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# Tutorial on Fourier Transformations and Wavelet Transformations in Clifford Geometric Algebra

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**Abstract**

First, the basic concept multivector functions and their vector derivative in geometric algebra (GA) is introduced. Second, beginning with the Fourier transform on a scalar function we generalize to a real Fourier transform on GA multivector-valued functions ($f : \mathbb{R}^3 \to Cl_{3,0}$). Third, we show a set of important properties of the Clifford Fourier transform (CFT) on $Cl_{3,0}$ such as differentiation properties, and the Plancherel theorem. We round off the treatment of the CFT (at the end of this tutorial) by applying the Clifford Fourier transform properties for proving an uncertainty principle for $Cl_{3,0}$ multivector functions.

For wavelets in GA it is shown how continuous Clifford $Cl_{3,0}$-valued admissible wavelets can be constructed using the similitude group *SIM(3)*, a subgroup of the affine group of $\mathbb{R}^3$. We express the admissibility condition in terms of the CFT and then derive a set of important properties such as dilation, translation and rotation covariance, a reproducing kernel, and show how to invert the Clifford wavelet transform of multivector functions. We explain (at the end of this tutorial) a generalized Clifford wavelet uncertainty principle. For scalar admissibility constant it sets bounds of accuracy in multivector wavelet signal and image processing. As concrete example we introduce multivector Clifford Gabor wavelets, and describe important properties such as the Clifford Gabor transform isometry, a reconstruction formula, and (at the end of this tutorial) an uncertainty principle for Clifford Gabor wavelets.

**Keywords**: vector derivative, multivector-valued function, Clifford (geometric) algebra, Clifford Fourier transform, uncertainty principle, similitude group, geometric algebra wavelet transform, geometric algebra Gabor wavelets.

# Contents

# 1 Introduction to geometric algebra Fourier transformation

In the field of applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. The Fourier transform provides also a technique for signal analysis where the signal from the original domain is transformed to the spectral or frequency domain. In the frequency domain many characteristics of the signal are revealed. But how to extend the Fourier transform to geometric algebra?

Brackx et al. [1] extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. A related applied approach for hypercomplex Clifford Fourier transformations in $Cl_{0,n}$ was followed by Bülow et. al. [12]. In [14], Li et. al. extended the Fourier transform holomorphically to a function of $m$ complex variables.

In this tutorial we adopt and expand the generalization of the Fourier transform in Clifford geometric algebra[1] $\mathcal{G}_3$ recently suggested by Ebling and Scheuermann [10]. We explicitly show detailed properties of the real[2]

[1] In the geometric algebra literature [4] instead of the mathematical notation $Cl_{p,q}$ the notation $\mathcal{G}_{p,q}$ is widely in use. It is convention to abbreviate $\mathcal{G}_{n,0}$ to $\mathcal{G}_n$. We will use the words Clifford algebra and geometric algebra interchangably, similarly the notions of geometric algebra FT and Clifford FT, and we will use both notations $Cl_{p,q}$ and $\mathcal{G}_{p,q}$. Nowadays geometric algebra is often understood as Clifford algebra together with geometric interpretation based on the underlying vector space and its subspaces.

[2] The meaning of *real* in this context is, that we use the three dimensional volume element $i_3 = \boldsymbol{e}_{123}$ of the geometric algebra $\mathcal{G}_3$ over the field of the reals $\mathbb{R}$ to construct

Clifford geometric algebra Fourier transform (CFT), which we subsequently use (at the end of this tutorial) to define and prove the uncertainty principle for $\mathcal{G}_3$ multivector functions. In this tutorial we closely follow the approach described in [15, 16, 17, 35].

In the next section we briefly review the basics of GA, including the definition of multivector functions. Then we briefly study the vector derivative for a multivector valued function. We demonstrate that with a little modification it obeys rules which resemble the rules for a scalar partial derivative. Having laid these foundations we define the CFT and study some of its basic properties. A thorough undertstanding of the CFT will be essential for constructing Clifford wavelets later in this tutorial.

# 2 Clifford's geometric algebra

In this section we introduce the axioms and the vector derivative of geometric algebra. Fore more details we refer the reader to [4, 7].

## 2.1 Axioms of geometric algebra

For $\mathcal{G}_n$ to be a Clifford geometric algebra over the real $n$-dimensional Euclidean vector space $\mathbb{R}^n$, the *geometric product* of elements A, B, C $\in \mathcal{G}_n$ must satisfy the following axioms:

**Axiom 2.1** *Addition is commutative:*

$$A + B = B + A.$$

**Axiom 2.2** *Addition and the geometric product are associative:*

$$(A + B) + C = A + (B + C), \qquad A(BC) = (AB)C,$$

*and distributive:*

$$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC.$$

**Axiom 2.3** *There exist unique additive and multiplicative identities 0 and 1 such that:*

$$A + 0 = A, \qquad 1A = A.$$

the kernel of the Clifford Fourier transformation of definition 3.3. This $i_3$ has a clear geometric interpretation.

**Axiom 2.4** *Every $A$ in $\mathcal{G}_n$ has an additive inverse:*

$$A + (-A) = 0.$$

**Axiom 2.5** *For any nonzero vector $\boldsymbol{a}$ in $\mathcal{G}_n$ the square of $\boldsymbol{a}$ is equal to a unique positive scalar $|\boldsymbol{a}|^2$, that is*

$$\boldsymbol{a}\boldsymbol{a} = \boldsymbol{a}^{\mathbf{2}} = |\boldsymbol{a}|^2 > 0.$$

Depending on the signature of the underlying vector space $\mathbb{R}^{p,q}$ zero and negative squares of vectors will also occur.

**Axiom 2.6** *Every k-vector, $A_k = \boldsymbol{a}_1\boldsymbol{a}_2 ... \boldsymbol{a}_k$, can be factorized into pairwise orthogonal vector factors, which satisfy:*

$$\boldsymbol{a}_i\boldsymbol{a}_j = -\boldsymbol{a}_j\boldsymbol{a}_i, \qquad i, j = 1, 2, ..., k \quad \textit{and} \quad i \neq j.$$

## 2.2 Clifford's geometric algebra $\mathcal{G}_3$ of $\mathbb{R}^3$

Let us consider an orthonormal vector basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the real 3D Euclidean vector space $\mathbb{R}^3$. The geometric algebra over $\mathbb{R}^3$ denoted by $\mathcal{G}_3$ then has the graded $2^3 = 8$-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \boldsymbol{e}_{123}\}, \tag{2.1}$$

where 1 is the real scalar identity element (grade 0) of Axiom 2.3, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the $\mathbb{R}^3$ basis vectors (grade 1), $\boldsymbol{e}_{12} = \boldsymbol{e}_1\boldsymbol{e}_2$, $\boldsymbol{e}_{31} = \boldsymbol{e}_3\boldsymbol{e}_1$, and $\boldsymbol{e}_{23} = \boldsymbol{e}_2\boldsymbol{e}_3$ are frequently used definitions for the basis bivectors (grade 2), and $\boldsymbol{e}_{123} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = i_3$ defines the unit oriented pseudoscalars[3] (grade 3), i.e. the highest grade blade element in $\mathcal{G}_3$.

The associative geometric multiplication of the basis vectors obeys according to the axioms

$$\begin{aligned} \boldsymbol{e}_k\,\boldsymbol{e}_l &= -\boldsymbol{e}_l\,\boldsymbol{e}_k && \text{for} \quad k \neq l, \qquad k, l = 1, 2, 3, \\ \boldsymbol{e}_k^2 &= 1 && \text{for} \qquad k = 1, 2, 3. \end{aligned}$$

Inner products obey therefore

$$\boldsymbol{e}_k \cdot \boldsymbol{e}_l = \frac{1}{2}(\boldsymbol{e}_k\boldsymbol{e}_l + \boldsymbol{e}_l\boldsymbol{e}_k) = \delta_{kl}, \qquad k, l = 1, 2, 3.$$

[3] Other names in use are *trivector* or *volume element*.

According to these rules the Clifford (geometric) product of two arbitrary grade 1 vectors $\boldsymbol{x}$, $\boldsymbol{y}$ comprises the *inner product* and the *outer product*, i.e. the symmetric scalar part and the antisymmetric bivector part:

$$\boldsymbol{xy} = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \wedge \boldsymbol{y},$$

where in coordinates

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= \frac{1}{2}(\boldsymbol{xy} + \boldsymbol{yx}) \\ &= (x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) \cdot (y_1\boldsymbol{e}_1 + y_2\boldsymbol{e}_2 + y_3\boldsymbol{e}_3) \\ &= x_1y_1 + x_2y_2 + x_3y_3, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{x} \wedge \boldsymbol{y} &= \frac{1}{2}(\boldsymbol{xy} - \boldsymbol{yx}) \\ &= (x_1y_2 - x_2y_1)\boldsymbol{e}_{12} + (x_3y_1 - x_1y_3)\boldsymbol{e}_{31} + (x_2x_3 - x_3x_2)\boldsymbol{e}_{23}. \end{aligned}$$

