\rangle|B_1\rangle|C_2\rangle \\
&+ (\sin\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\cos\gamma)|A_1\rangle|B_2\rangle|C_1\rangle \\
&+ (\cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\beta\cos\gamma)|A_2\rangle|B_1\rangle|C_1\rangle \\
&+ (\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\beta\cos\gamma)|A_1\rangle|B_2\rangle|C_2\rangle \\
&+ (\sin\alpha\cos\beta\sin\gamma + \cos\alpha\sin\beta\cos\gamma)|A_2\rangle|B_1\rangle|C_2\rangle \\
&+ (\sin\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma)|A_2\rangle|B_2\rangle|C_1\rangle \\
&+ (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma)|A_2\rangle|B_2\rangle|C_2\rangle].
\end{aligned} \tag{39}$$

For calculating the joint probability of Alice finding eigenvalues (*i.e.* registration by detectors numbered) $b$ and $c$ $(b, c = 1, 2)$, given angles $\alpha$, $\beta$ and $\gamma$, an extension of equation (3) is used, since Bob will be measuring "which-way" information on his end:

$$\begin{aligned} p(b \wedge c|\alpha \wedge \beta \wedge \gamma) &= \sum_j |\langle A_j, B_b, C_c|\psi\rangle|^2 \\ &= \langle\psi|P_{bc}|\psi\rangle \end{aligned} \tag{40}$$

where the projector $P_{bc} = \sum_j |A_j, B_b, C_c\rangle\langle A_j, B_b, C_c|$. Applying the state (39) to equation (40), one finds the probability for example, of both photons $B$ and $C$ being detected by detectors numbered 1, given angles $\alpha$, $\beta$ and $\gamma$:

$$p(1 \wedge 1|\alpha \wedge \beta \wedge \gamma) = \frac{1}{2}[\cos^2\beta\cos^2\gamma + \sin^2\beta\sin^2\gamma]. \tag{41}$$

Note that probability (41) is independent of Bobs polarizer angle $\alpha$; therefore $\alpha$ is dropped hereafter on the left hand side. To continue, define the relative angle $\theta$ between Alices angles $\beta$ and $\gamma$ as:

$$\theta = \beta - \gamma. \tag{42}$$

Applying equation (42), probability (41) becomes

$$\begin{aligned} p(1 \wedge 1|\beta \wedge \gamma) &= p(1 \wedge 1|\beta \wedge \theta) \\ &= \frac{1}{4} + \frac{1}{8}[\cos 2\theta + \cos(4\beta - 2\theta)]. \end{aligned} \tag{43}$$

To get the probability of measuring $b = c = 1$ given relative angle $\theta$ and *any* absolute angle $\beta$, probability (43) is integrated over the domain of the uniform random variable $\beta$ and normalized:

$$\begin{aligned} p(1 \wedge 1|\theta) &= \frac{\int_0^\pi p(1 \wedge 1|\beta \wedge \theta)\mathrm{d}\beta}{\int_0^\pi \mathrm{d}\beta} \\ &= \frac{1}{8} + \frac{1}{4}\cos^2\theta. \end{aligned} \tag{44}$$

Similarly,

$$\begin{aligned} &p(1 \wedge 2|\theta) = p(2 \wedge 1|\theta) = \frac{1}{8} + \frac{1}{4}\sin^2\theta, \\ &p(2 \wedge 2|\theta) = \frac{1}{8} + \frac{1}{4}\cos^2\theta. \end{aligned} \tag{45}$$