The general elements of a geometric algebra are called multivectors. Every multivector $M$ can be represented as a linear combination of $k$-grade elements ($k = 0,\ 1,\ 2,\ 3$). It means that in $\mathcal{G}_3$ a multivector can be expressed as

$$\begin{aligned} M = \sum_A \alpha_A \boldsymbol{e}_A \quad &= \quad \underbrace{\alpha_0}_{\text{scalar part}} + \underbrace{\alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 + \alpha_3\boldsymbol{e}_3}_{\text{vector part}} + \\ &+ \underbrace{\alpha_{12}\boldsymbol{e}_{12} + \alpha_{31}\boldsymbol{e}_{31} + \alpha_{23}\boldsymbol{e}_{23}}_{\text{bivector part}} + \underbrace{\alpha_{123}\boldsymbol{e}_{123}}_{\text{trivector part}}, \end{aligned} \tag{2.2}$$

where $A \in \{0, 1, 2, 3, 12, 31, 23, 123\}$, and $\alpha_A \in \mathbb{R}$. Note that $i_3 = \boldsymbol{e}_{123}$ commutes with all other elements of $\mathcal{G}_3$ and squares to $i_3^2$ = -1. The *grade selector* is defined as $\langle M \rangle_k$ for the $k$-vector part of $M$, especially $\langle M \rangle = \langle M \rangle_0$. Then equation (2.2) can be rewritten as

$$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3. \tag{2.3}$$

The *reverse* of $M$ is defined by the anti-automorphism

$$\widetilde{M} = \langle M \rangle + \langle M \rangle_1 - \langle M \rangle_2 - \langle M \rangle_3, \tag{2.4}$$

which fulfils $\widetilde{(MN)} = \widetilde{N}\widetilde{M}$ for every $M, N \in \mathcal{G}_3$. The *square norm* of $M$ is defined by

$$\|M\|^2 = \langle M\widetilde{M} \rangle, \tag{2.5}$$

Table 2.1: Multiplication table of $\mathcal{G}_3$ basis elements.

| | 1 | $\boldsymbol{e}_1$ | $\boldsymbol{e}_2$ | $\boldsymbol{e}_3$ | $\boldsymbol{e}_{12}$ | $\boldsymbol{e}_{31}$ | $\boldsymbol{e}_{23}$ | $\boldsymbol{e}_{123}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | $\boldsymbol{e}_1$ | $\boldsymbol{e}_2$ | $\boldsymbol{e}_3$ | $\boldsymbol{e}_{12}$ | $\boldsymbol{e}_{13}$ | $\boldsymbol{e}_{23}$ | $\boldsymbol{e}_{123}$ |
| $\boldsymbol{e}_1$ | $\boldsymbol{e}_1$ | 1 | $\boldsymbol{e}_{12}$ | $-\boldsymbol{e}_{31}$ | $\boldsymbol{e}_2$ | $-\boldsymbol{e}_3$ | $\boldsymbol{e}_{123}$ | $\boldsymbol{e}_{23}$ |
| $\boldsymbol{e}_2$ | $\boldsymbol{e}_2$ | -$\boldsymbol{e}_{12}$ | 1 | $\boldsymbol{e}_{23}$ | -$\boldsymbol{e}_1$ | $\boldsymbol{e}_{123}$ | $\boldsymbol{e}_3$ | $\boldsymbol{e}_{31}$ |
| $\boldsymbol{e}_3$ | $\boldsymbol{e}_3$ | $\boldsymbol{e}_{31}$ | $-\boldsymbol{e}_{23}$ | 1 | $\boldsymbol{e}_{123}$ | $\boldsymbol{e}_1$ | $-\boldsymbol{e}_2$ | $\boldsymbol{e}_{12}$ |
| $\boldsymbol{e}_{12}$ | $\boldsymbol{e}_{12}$ | $-\boldsymbol{e}_2$ | $\boldsymbol{e}_1$ | $\boldsymbol{e}_{123}$ | -1 | $\boldsymbol{e}_{23}$ | $-\boldsymbol{e}_{31}$ | $-\boldsymbol{e}_3$ |
| $\boldsymbol{e}_{31}$ | $\boldsymbol{e}_{31}$ | $\boldsymbol{e}_3$ | $\boldsymbol{e}_{123}$ | $-\boldsymbol{e}_1$ | $-\boldsymbol{e}_{23}$ | -1 | $\boldsymbol{e}_{12}$ | $-\boldsymbol{e}_2$ |
| $\boldsymbol{e}_{23}$ | $\boldsymbol{e}_{23}$ | $\boldsymbol{e}_{123}$ | $-\boldsymbol{e}_3$ | $\boldsymbol{e}_2$ | $\boldsymbol{e}_{31}$ | $-\boldsymbol{e}_{12}$ | -1 | $-\boldsymbol{e}_1$ |
| $\boldsymbol{e}_{123}$ | $\boldsymbol{e}_{123}$ | $\boldsymbol{e}_{23}$ | $\boldsymbol{e}_{31}$ | $\boldsymbol{e}_{12}$ | $-\boldsymbol{e}_3$ | $-\boldsymbol{e}_2$ | $-\boldsymbol{e}_1$ | -1 |

where

$$\langle M\widetilde{N}\rangle = M * \widetilde{N} = \sum_A \alpha_A \beta_A \tag{2.6}$$

is a real valued (inner) *scalar product* for any $M, N$ in $\mathcal{G}_3$ with $M$ of equation (2.2) and $N = \sum_A \beta_A \boldsymbol{e}_A$. Note that

$$\langle M\ N\rangle \ = \ \langle N\ M\rangle \ = \ \langle \widetilde{M}\ \widetilde{N}\rangle \ = \ \langle \widetilde{N}\ \widetilde{M}\rangle, \tag{2.7}$$

and that

$$\boldsymbol{x}^2\ \|M\|^2 = \|\boldsymbol{x}\|^2 \|M\|^2 = \|\boldsymbol{x}M\|^2, \qquad \boldsymbol{x} \in \mathbb{R}^3 \tag{2.8}$$

For $N = M$ in (2.6) we can re-express (2.5) as

$$\|M\|^2 = \sum_A \alpha_A^2. \tag{2.9}$$

We can therefore show that the norm satisfies[4] the inequality

$$\langle M\widetilde{N}\rangle \ \leq \ \|M\|\ \|N\| \qquad \text{for all} \quad M, N \in \mathcal{G}_3. \tag{2.10}$$

As a consequence of equation (2.10) we obtain the *multivector Cauchy-Schwarz inequality*

$$|\langle M\widetilde{N}\rangle|^2 \ \leq \ \|M\|^2\ \|N\|^2 \qquad \text{for all} \quad M, N \in \mathcal{G}_3. \tag{2.11}$$

[4]Compare appendix A for the proof of (2.10) and (2.11).

## 2.3 Multivector functions, vector differential and vector derivative

Let $f = f(\boldsymbol{x})$ be a multivector-valued function of a vector variable $\boldsymbol{x}$ in $\mathcal{G}_3$ (compare the expansion of $f$ in the basis (2.1) as given in (3.4)). For an arbitrary vector $\boldsymbol{a} \in \mathbb{R}^3$ we define[5] the *vector differential* in the $\boldsymbol{a}$ direction as

$$\boldsymbol{a} \cdot \nabla f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{f(\boldsymbol{x} + \epsilon \boldsymbol{a}) - f(\boldsymbol{x})}{\epsilon} \tag{2.12}$$

provided this limit exists and is well defined. The basis independent *vector derivative* $\nabla$ defined in [4, 7] obeys equation (2.12) for all vectors $\boldsymbol{a}$ and can be expanded as

$$\nabla = \boldsymbol{e}_k \partial_k = \boldsymbol{e}_1 \partial_1 + \boldsymbol{e}_2 \partial_2 + \boldsymbol{e}_3 \partial_3, \tag{2.13}$$

where

$$\partial_k = \boldsymbol{e}_k \cdot \nabla = \frac{\partial}{\partial x_k}, \qquad k = 1, 2, 3 \tag{2.14}$$

is the scalar partial derivative with respect to the $k^{th}$ coordinate $x_k = \boldsymbol{x} \cdot \boldsymbol{e}_k$.

The properties of a vector differential applied to multivector functions resemble much that of one dimensional scalar differentiation sum, constant multiple, product, and chain rules. For example, if $f$ and $g$ are multivector functions of $\boldsymbol{x}$, then the *sum rule* gives

$$\boldsymbol{a} \cdot \nabla (f + g) = \boldsymbol{a} \cdot \nabla f + \boldsymbol{a} \cdot \nabla g, \tag{2.15}$$

and the *product rule* gives

$$\boldsymbol{a} \cdot \nabla (fg) = (\boldsymbol{a} \cdot \nabla f) g + f \boldsymbol{a} \cdot \nabla g. \tag{2.16}$$

If $\alpha$ is a real scalar constant, the *constant multiple rule* yields

$$\boldsymbol{a} \cdot \nabla (\alpha f) = \alpha (\boldsymbol{a} \cdot \nabla f). \tag{2.17}$$

Finally, if $f = f(\lambda(\boldsymbol{x}))$ where $\lambda = \lambda(\boldsymbol{x})$ is a scalar function of $\boldsymbol{x}$, then the *chain rule* leads to

$$\boldsymbol{a} \cdot \nabla f = (\boldsymbol{a} \cdot \nabla \lambda) \frac{\partial f}{\partial \lambda}. \tag{2.18}$$

By using (2.12) and definition 17 of [7] we can derive the general rules[6] for vector differentiation from the corresponding rules for the vector differential as follows:

[5] Bracket convention: $A \cdot BC = (A \cdot B)C \neq A \cdot (BC)$ and $A \wedge BC = (A \wedge B)C \neq A \wedge (BC)$ for multivectors $A, B, C \in \mathcal{G}_{p,q}$. The vector variable index $\boldsymbol{x}$ of the vector derivative is dropped: $\boldsymbol{\nabla_x} = \boldsymbol{\nabla}$ and $\boldsymbol{a} \cdot \boldsymbol{\nabla_x} = \boldsymbol{a} \cdot \boldsymbol{\nabla}$, but not when differentiating with respect to a different vector variable (compare e.g. proposition 2.10).

[6] Compare [4, 7] for the frame (basis) independent proofs of these propositions.

**Proposition 2.7** $\nabla(f+g) = \nabla f + \nabla g$.

**Proposition 2.8** $\nabla(fg) = (\dot{\nabla}\dot{f})g + \dot{\nabla}f\dot{g} = (\dot{\nabla}\dot{f})g + \sum_{k=1}^{n} \boldsymbol{e}_k f(\partial_k g)$.
*(Multivector functions $f$ and $g$ do not necessarily commute.)*

**Proposition 2.9** *For* $f(\boldsymbol{x}) = g(\lambda(\boldsymbol{x}))$, $\lambda(\boldsymbol{x}) \in \mathbb{R}$,

$$\boldsymbol{a} \cdot \nabla f = \{\boldsymbol{a} \cdot \nabla \lambda(\boldsymbol{x})\} \frac{\partial g}{\partial \lambda}$$

**Proposition 2.10** $\nabla f = \nabla_{\boldsymbol{a}} (\boldsymbol{a} \cdot \nabla f)$ *(derivative from differential)*

Differentiating twice with the vector derivative, we get the differential Laplacian operator $\nabla^2$. We can write $\nabla^2 = \nabla \cdot \nabla + \nabla \wedge \nabla$. But for integrable functions $\nabla \wedge \nabla = 0$. In this case we have $\nabla^2 = \nabla \cdot \nabla$.

**Proposition 2.11** *(integration of parts)*

$$\int_{\mathbb{R}^3} g(\boldsymbol{x})[\boldsymbol{a}\cdot\nabla h(\boldsymbol{x})] d^3\boldsymbol{x} = \left[\int_{\mathbb{R}^2} g(\boldsymbol{x})h(\boldsymbol{x})d^2\boldsymbol{x}\right]_{a\cdot x=-\infty}^{a\cdot x=\infty} - \int_{\mathbb{R}^3} [\boldsymbol{a}\cdot\nabla g(\boldsymbol{x})]h(\boldsymbol{x})d^3\boldsymbol{x}$$

We illustrate proposition 2.11 by inserting $\boldsymbol{a} = \boldsymbol{e}_3$, i.e.

$$\int_{\mathbb{R}^3} g(\boldsymbol{x})[\partial_3 h(\boldsymbol{x})] d^3\boldsymbol{x} = \left[\int_{\mathbb{R}^2} g(\boldsymbol{x})h(\boldsymbol{x})dx_1 dx_2\right]_{x_3=-\infty}^{x_3=\infty} - \int_{\mathbb{R}^3} [\partial_3 g(\boldsymbol{x})]h(\boldsymbol{x})d^3\boldsymbol{x},$$

which is nothing but the usual integration of parts formula for the partial derivative $\partial_3 h(\boldsymbol{x})$.

It is convenient to introduce an inner product of $\mathbb{R}^3 \to Cl_{3,0}$ functions $f, g$ as follows

$$(f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})} = \int_{\mathbb{R}^3} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\, d^3\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A \widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^3} f_A(\boldsymbol{x}) g_B(\boldsymbol{x})\, d^3\boldsymbol{x}. \tag{2.19}$$

In (2.19) the inner product $(\ ,\ )_{L^2(\mathbb{R}^3;Cl_{3,0})}$ satisfies the following conditions[1]

$$\begin{aligned}
(f, g+h)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= (f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})} + (f,h)_{L^2(\mathbb{R}^3;Cl_{3,0})}, \\
(f, \lambda g)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= (f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})}\tilde{\lambda}, \\
(f\lambda, g)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= (f, g\tilde{\lambda})_{L^2(\mathbb{R}^3;Cl_{3,0})}, \\
(f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= \widetilde{(g,f)}_{L^2(\mathbb{R}^3;Cl_{3,0})}.
\end{aligned} \tag{2.20}$$

where $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$, and the constant multivector $\lambda \in Cl_{3,0}$. The scalar part of the inner product gives the $L^2$-norm

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} &= \left\langle (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})} \right\rangle \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{x}) * \tilde{f}(\boldsymbol{x}) d^3\boldsymbol{x} \overset{(2.6)}{=} \int_{\mathbb{R}^3} \sum_A f_A^2(\boldsymbol{x}) d^3\boldsymbol{x}. \end{aligned} \tag{2.21}$$

In particular for $g = \boldsymbol{a}f,\ f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$, $\boldsymbol{a} \in \mathbb{R}^3$ we get because of $\langle \boldsymbol{a}f\widetilde{\boldsymbol{a}f}\rangle_0 = \langle \boldsymbol{a}f\tilde{f}\boldsymbol{a}\rangle_0 = \langle \boldsymbol{a}^2 f\tilde{f}\rangle_0 = \boldsymbol{a}^2 f * \tilde{f}$

$$\|\boldsymbol{a}f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} = \int_{\mathbb{R}^3} \boldsymbol{a}^2 f(\boldsymbol{x}) * \tilde{f}(\boldsymbol{x}) d^3\boldsymbol{x} = \int_{\mathbb{R}^3} \boldsymbol{a}^2 \sum_A f_A^2(\boldsymbol{x}) d^3\boldsymbol{x}. \tag{2.22}$$

**Definition 2.12 (Clifford module)** *Let $Cl_{3,0}$ be the real Clifford algebra of 3D Euclidean space $\mathbb{R}^3$. A Clifford algebra module $L^2(\mathbb{R}^3; Cl_{3,0})$ is defined by*

$$L^2(\mathbb{R}^3; Cl_{3,0}) = \{f : \mathbb{R}^3 \longrightarrow Cl_{3,0} \mid \|f\|_{L^2(\mathbb{R}^3;Cl_{3,0})} < \infty\}. \tag{2.23}$$

# 3 Clifford Fourier transform

In this section we present the Fourier transform in $\mathbb{R}$ and generalize it to Clifford's geometric algebra $\mathcal{G}_3$. Generalizations to other dimensions can be found in [16, 17, 18].

## 3.1 Fourier transform in $\mathbb{R}$

Popoulis [8] defined the Fourier transform and its inverse as follows:

**Definition 3.1** *For an integrable function $f \in L^2(\mathbb{R})$, the Fourier transform of $f$ is the function $\mathcal{F}\{f\}$: $\mathbb{R} \to \mathbb{C}$ given by*

$$\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-i\omega x}\, dx, \tag{3.1}$$

*where $i^2$=-1 is the unit imaginary, and* $\exp(-i\omega x) = \cos(\omega x) + i\sin(\omega x)$.

The function $\mathcal{F}\{f\}(\omega)$ has the general form

$$\mathcal{F}\{f\}(\omega) = A(\omega) + iB(\omega) = C(\omega)e^{i\phi(\omega)}. \tag{3.2}$$

$C(\omega)$ is called the Fourier spectrum of $f(t)$, $C^2(\omega)$ its energy spectrum, and $\phi(\omega)$ its phase angle.

Table 3.1: Properties of the traditional Fourier transform

| **Property** | **Function** | **Fourier Transform** |
|---|---|---|
| Linearity | $\alpha f(x)+\beta g(x)$ | $\alpha\mathcal{F}\{f\}(\omega)+\ \beta\mathcal{F}\{g\}(\omega)$ |
| Delay | $f(x-a)$ | $e^{-i\omega a}\mathcal{F}\{f\}(\omega)$ |
| Shift | $e^{i\omega_0 x}f(x)$ | $\mathcal{F}\{f\}(\omega-\omega_0)$ |
| Scaling | $f(ax)$ | $\frac{1}{\lvert a\rvert}\mathcal{F}\{f\}(\frac{\omega}{a})$ |
| Convolution | $(f\star g)(x)$ | $\mathcal{F}\{f\}(\omega)\,\mathcal{F}\{g\}(\omega)$ |
| Derivative | $f^{(n)}(x)$ | $(i\omega)^n\mathcal{F}\{f\}(\omega)$ |
| Parseval theorem | $\int_{\mathbb{R}}\lvert f(x)\rvert^2\,dx$ | $\frac{1}{2\pi}\int_{\mathbb{R}}\lvert\mathcal{F}\{f\}(\omega)\rvert^2\,d\omega$ |

**Definition 3.2** *If $\mathcal{F}\{f\}(\omega)\in L^2(\mathbb{R})$ and $f\in L^2(\mathbb{R})$, the inverse Fourier transform is given by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}(\omega)] = f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}\{f\}(\omega)\,e^{i\omega x}\,d\omega. \tag{3.3}$$

The following table 3.1 summarizes some basic properties of the Fourier transform.