Note that probabilities (44), (45) sum to unity. This is to be expected, since the von Neumann projectors $P_{bc}$ are complete. Next, we claim that equations (44), (45) *obey* the Bell inequality [9]: given four variables $b_1$, $b_2$, $c_1$ and $c_2$ each with domain $\{-1, 1\}$, the function

$$\Gamma = b_1c_1 + b_1c_2 + b_2c_1 - b_2c_2 \tag{46}$$

must have range $\{-2, 2\}$ and hence the average $\langle\Gamma\rangle$ over many trials must obey

$$|\langle\Gamma\rangle| \leq 2. \tag{47}$$

Suppose then that $\{b_1, b_2\}$ and $\{c_1, c_2\}$ are sets of outcomes of photons $B$ and $C$ at angles $(\beta =)\beta_1, \beta_2$ and $(\gamma =)\gamma_1, \gamma_2$ respectively. These outcomes each are either 1 or $-1$, representing detector 1 or 2 registration, respectively. Further, suppose that

$$|\beta_2 - \gamma_1| = \varphi = |\beta_1 - \gamma_1| = |\beta_1 - \gamma_2| = \frac{1}{3}|\beta_2 - \gamma_2|; \tag{48}$$

as illustrated in figure 4.

Consider the two angles $\beta$ and $\gamma$ set to $\beta_1$ and $\gamma_1$ respectively. For photon $B$, the outcome $b_1$ can be 1 or $-1$. Likewise, for photon $C$, the outcome $c_1$ can be 1 or $-1$. Then, the expected value of the product $b_1c_1$ is

$$\begin{aligned}
\langle b_1c_1\rangle &= \sum_{b_1,c_1=-1}^{1} p(b_1, c_1)b_1c_1 \\
&= p(-1,-1) - p(1,-1) - p(-1,1) + p(1,1) \\
&= p(2 \wedge 2|\varphi) - p(1 \wedge 2|\varphi) - p(2 \wedge 1|\varphi) + p(1 \wedge 1|\varphi) \\
&= \frac{1}{2}\cos 2\varphi.
\end{aligned} \tag{49}$$

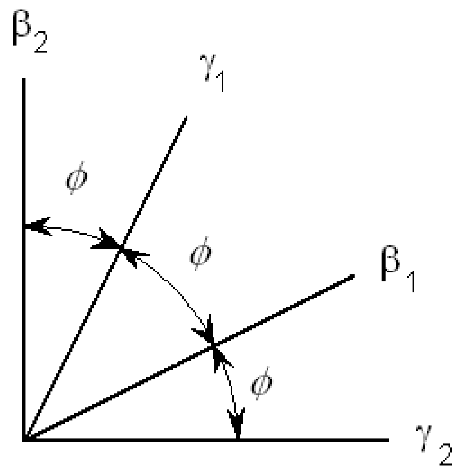


Fig 4. Diagram showing the relative angles between Alices polarizer settings $\beta_1, \beta_2$ and $\gamma_1, \gamma_2$.

Similarly,

$$\begin{aligned}
\langle b_1c_2\rangle = \langle b_2c_1\rangle &= \frac{1}{2}\cos 2\varphi, \\
\langle b_2c_2\rangle &= \frac{1}{2}\cos 6\varphi.
\end{aligned} \tag{50}$$

Averaging equation (46), utilizing linearity of $\langle\cdot\rangle$ and applying equations (49), (50) results in:

$$\begin{aligned}\langle\Gamma\rangle &= \langle b_1c_1\rangle + \langle b_1c_2\rangle + \langle b_2c_1\rangle - \langle b_2c_2\rangle \\ &= \frac{3}{2}\cos 2\varphi - \frac{1}{2}\cos 6\varphi.\end{aligned} \tag{51}$$

Since

$$|\langle\Gamma\rangle| = |\frac{3}{2}\cos 2\varphi - \frac{1}{2}\cos 6\varphi| \leq 2, \tag{52}$$

it follows from inequality (52) that

Theorem D. *In the GHZ gedanken experiment, Bells inequality is satisfied for Alice if Bob measures eigenvalue or which-way information; i.e. if von Neumann projectors* $P_{bc}$ *are used in the calculation.*