## 3.2 Clifford geometric algebra Fourier transform in $\mathcal{G}_3$

Consider a multivector valued function $f(\boldsymbol{x})$ in $\mathcal{G}_3$, i.e. $f\colon\,\mathbb{R}^3\to\mathcal{G}_3$ where $\boldsymbol{x}$ is a vector variable. With the help of equation (2.2) $f(\boldsymbol{x})$ can be decomposed as

$$\begin{aligned} f(\boldsymbol{x}) = \sum_A f_A(\boldsymbol{x})\boldsymbol{e}_A \quad &= \quad f_0(\boldsymbol{x}) + f_1(\boldsymbol{x})\boldsymbol{e}_1 + f_2(\boldsymbol{x})\boldsymbol{e}_2 + f_3(\boldsymbol{x})\boldsymbol{e}_3 \\ &\quad + f_{12}(\boldsymbol{x})\,\boldsymbol{e}_{12} + f_{31}(\boldsymbol{x})e_{31} + f_{23}(\boldsymbol{x})\boldsymbol{e}_{23} + f_{123}(\boldsymbol{x})\boldsymbol{e}_{123}, \end{aligned} \tag{3.4}$$

where the $f_A$ are eight real-valued functions. Equation (3.4) can also be written as (compare table 2.1)

$$\begin{aligned} f(\boldsymbol{x}) \quad &= \quad [f_0(\boldsymbol{x}) + f_{123}(\boldsymbol{x})i_3] + [f_1(\boldsymbol{x}) + f_{23}(\boldsymbol{x})i_3]\,\boldsymbol{e}_1 \\ &\quad + [f_2(\boldsymbol{x}) + f_{31}(\boldsymbol{x})i_3]\,\boldsymbol{e}_2 + [f_3(\boldsymbol{x}) + f_{12}(\boldsymbol{x})i_3]\,\boldsymbol{e}_3. \end{aligned} \tag{3.5}$$

Equation (3.5) can be regarded as a set of four complex functions. This motivates the extension of the Fourier transform to $\mathcal{G}_3$ multivector functions $f$. We will call this the Clifford Fourier transform (CFT).

Alternatively to (3.7), Bülow et. al. [12] extended the real Fourier transform to the n-dimensional geometric algebra $\mathcal{G}_{0,n}$. This variant of the Clifford Fourier transform of a multivector valued function in $\mathcal{G}_{0,n}$ is given by

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \prod_{k=1}^{n} e^{-\boldsymbol{e}_k 2\pi \omega_k x_k} d^n \boldsymbol{x}, \tag{3.6}$$

where

$$\boldsymbol{x} = \sum_{k=1}^{k=n} x_k \boldsymbol{e}_k, \quad \boldsymbol{\omega} = \sum_{k=1}^{k=n} \omega_k \boldsymbol{e}_k, \quad \text{and} \quad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = -\delta_{ij} \quad i, j = 1, 2, ..., n.$$

Yet in the following we will adopt (compare [10])

**Definition 3.3** *The Clifford Fourier transform of $f(\boldsymbol{x})$ is the function $\mathcal{F}\{f\}$: $\mathbb{R}^3 \to \mathcal{G}_3$ given by*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{x}, \tag{3.7}$$

where we can write $\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3$, $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3$ with $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ the basis vectors of $\mathbb{R}^3$. Note that[7]

$$d^3\boldsymbol{x} = \frac{d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge d\boldsymbol{x_3}}{i_3} \tag{3.8}$$

is scalar valued ($d\boldsymbol{x_k} = dx_k \boldsymbol{e_k}$, $k = 1, 2, 3$, no summation). Because $i_3$ commutes with every element of $\mathcal{G}_3$, the Clifford Fourier kernel $e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}$ will also commute with every element of $\mathcal{G}_3$.

**Theorem 3.4** *The Clifford Fourier transform $\mathcal{F}\{f\}$ of $f \in L^2(\mathbb{R}^3, \mathcal{G}_3)$, $\int_{\mathbb{R}^3} \|f\|^2 d^3\boldsymbol{x} < \infty$ is invertible and its inverse is calculated by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{\omega}. \tag{3.9}$$

[7] The division by the geometric algebra unit volume element $i_3$ in (3.8) to obtain a scalar infinitesimal volume is a matter of choice. Defining instead the pseudoscalar $d^3\boldsymbol{x}_p = d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge d\boldsymbol{x_3}$ would work equally well. It would simply mean, that all integrals using $d^3\boldsymbol{x}_p$ instead of $d^3\boldsymbol{x}$ in this paper would have to be multiplied by $-i_3 = \frac{1}{i_3}$, which commutes with every multivector.

**Proof** Substituting equation (3.7) in equation (3.9) gives

$$\begin{aligned}
\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\boldsymbol{y})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d^3\boldsymbol{y}\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})\, \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i_3(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\omega}}\, d^3\boldsymbol{\omega}\, d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})\, \boldsymbol{\delta}(\boldsymbol{x}-\boldsymbol{y})\, d^3\boldsymbol{y} \\
&= f(\boldsymbol{x}).
\end{aligned}$$

Equation (3.9) is called the Clifford Fourier integral theorem. It describes how to get from the transform back to the original function $f$.

# 4 Basic properties of Clifford Fourier transform

We summarize some important properties of the Clifford Fourier transform which are similar to the traditional scalar Fourier transform properties. Most can be proved via substitution of variables.

## 4.1 Linearity

If $f(\boldsymbol{x}) = \alpha f_1(\boldsymbol{x}) + \beta f_2(\boldsymbol{x})$ for constants $\alpha$ and $\beta$, $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x}) \in \mathcal{G}_3$ then by construction

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \alpha\mathcal{F}\{f_1\}(\boldsymbol{\omega}) + \beta\mathcal{F}\{f_2\}(\boldsymbol{\omega}). \tag{4.1}$$

## 4.2 Delay property

If the argument of $f(\boldsymbol{x})$ is offset by a constant vector $\boldsymbol{a}$, i.e. $f_d(\boldsymbol{x}) = f(\boldsymbol{x}-\boldsymbol{a})$, then

$$\mathcal{F}\{f_d\}(\boldsymbol{\omega}) = e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{4.2}$$

**Proof** Equation (3.7) gives

$$\mathcal{F}\{f_d\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x}-\boldsymbol{a})e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}.$$

We substitute $\boldsymbol{t}$ for $\boldsymbol{x}-\boldsymbol{a}$ in the above expression, and get with $d^3\boldsymbol{x}= d^3\boldsymbol{t}$

$$\begin{aligned}
\mathcal{F}\{f_d\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(\boldsymbol{t})e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{t}}\, d^3\boldsymbol{t} \\
&= e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega}).
\end{aligned}$$

This proves (4.2).

### 4.3 Scaling property

Let $a$ be a positive scalar constant, then the Clifford Fourier transform of the function $f_a(\boldsymbol{x}) = f(a\boldsymbol{x})$ becomes

$$\mathcal{F}\{f_a\}(\boldsymbol{\omega}) = \frac{1}{a^3}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a}). \tag{4.3}$$

**Proof**. Equation (3.7) gives

$$\mathcal{F}\{f_a\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(a\boldsymbol{x})e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}.$$

We substitute $\boldsymbol{u}$ for $a\boldsymbol{x}$, and get

$$\begin{aligned}
\mathcal{F}\{f_a\}(\boldsymbol{\omega}) &= \frac{1}{a^3}\int_{\mathbb{R}^3} f(\boldsymbol{u})e^{-i_3\left(\frac{\boldsymbol{\omega}}{a}\cdot\boldsymbol{u}\right)} d^3\boldsymbol{u} \\
&= \frac{1}{a^3}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a}).
\end{aligned}$$

### 4.4 Shift property

If $\boldsymbol{\omega}_0 \in \mathbb{R}^3$ and $f_0(\boldsymbol{x}) = f(\boldsymbol{x})e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}}$, then

$$\mathcal{F}\{f_0\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0) \tag{4.4}$$

**Proof** Using equation (3.7) and simplifying it we obtain

$$\begin{aligned}
\mathcal{F}\{f_0\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(\boldsymbol{x})e^{-i_3(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= \mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0).
\end{aligned}$$

The shift property shows that the multiplication by $e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}}$ shifts the CFT of the multivector function $f(\boldsymbol{x})$ so that it becomes centered on the point $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ in the frequency domain.

## 5 Differentiation of Clifford Fourier transform

The CFT differentiation properties also resemble that of the traditional scalar Fourier transform of table 3.1.

## 5.1 Vector differential and partial differentiation

The Clifford Fourier transform of the vector differential of $f(\boldsymbol{x})$ is

$$\mathcal{F}\{\boldsymbol{a}\cdot\nabla f\}(\boldsymbol{\omega}) = i_3\,\boldsymbol{a}\cdot\boldsymbol{\omega}\,\mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{5.1}$$

The vector differential in the $\boldsymbol{a}$ direction is

$$\begin{aligned}
\boldsymbol{a}\cdot\nabla f(\boldsymbol{x}) &= \boldsymbol{a}\cdot\nabla\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f\}(\boldsymbol{\omega})\,e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d^3\boldsymbol{\omega}\\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f\}(\boldsymbol{\omega})\,(\boldsymbol{a}\cdot\nabla e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}})\,d^3\boldsymbol{\omega}\\
&\overset{\text{Prop. 2.9}}{=} \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f\}(\boldsymbol{\omega})\,(i_3\,\boldsymbol{a}\cdot\boldsymbol{\omega})e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d^3\boldsymbol{\omega}\\
&= \mathcal{F}^{-1}[i_3\;\boldsymbol{a}\cdot\boldsymbol{\omega}\mathcal{F}\{f\}](\boldsymbol{x}).
\end{aligned}$$

This proves (5.1). Setting $\boldsymbol{a}=\boldsymbol{e}_k$ we get for a partial derivative of $f(\boldsymbol{x})$

$$\mathcal{F}\{\partial_k f\}(\boldsymbol{\omega}) = i_3\omega_k\mathcal{F}\{f\}(\boldsymbol{\omega}), \qquad k=1,2,3. \tag{5.2}$$

By a similar calculation we can find the derivatives of second order, i.e.

$$\mathcal{F}\{\boldsymbol{a}\cdot\nabla\;\boldsymbol{b}\cdot\nabla f\}(\boldsymbol{\omega}) = -\boldsymbol{a}\cdot\boldsymbol{\omega}\;\boldsymbol{b}\cdot\boldsymbol{\omega}\;\mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{5.3}$$

For $\boldsymbol{a}=\boldsymbol{e}_k, \boldsymbol{b}=\boldsymbol{e}_l$ we therefore get

$$\mathcal{F}\{\partial_k\partial_l\}(\boldsymbol{\omega}) = -\omega_k\omega_l\mathcal{F}\{f\}(\boldsymbol{\omega}) \qquad k,l=1,2,3. \tag{5.4}$$

If $\boldsymbol{x}$ is a vector variable, then

$$\mathcal{F}\{\boldsymbol{x}f(\boldsymbol{x})\}(\boldsymbol{\omega}) = i_3\nabla_{\boldsymbol{\omega}}\mathcal{F}\{f\}(\boldsymbol{\omega}) \tag{5.5}$$

**Proof** Direct calculation gives

$$\begin{aligned}
\mathcal{F}\{\boldsymbol{x}f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3}\boldsymbol{x}f(\boldsymbol{x})\,e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d^3\boldsymbol{x}\\
&= \int_{\mathbb{R}^3}\boldsymbol{x}\,e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}f(\boldsymbol{x})\,d^3\boldsymbol{x}\\
&= \int_{\mathbb{R}^3}i_3\;\nabla_{\boldsymbol{\omega}}\;e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}f(\boldsymbol{x})\,d^3\boldsymbol{x}\\
&= i_3\;\nabla_{\boldsymbol{\omega}}\int_{\mathbb{R}^3}f(\boldsymbol{x})\;e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d^3\boldsymbol{x}\\
&= i_3\nabla_{\boldsymbol{\omega}}\mathcal{F}\{f\}(\boldsymbol{\omega}),
\end{aligned}$$

because we get with propositions 2.9 and 2.10

$$\nabla_{\boldsymbol{\omega}}\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}} = -i_3\left[\nabla_{\boldsymbol{\omega}}\,(\boldsymbol{\omega}\cdot\boldsymbol{x})\right]\; e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}} = -i_3\boldsymbol{x}e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}. \tag{5.6}$$

The Clifford Fourier transform of $\boldsymbol{a}\cdot\boldsymbol{x}\; f(\boldsymbol{x})$ gives

$$\begin{aligned}
\mathcal{F}\{\boldsymbol{a}\cdot\boldsymbol{x}\; f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x}\; f(\boldsymbol{x})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{x})\;\boldsymbol{a}\cdot\boldsymbol{x}\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\
&\overset{\text{Prop. 2.9}}{=} \int_{\mathbb{R}^3} f(\boldsymbol{x})\, i_3\;\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\; e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= i_3\;\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\int_{\mathbb{R}^3} f(\boldsymbol{x})\; e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\; d^3\boldsymbol{x} \\
&= i_3\,\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\mathcal{F}\{f\}(\boldsymbol{\omega}).
\end{aligned} \tag{5.7}$$

For $\boldsymbol{a} = e_k$ $(k = 1,\, 2,\, 3)$ we get

$$\mathcal{F}\{x_k\; f(\boldsymbol{x})\}(\boldsymbol{\omega}) = i_3\;\frac{\partial}{\partial\omega_k}\mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{5.8}$$

## 5.2 Vector derivative and Laplace operator

The Clifford Fourier transform of the vector derivative is

$$\mathcal{F}\{\nabla f\}(\boldsymbol{\omega}) = i_3\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega}) \tag{5.9}$$

and of the Laplace operator

$$\mathcal{F}\{\nabla^2 f\}(\boldsymbol{\omega}) = -\boldsymbol{\omega}^2\mathcal{F}\{f\}(\boldsymbol{\omega}) \tag{5.10}$$

**Proof** For $g(\boldsymbol{x})$= $e^{i_3\lambda(\boldsymbol{x})}$, $\lambda(\boldsymbol{x}) = \boldsymbol{\omega}\cdot\boldsymbol{x}$ reference [7] gives

$$\boldsymbol{a}\cdot\nabla g = \boldsymbol{a}\cdot\boldsymbol{\omega}\, i_3\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}},$$

where we used proposition 2.9 and $\boldsymbol{a}\cdot\nabla(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \boldsymbol{a}\cdot\boldsymbol{\omega}$. Applying proposition 2.10, we obtain

$$\begin{aligned}
\nabla g &= \nabla_{\boldsymbol{a}}\,(\boldsymbol{a}\cdot\nabla g) \\
&= \nabla_{\boldsymbol{a}}\,\{\boldsymbol{a}\cdot\boldsymbol{\omega}\, i_3\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\} \\
&= \nabla_{\boldsymbol{a}}\,\{\boldsymbol{a}\cdot\boldsymbol{\omega}\}\, i_3\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}} \\
&= i_3\,\boldsymbol{\omega}\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}.
\end{aligned} \tag{5.11}$$

According to proposition 72 of [7], the application of (5.11) leads to

$$\begin{aligned}\nabla f(\boldsymbol{x}) &= \nabla \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathcal{F}\{f\}(\boldsymbol{\omega}) e^{i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}} d^3\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \nabla\, e^{i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}} \mathcal{F}\{f\}(\boldsymbol{\omega})\, d^3\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} i_3\boldsymbol{\omega}\, e^{i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}} \mathcal{F}\{f\}(\boldsymbol{\omega})\, d^3\boldsymbol{\omega} \\ &= \mathcal{F}^{-1}[i_3\boldsymbol{\omega}\mathcal{F}\{f\}](\boldsymbol{x}), \end{aligned} \tag{5.12}$$

and therefore

$$\mathcal{F}\{\nabla f\}(\boldsymbol{\omega}) = i_3\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega}).$$

Vector differentiating equation (5.12) once more we get

$$\begin{aligned}\mathcal{F}\{\nabla^2 f\} &= \mathcal{F}\{\nabla(\nabla f)\} \\ &= i_3\boldsymbol{\omega}\mathcal{F}\{\nabla f\}(\boldsymbol{\omega}) \\ &= -\boldsymbol{\omega}^2\mathcal{F}\{f\}(\boldsymbol{\omega}). \end{aligned} \tag{5.13}$$

In general[8] we get

$$\mathcal{F}\{\nabla^m f\} = (i_3\boldsymbol{\omega})^m\mathcal{F}\{f\}(\boldsymbol{\omega}), \qquad m \in \mathbb{N}. \tag{5.14}$$

# 6 Convolution and CFT

The most important property of the Clifford Fourier tansform for signal processing applications is the convolution theorem. Because of the non-Abelian geometric product we have the following definition:

**Definition 6.1** *Let $f$ and $g$ be multivector valued functions and both have Clifford Fourier transforms, then the convolution of $f$ and $g$ is denoted $f\star g$, and defined by*

$$(f\star g)(\boldsymbol{x}) = \int_{\mathbb{R}^3} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})\, d^3\boldsymbol{y}, \tag{6.1}$$

**Theorem 6.2** *The Clifford Fourier transform of the convolution of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ is equal to the product of the Clifford Fourier transforms of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, i.e*

$$\mathcal{F}\{f\star g\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega})\mathcal{F}\{g\}(\boldsymbol{\omega}), \tag{6.2}$$

[8] This general formula should prove very useful for transforming partial differential equations (more precisely: vector derivative equations) into algebraic equations.

**Proof** Let $\mathcal{F}\{f\}(\boldsymbol{\omega})$ and $\mathcal{F}\{g\}(\boldsymbol{\omega})$ denote the Clifford Fourier transforms of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ respectively. Transforming equation (6.1), we get

$$\begin{aligned}\mathcal{F}\{f\star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3}[\int_{\mathbb{R}^3} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})d^3\boldsymbol{y}]\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{y})[\int_{\mathbb{R}^3} g(\boldsymbol{x}-\boldsymbol{y})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}]\, d^3\boldsymbol{y}.\end{aligned}$$

By introducing the vector $\boldsymbol{z}=\boldsymbol{x}-\boldsymbol{y}$, the transform can be reexpressed as

$$\begin{aligned}\mathcal{F}\{f\star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(\boldsymbol{y})[\int_{\mathbb{R}^3} g(\boldsymbol{z})\, e^{-i_3[\boldsymbol{\omega}\cdot(\boldsymbol{y}+\boldsymbol{z})]}\, d^3\boldsymbol{z}]\, d^3\boldsymbol{y} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{y})[\int_{\mathbb{R}^3} g(\boldsymbol{z})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d^3\boldsymbol{z}]\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d^3\boldsymbol{y} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{y})e^{-i_3(\boldsymbol{\omega}\cdot\boldsymbol{y})}\, d^3\boldsymbol{y}\mathcal{F}\{g\}(\boldsymbol{\omega}) \\ &= \mathcal{F}\{f\}(\boldsymbol{\omega})\mathcal{F}\{g\}(\boldsymbol{\omega}).\end{aligned}$$

## 7 Plancherel and Parseval theorems

Just as in the case of the traditional scalar Fourier transform, the **Plancherel theorem** in the geometric algebra $\mathcal{G}_3$ relates two multivector functions with their Clifford Fourier transforms.