For the case where Bob destroys which-way information, Alices probabilities are predicted using an extension of equation (13). To make the calculations easier, Bob fixes his polarizer angle to $\alpha = 0$. (Since Bob needs to destroy which-way information, he might set up an MZ to accomplish this with the plane of MZ set to this angle.) This reduces state (39) to

$$\begin{aligned}|\psi\rangle = \frac{1}{\sqrt{2}}[&\cos\beta\cos\gamma|A_1\rangle|B_1\rangle|C_1\rangle - \cos\beta\sin\gamma|A_1\rangle|B_1\rangle|C_2\rangle \\ &- \sin\beta\cos\gamma|A_1\rangle|B_2\rangle|C_1\rangle + \sin\beta\sin\gamma|A_2\rangle|B_1\rangle|C_1\rangle \\ &+ \sin\beta\sin\gamma|A_1\rangle|B_2\rangle|C_2\rangle + \sin\beta\cos\gamma|A_2\rangle|B_1\rangle|C_2\rangle \\ &+ \cos\beta\sin\gamma|A_2\rangle|B_2\rangle|C_1\rangle + \cos\beta\cos\gamma|A_2\rangle|B_2\rangle|C_2\rangle].\end{aligned} \tag{53}$$

From state (53), Alices probability of measuring 1 from both photons given angles $\beta$ and $\theta$ is then:

$$\begin{aligned}\tilde{p}(1\wedge 1|\beta\wedge\theta) &= |\sum_j \langle A_j, B_1, C_1|\psi\rangle|^2 \\ &= \frac{1}{2}\cos^2\theta.\end{aligned} \tag{54}$$

Since probability (54) is independent of angle $\beta$, it follows that

$$\tilde{p}(1\wedge 1|\theta) = \frac{1}{2}\cos^2\theta. \tag{55}$$

Similarly,

$$\begin{aligned}&\tilde{p}(1\wedge 2|\theta) = \tilde{p}(2\wedge 1|\theta) = \frac{1}{2}\sin^2\theta, \\ &\tilde{p}(2\wedge 2|\theta) = \frac{1}{2}\cos^2\theta.\end{aligned} \tag{56}$$

Note that probabilities (55), (56) sum to unity, again, as expected from a measurement done in a complete basis. In fact, it is a simple exercise in notation to show that their associated projection operators $\tilde{P}_{bc} = \sum_{j,k} |A_j, B_b, C_c\rangle\langle A_k, B_b, C_c|$ form a conditional POVM, using the results in the previous section. Further, it is straightforward to show using the method above that the Bell correlation function $\tilde{\Gamma}$ derived from probabilities (55), (56) is related to correlation function (51), upon averaging, by a constant; *i.e.*

$$\langle\tilde{\Gamma}\rangle = 3\cos 2\varphi - \cos 6\varphi = 2\langle\Gamma\rangle. \tag{57}$$

From equation (57) it is evident that probabilities (55), (56) give rise to a *violation* of the Bell inequality (47), in particular, at $\varphi = \pi/8$. Hence:

Theorem E. *If Bob destroys which-way information, i.e. if the conditional POVM $\tilde{P}_{bc}$ is used in the calculation, then Alice will determine a violation of Bells inequality in the GHZ gedanken experiment.*

## 5. Conclusion

It has been shown that there exists experimental evidence of a non-globally trace-preserving POVM from the Aspect experimental data involving entangled pairs of photons. The data cannot be modeled using von Neumann projectors. The reason for this has to do with interference effects which the POVM accounts for, and the von Neumann projectors do not. The interference effects arise since the second photons "which-way" information is not measured as the primary photon passes through the Mach-Zehnder interferometer. Had the which-way information been measured, then there would be no interference effects in the data and hence the data could be modeled with von Neumann projectors. Since the POVM is conditional, it is not a POVM on the entire space, but only a proper subset. This is due entirely to completeness; the operators are positive semidefinite observables over the entire space. It is not known whether the conditional POVM can be applied to states outside this subset; it is generally accepted that completeness is necessary in quantum theory.

The conditional POVM is also applied to a proposed experiment involving three-particle entangled Greenberger-Horne-Zeilinger states. In this experiment, Alice receives two photons from every entangled triple, and Bob, one. Alice then measures correlations between her two photons, while Bob chooses to measure which-way information or destroy it. The von Neumann projectors are applicable to the former case, and the POVM to the latter. No violation of Bells inequality on Alices end occurs in the former, but does occur in the case of the latter.

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