**Theorem 7.1** *Assume that $f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) \in \mathcal{G}_3$ with Clifford Fourier transform $\mathcal{F}\{f_1\}(\boldsymbol{\omega})$ and $\mathcal{F}\{f_2\}(\boldsymbol{\omega})$ respectively, then*

$$\langle f_1(\boldsymbol{x})\ \widetilde{f_2(\boldsymbol{x})}\rangle_V = \frac{1}{(2\pi)^3}\langle \mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V, \tag{7.1}$$

*where we define the volume integral*

$$\langle f_1(\boldsymbol{x})\ \widetilde{f_2(\boldsymbol{x})}\rangle_V = \int_{\mathbb{R}^3} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}d^3\boldsymbol{x} \tag{7.2}$$

**Proof** Direct calculation yields

$$\begin{aligned}
\langle f_1(\boldsymbol{x})\,\widetilde{f_2(\boldsymbol{x})}\rangle_V &= \int_{\mathbb{R}^3} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}d^3\boldsymbol{x} \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}[\int_{\mathbb{R}^3}\mathcal{F}\{f_1\}(\boldsymbol{\omega})\,e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}d^3\boldsymbol{\omega}]\,\widetilde{f_2(\boldsymbol{x})}d^3\boldsymbol{x} \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f_1\}(\boldsymbol{\omega})\left[\int_{\mathbb{R}^3}\widetilde{f_2(\boldsymbol{x})\,e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}}\,d^3\boldsymbol{x}\right]d^3\boldsymbol{\omega}. \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}d^3\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^3}\langle\mathcal{F}\{f_1\}(\boldsymbol{\omega})\,\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V.
\end{aligned}$$

In particular, with $f_1(\boldsymbol{x}) = f_2(\boldsymbol{x}) = f(\boldsymbol{x})$, we get the (multivector) **Parseval theorem**, i.e.

$$\langle f(\boldsymbol{x})\,\widetilde{f(\boldsymbol{x})}\rangle_V = \frac{1}{(2\pi)^3}\langle\mathcal{F}\{f\}(\boldsymbol{\omega})\,\widetilde{\mathcal{F}\{f\}(\boldsymbol{\omega})}\rangle_V, \tag{7.3}$$

Note that equation (7.1) is multivector valued. This theorem holds for each grade $k$ of the multivectors on both sides of equation (7.1)

$$\langle\langle f_1(\boldsymbol{x})\,\widetilde{f_2(\boldsymbol{x})}\rangle_V\rangle_k = \frac{1}{(2\pi)^3}\langle\langle\mathcal{F}\{f_1\}(\boldsymbol{\omega})\,\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V\rangle_k, \quad k = 0, 1, 2, 3. \tag{7.4}$$

For $k = 0$ and according to equations (7.2) and (2.5), the (scalar) Parseval theorem becomes

$$\int_{\mathbb{R}^3} \|f(\boldsymbol{x})\|^2\; d^3\boldsymbol{x} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\; d^3\boldsymbol{\omega}. \tag{7.5}$$

Because of the similarity with equation (3.2) we call $\int_{\mathbb{R}^3}\|f(\boldsymbol{x})\|^2\; d^3\boldsymbol{x}$ the energy of $f$. Finally, we summarize the properties of the Clifford Fourier transform (CFT) in table 7.1.

# 8 Introduction to the geometric algebra treatment of wavelets

Transformations such as the Fourier transformation are powerful methods for signal representations and feature detection in signals. The signals are transformed from the original domain to the spectral or frequency domain. In the frequency domain many characteristics of a signal are seen more

Table 7.1: Properties of the Clifford Fourier transform (CFT)

| **Property** | **Multivector Function** | **CFT** |
|---|---|---|
| Linearity | $\alpha f(\boldsymbol{x})+\beta\, g(\boldsymbol{x})$ | $\alpha\mathcal{F}\{f\}(\boldsymbol{\omega})+\,\beta\mathcal{F}\{g\}(\boldsymbol{\omega})$ |
| Delay | $f(\boldsymbol{x}-\boldsymbol{a})$ | $e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega})$ |
| Shift | $e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}}f(\boldsymbol{x})$ | $\mathcal{F}\{f\}(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$ |
| Scaling | $f(a\boldsymbol{x})$ | $\frac{1}{a^3}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$ |
| Convolution | $(f\star g)(\boldsymbol{x})$ | $\mathcal{F}\{f\}(\boldsymbol{\omega})\,\mathcal{F}\{g\}(\boldsymbol{\omega})$ |
| Vec. diff. | $\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})$ | $i_3\;\boldsymbol{a}\cdot\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega})$ |
| | $\boldsymbol{a}\cdot\boldsymbol{x}\;f(\boldsymbol{x})$ | $i_3\;\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\;\mathcal{F}\{f\}(\boldsymbol{\omega})$ |
| | $\boldsymbol{x}f(\boldsymbol{x})$ | $i_3\;\nabla_{\boldsymbol{\omega}}\;\mathcal{F}\{f\}(\boldsymbol{\omega})$ |
| Vec. deriv. | $\nabla^m f(\boldsymbol{x})$ | $(i_3\;\boldsymbol{\omega})^m\mathcal{F}\{f\}(\boldsymbol{\omega})$ |
| Plancherel T. | $\langle f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}\rangle_V$ | $\frac{1}{(2\pi)^3}\langle\mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V$ |
| sc. Parseval T. | $\int_{\mathbb{R}^3}\;\Vert f(\boldsymbol{x})\Vert^2\,d^3\boldsymbol{x}$ | $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\;\Vert\mathcal{F}\{f\}(\boldsymbol{\omega})\Vert^2\,d^3\boldsymbol{\omega}$ |

clearly. In contrast to the Fourier kernel, wavelet basis functions are localized in both spatial and frequency domains and thus yield very sparse and well-structured representations of *piecewise smooth signals* (signals that are smooth except for a finite number of discontinuous jumps), important facts from a signal processing point of view.

On the other hand Clifford geometric algebra leads to the consequent generalization[1] of real and harmonic analysis to higher dimensions. Clifford algebra accurately treats geometric entities depending on their dimension as scalars, vectors, bivectors (plane area elements), and volume elements, etc. The distinction of axial and polar vectors in physics, e.g. is resolved in the form of vectors and bivectors. The quaternion description of rotations[6] is fully incorporated in the form of rotors. With respect to the geometric product of vectors division by non-zero vectors is defined. Clifford algebra has applications in signal and image processing.[13]

This motivated Mitrea[29] to generalize discrete real wavelets to discrete Clifford algebra wavelets. Some properties of these extended wavelets were also demonstrated. This first work was then followed by Brackx and Sommen[30, 31] who proposed an extension of real wavelets to the Clifford algebra $Cl_{0,n}$ called the continuous Clifford wavelet transform. This approach used a group composed of dilations, translations and the Spin-group. Quaternion ($Cl_{0,2}$) wavelets have been studied by Zhao and Peng, [32] and applied by Bayro-Corrochano.[34] Zhao[33] also constructed continuous Clifford algebra $Cl_{0,n}$-valued wavelets using the semi-direct product of closed $GL(n, \mathbb{R})$ subgroups with the translation subgroup of $\mathbb{R}^n$. Some properties of these extended wavelets were investigated using the classical Fourier transform.

The purpose in our tutorial is to construct Clifford algebra $Cl_{3,0}$-valued wavelets using the similitude group *SIM(3)* and then give a detailed explanation of their properties using the Clifford Fourier transform (CFT).[15, 16]

Based on the uncertainty principle for the CFT we derive a generalized Clifford wavelet uncertainty principle. For scalar admissibility constant the interpretation of this uncertainty principle proceeds as usual.

As a concrete example we generalize complex Gabor wavelets to multivector Clifford Gabor wavelets. Next, we describe some of their important properties and we consequently establish an uncertainty principle for Clifford Gabor wavelets.

The outline of the next part of the tutorial is as follows. In section 9, we briefly review Clifford algebra, the CFT, and the similitude group $SIM(3)$. In section 10, we discuss the basic ideas for constructing the Clifford algebra wavelet transform. We then derive important properties of our newly constructed wavelet transform. In section 11, we present the example of multivector Gabor wavelets and show to what extent the properties of these Clifford Gabor wavelets resemble that of real wavelets.

Section 12 is devoted to uncertainty principles (minima of products of variances) for the CFT and GA wavelets. For geometric algebra wavelets we show a generalized Clifford wavelet uncertainty principle. This leads to the uncertainty principle for the Clifford Gabor wavelet transform.

# 9 Wavelet basics: Similitude group

This section recalls the similitude group $SIM(3)$ and its properties from the viewpoint of wavelets.

We consider *the similitude* group *SIM (3)* denoted by $\mathcal{G}$, a subgroup of the affine group of motion on $\mathbb{R}^3$ associated with wavelets as follows (for more details see [22])

$$\mathcal{G} = \mathbb{R}^+ \times SO(3) \otimes \mathbb{R}^3 = \{(a, r_{\boldsymbol{\theta}}, \boldsymbol{b}) | a \in \mathbb{R}^+, r_{\boldsymbol{\theta}} \in SO(3), \boldsymbol{b} \in \mathbb{R}^3\}, \quad (9.1)$$

where $SO(3)$ is the special orthogonal group of $\mathbb{R}^3$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ with $\theta_1 \in [0, \pi], \theta_2, \theta_3 \in [0, 2\pi]$. Instead of $(a, r_{\boldsymbol{\theta}}, \boldsymbol{b})$ we often write simply $(a, \boldsymbol{\theta}, \boldsymbol{b})$. More precisely, we represent $SO(3)$ of $\mathbb{R}^3$ by rotors $R$

$$SO(3) = \{r \,|\, r(\boldsymbol{x}) = \tilde{R}\boldsymbol{x}R, R \in Cl^+_{3,0}, \tilde{R}R = R\tilde{R} = 1\}. \quad (9.2)$$

Any $r \in SO(3)$ has a unique Euler angle representation with rotors of the form

$$R = R_z(\theta_3)R_y(\theta_1)R_z(\theta_2), \tag{9.3}$$

where $R_z, R_y$ denote rotors about the $z$- and $y$-axes, respectively. Note that the group $\mathcal{G}$ includes dilations, rotations and translations. The representation defined by (9.1) is consistent with the group action $(a, \boldsymbol{\theta}, \boldsymbol{b})$ on $\mathbb{R}^3$ as follows

$$\begin{aligned} (a, \boldsymbol{\theta}, \boldsymbol{b}): \quad \mathbb{R}^3 & \to \mathbb{R}^3 \\ \boldsymbol{x} & \mapsto a\tilde{R}(\boldsymbol{\theta})\boldsymbol{x}R(\boldsymbol{\theta}) + \boldsymbol{b}. \end{aligned} \tag{9.4}$$

The above leads to two important propositions.

**Proposition 9.1** *With respect to the representation defined by (9.1), $\mathcal{G}$ is a non-abelian group in which $(1, 1, 0)$ and $(a^{-1}, r^{-1}, -a^{-1}r^{-1}(\boldsymbol{b}) = -R\boldsymbol{b}\tilde{R}/a)$ are its identity element and inverse element, respectively.*

**Proposition 9.2** *The left Haar measure[9] on $\mathcal{G}$ (see [23]) is given by*

$$d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b}) = d\mu(a, \boldsymbol{\theta})d^3\boldsymbol{b},$$
$$d\mu(a, \boldsymbol{\theta}) = \frac{dad\boldsymbol{\theta}}{a^4}, \quad d\boldsymbol{\theta} = \frac{1}{8\pi^2}\sin\theta_1 d\theta_1 d\theta_2 d\theta_3, \tag{9.5}$$

*where $d\boldsymbol{\theta}$ is the Haar measure on SO(3) (see [24]).*

We often abbreviate $d\mu = d\mu(a, \boldsymbol{\theta}),\ \ d\lambda = d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b})$. Similar to (7.2) the inner product of $f(a, \boldsymbol{\theta}, \boldsymbol{b}), g(a, \boldsymbol{\theta}, \boldsymbol{b}) \in L^2(\mathcal{G}; Cl_{3,0})$ is defined by

$$(f, g)_{L^2(\mathcal{G};Cl_{3,0})} = \int_{\mathcal{G}} f(a, \boldsymbol{\theta}, \boldsymbol{b})\widetilde{g(a, \boldsymbol{\theta}, \boldsymbol{b})}\, d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b}), \tag{9.6}$$

and its associated scalar norm

$$\|f\|^2_{L^2(\mathcal{G};Cl_{3,0})} = \left\langle (f, f)_{L^2(\mathcal{G};Cl_{3,0})} \right\rangle = \int_{\mathcal{G}} f(a, \boldsymbol{\theta}, \boldsymbol{b}) * \tilde{f}(a, \boldsymbol{\theta}, \boldsymbol{b})d\boldsymbol{\mu}. \tag{9.7}$$

# 10 Clifford algebra $Cl_{3,0}$-valued wavelet transform

## 10.1 Wavelet family and Fourier transform

Based on the concepts of Clifford algebra, one can extend the real continuous wavelet transform to a continuous Clifford wavelet transform. This

[9] The right Haar measure on $\mathcal{G}$ is $d\delta(a, \boldsymbol{\theta}, \boldsymbol{b}) = \frac{dad\boldsymbol{\theta}}{a}d^3\boldsymbol{b}$.

section constructs the Clifford algebra $Cl_{3,0}$-valued wavelets from a group theoretical point of view. We will see how some properties of the classical wavelet transform are extended in the new construction. In particular we look at the admissibility condition, inner product and norm identities, and a reproducing kernel. We define the unitary linear operator

$$\begin{aligned} U_{a,\boldsymbol{\theta},\boldsymbol{b}} : L^2(\mathbb{R}^3; Cl_{3,0}) &\longrightarrow L^2(\mathcal{G}; Cl_{3,0}) \\ \psi(\boldsymbol{x}) &\longrightarrow U_{a,\boldsymbol{\theta},\boldsymbol{b}}\, \psi(\boldsymbol{x}) = \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) \\ &= \frac{1}{a^{3/2}} \psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-\boldsymbol{b}}{a})). \end{aligned} \tag{10.1}$$

The family of wavelets $\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}$ are so-called *daughter Clifford wavelets* with $a \in \mathbb{R}^+$ as dilation parameter, $\boldsymbol{b} \in \mathbb{R}^3$ as the translation vector parameter, and $\boldsymbol{\theta}$ as the $SO(3)$ rotation parameters. The normalization constant $a^{-3/2}$ ensures that the norm of $\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}$ is independent of $a$, i.e.

$$\|\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\|_{L^2(\mathbb{R}^3;Cl_{3,0})} = ||\psi||_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{10.2}$$

This can be seen from

$$\begin{aligned} \|\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} &= \int_{\mathbb{R}^3} \sum_A \frac{1}{a^3} \psi_A^2(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}))\, d^3\boldsymbol{x} \\ &= \frac{1}{a^3} \int_{\mathbb{R}^3} \sum_A \psi_A^2(\boldsymbol{z}) a^3 \det(r_{\boldsymbol{\theta}})\, d^3\boldsymbol{z} \\ &= \int_{\mathbb{R}^3} \sum_A \psi_A^2(\boldsymbol{z})\, d^3\boldsymbol{z}. \end{aligned} \tag{10.3}$$

Applying (2.21) to the last line of (10.3), we obtain the desired result. $\square$

In the $Cl_{3,0}$ Clifford Fourier domain, equation (10.1) can be represented in the form

$$\mathcal{F}\{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\}(\boldsymbol{\omega}) = e^{-i_3 \boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})). \tag{10.4}$$

Substituting $(\boldsymbol{x}-\boldsymbol{b})/a = \boldsymbol{y}$ for the argument of (10.1) under the CFT integral of (10.4) gives

$$\begin{aligned} \mathcal{F}\{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} \frac{1}{a^{\frac{3}{2}}} \psi(r_{\boldsymbol{\theta}}^{-1}\boldsymbol{y}) e^{-i_3 \boldsymbol{\omega}\cdot(\boldsymbol{b}+ a\boldsymbol{y})} a^3\, d^3\boldsymbol{y} \\ &= e^{-i_3 \boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \int_{\mathbb{R}^3} \psi(r_{\boldsymbol{\theta}}^{-1}\boldsymbol{y}) e^{-i_3 a\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d^3\boldsymbol{y} \\ &= e^{-i_3 \boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})). \end{aligned}$$

$\square$

### 10.2 Admissibility

Following Zhao[33] we call $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ admissible wavelet if

$$C_\psi = \int_{\mathbb{R}^+} \int_{S0(3)} a^3 \{\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \, \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})) \, d\mu, \tag{10.5}$$

is an invertible multivector constant and finite at a.e. $\boldsymbol{\omega} \in \mathbb{R}^3$. The admissibility condition is important to guarantee that the Clifford wavelet transform is invertible as we will see later. We notice that for $\boldsymbol{\omega} = 0$ we get $\hat{\psi}(0) = \int_{\mathbb{R}^3} \psi(\boldsymbol{x}) e^{i_3 0 \cdot \boldsymbol{x}} \, d^3\boldsymbol{x} = 0$ for the scalar part of $C_\psi$ to be finite. Therefore, like classical wavelets (see [25]), an admissible Clifford-valued mother wavelet $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ has to satisfy

$$\int_{\mathbb{R}^3} \psi(\boldsymbol{x}) \, d^3\boldsymbol{x} = \int_{\mathbb{R}^3} \psi_A(\boldsymbol{x}) \boldsymbol{e}_A \, d^3\boldsymbol{x} = 0, \tag{10.6}$$

where $\psi_A(\boldsymbol{x})$ are real-valued wavelets. It means that the integral of every component of the Clifford mother wavelet is zero. The admissibility constant (10.5) can be simplified to

$$C_{\psi^c} = \int_{\mathbb{R}^3} \frac{\widetilde{\widehat{\psi}(\boldsymbol{\xi})} \widehat{\psi}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^3} \, d^3\boldsymbol{\xi}. \tag{10.7}$$

According to (10.5) it is not difficult to see that $C_\psi = \widetilde{C_\psi}$. Consequently, we have

$$C_\psi = \langle C_\psi \rangle + \langle C_\psi \rangle_1, \tag{10.8}$$

with positive scalar part ($\langle C_\psi \rangle > 0$)

$$\begin{aligned} \langle C_\psi \rangle &= \int_{\mathbb{R}^3} \langle \{\widehat{\psi}(\boldsymbol{\xi})\}^{\sim} \widehat{\psi}(\boldsymbol{\xi}) \rangle \, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3 = \left\| \, |\boldsymbol{\xi}|^{-3/2} \, \widehat{\psi}(\boldsymbol{\xi}) \right\|_{L^2(\mathbb{R}^3; Cl_{3,0})} \\ &= \int_{\mathbb{R}^3} [\langle \widehat{\psi}(\boldsymbol{\xi}) \rangle^2 + \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_1^2 - \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_2^2 - \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_3^2] \, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3, \end{aligned} \tag{10.9}$$

and vector part

$$\begin{aligned} \langle C_\psi \rangle_1 &= \int_{\mathbb{R}^3} \langle \{\widehat{\psi}(\boldsymbol{\xi})\}^{\sim} \widehat{\psi}(\boldsymbol{\xi}) \rangle_1 \, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3 \\ &= \int_{\mathbb{R}^3} [\langle \widehat{\psi}(\boldsymbol{\xi}) \rangle \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_1 + \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_1 \cdot \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_2 - \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_2 \cdot \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_3] \, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3. \end{aligned} \tag{10.10}$$

The inverse of $C_\psi$ is given by

$$C_\psi^{-1} = \frac{\langle C_\psi \rangle - \langle C_\psi \rangle_1}{\langle C_\psi \rangle^2 - \langle C_\psi \rangle_1^2}. \tag{10.11}$$

The inverse exists therefore if and only if $\langle C_\psi \rangle_1^2 \neq \langle C_\psi \rangle^2$.

### 10.3 Clifford wavelet transform

**Definition 10.1 (Clifford wavelet transform)** *We define the Clifford wavelet transform with respect to the mother wavelet* $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ *as follows*

$$\begin{aligned} T_\psi : L^2(\mathbb{R}^3; Cl_{3,0}) &\rightarrow L^2(\mathcal{G}; Cl_{3,0}) \\ f &\rightarrow T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\, d^3\boldsymbol{x} \\ &= (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3; Cl_{3,0})}. \end{aligned} \tag{10.12}$$

Note that in general the order of (10.12) is fixed because Clifford multiplication is non-commutative. Alternatively, we may use a convolution ($\star$) to express (10.12) by

$$T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\, d^3\boldsymbol{x} = (f \star \psi_{a,\boldsymbol{\theta}})(\boldsymbol{b}) \tag{10.13}$$

where

$$\psi_{a,\boldsymbol{\theta}}(\boldsymbol{x}) = \frac{1}{a^{\frac{3}{2}}} \psi\{(r_{\boldsymbol{\theta}}^{-1}(\frac{-\boldsymbol{x}}{a}))\}^{\sim}.$$

The Clifford wavelet transform (10.12) has a Clifford Fourier representation of the form

$$T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \widehat{f}(\boldsymbol{\omega})\, e^{i_3 \boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \{\widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim}\, d^3\boldsymbol{\omega} \tag{10.14}$$

**Proof** We have

$$\begin{aligned} &T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) \\ &\overset{(10.12)}{=} \langle f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} \rangle_{L^2(\mathbb{R}^3; Cl_{3,0})} \\ &\overset{Planc.T.}{=} \frac{1}{(2\pi)^3} \langle \widehat{f}, \widehat{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}} \rangle_{L^2(\mathbb{R}^3; Cl_{3,0})} \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{f}(\boldsymbol{\omega}) \left[\widehat{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}}(\boldsymbol{\omega})\right]^{\sim} d^3\boldsymbol{\omega} \\ &\overset{(10.4)}{=} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{f}(\boldsymbol{\omega})\, e^{i_3 \boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \left[\widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\right]^{\sim} d^3\boldsymbol{\omega}. \end{aligned}$$

This proves (10.14). ☐

With the inverse CFT (10.14) becomes

$$T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) = \mathcal{F}^{-1} \left\{ a^{\frac{3}{2}} \widehat{f}(.)[\widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(.))]^{\sim} \right\} (\boldsymbol{b}), \tag{10.15}$$

or equivalently

$$\mathcal{F}(T_\psi f(a,\boldsymbol{\theta},.))(\boldsymbol{\omega}) = a^{\frac{3}{2}}\widehat{f}(\boldsymbol{\omega})\{\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim}. \tag{10.16}$$

## 10.4 Properties of the Clifford wavelet transform

**Theorem 10.2 (Left linearity)** *Let $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$ and $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford mother wavelet. The Clifford wavelet transform $T_\psi$ is a linear operator, i.e.,*

$$[T_\psi(\lambda f + \mu g)](a,\boldsymbol{\theta},\boldsymbol{b}) = \lambda T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) + \mu T_\psi g(a,\boldsymbol{\theta},\boldsymbol{b}), \tag{10.17}$$

*with multivector constants $\lambda, \mu$ in $Cl_{3,0}$.*

**Theorem 10.3 (Translation covariance)** *Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford mother wavelet. If the argument of $T_\psi f(\boldsymbol{x})$ is translated by a constant $\boldsymbol{x}_0 \in \mathbb{R}^3$ then*

$$[T_\psi f(\cdot - \boldsymbol{x}_0)](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b} - \boldsymbol{x}_0). \tag{10.18}$$

**Proof** Equation (10.12) gives

$$\begin{aligned}
[T_\psi f(\cdot - \boldsymbol{x}_0)](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}} f(\boldsymbol{x} - \boldsymbol{x}_0)\widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}} f(\boldsymbol{x} - \boldsymbol{x}_0)\frac{1}{a^{3/2}}\left[\psi(a^{-1}r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{x} - \boldsymbol{b})\right]^{\sim} d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})\frac{1}{a^{3/2}}\left[\psi\left(a^{-1}r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{y} - (\boldsymbol{b} - \boldsymbol{x}_0))\right)\right]^{\sim} d^3\boldsymbol{y} \\
&= T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b} - \boldsymbol{x}_0).
\end{aligned}$$

**Theorem 10.4 (Dilation covariance)** *Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford mother wavelet. If $c$ is a real positive constant, then*

$$[T_\psi f(c\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) = \frac{1}{c^{3/2}}T_\psi f(ac,\boldsymbol{\theta},\boldsymbol{b}c). \tag{10.19}$$

**Proof** Equation (10.12) gives again

$$\begin{aligned}
[T_\psi f(c\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}^3} f(c\boldsymbol{x})\frac{1}{a^{3/2}}\left[\psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x} - b}{a}))\right]^{\sim} d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})\frac{1}{a^{3/2}}\left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{y}/c - \boldsymbol{b}}{a})\right)\right]^{\sim} \frac{1}{c^3}\, d^3\boldsymbol{y} \\
&= \frac{1}{c^{\frac{3}{2}}}\int_{\mathbb{R}^3} f(\boldsymbol{y})\frac{1}{(ac)^{3/2}}\left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{y} - \boldsymbol{b}c}{ac})\right)\right]^{\sim} d^3\boldsymbol{y} \\
&= \frac{1}{c^{3/2}}T_\psi f(ac,\boldsymbol{\theta},\boldsymbol{b}c).
\end{aligned}$$

**Theorem 10.5 (Rotation covariance)** *Let* $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ *be a Clifford mother wavelet. If* $r_{\boldsymbol{\theta}}$ *and* $r_{\boldsymbol{\theta}_0}$ *are both rotations, then*

$$[T_\psi f(r_{\boldsymbol{\theta}_0}\cdot)](a, \boldsymbol{\theta}, \boldsymbol{b}) = T_\psi f(a, \boldsymbol{\theta}', r_{\boldsymbol{\theta}_0}\boldsymbol{b}), \tag{10.20}$$

*with rotors* $R_{\boldsymbol{\theta}'} = R_{\boldsymbol{\theta}_0} R_{\boldsymbol{\theta}}$.

**Proof** Applying equation (10.12) and using the fact that the product of two rotations is always a rotation,[3] we obtain

$$\begin{aligned}
[T_\psi f(r_{\boldsymbol{\theta}_0}\cdot)](a, \boldsymbol{\theta}, \boldsymbol{b}) &= \int_{\mathbb{R}^3} f(r_{\boldsymbol{\theta}_0}\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(r_{\boldsymbol{\theta}_0}\boldsymbol{x}) \left[\psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-b}{a}))\right]^{\sim} d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y}) \left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{r_{\boldsymbol{\theta}_0}^{-1}\boldsymbol{y}-\boldsymbol{b}}{a})\right)\right]^{\sim} \det{}^{-1}(r_{\boldsymbol{\theta}})\, d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y}) \left[\psi\left(r_{\boldsymbol{\theta}}^{-1} r_{\boldsymbol{\theta}_0}^{-1}(\frac{\boldsymbol{y}-r_{\boldsymbol{\theta}_0}\boldsymbol{b}}{a})\right)\right]^{\sim} d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y}) \left[\psi\left((r_{\boldsymbol{\theta}_0} r_{\boldsymbol{\theta}})^{-1}(\frac{\boldsymbol{y}-r_{\boldsymbol{\theta}_0}\boldsymbol{b}}{a})\right)\right]^{\sim} d^3\boldsymbol{y} \\
&= T_\psi f(a, \boldsymbol{\theta}', r_{\boldsymbol{\theta}_0}\boldsymbol{b}),
\end{aligned}$$

where we omit brackets like $r_{\boldsymbol{\theta}_0}\boldsymbol{x} = r_{\boldsymbol{\theta}_0}(\boldsymbol{x})$. This proves (10.20). □
These four properties above correspond to classical wavelet transform properties. Now we will see the differences between the Clifford and the classical wavelet transforms.

**Theorem 10.6 (Inner product relation)** *Let* $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ *be an admissible Clifford mother wavelet and* $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$ *arbitrary. Then we have*

$$\begin{aligned}
(T_\psi f, T_\psi g)_{L^2(\mathcal{G}; Cl_{3,0})} &= (f C_\psi, g)_{L^2(\mathbb{R}^3; Cl_{3,0})} \\
&= \langle C_\psi \rangle (f, g)_{L^2(\mathbb{R}^3; Cl_{3,0})} + (f \langle C_\psi \rangle_1, g)_{L^2(\mathbb{R}^3; Cl_{3,0})}.
\end{aligned} \tag{10.21}$$

Before proving theorem 10.6 we remark that for $\langle C_\psi \rangle_1 = 0$ the operator $\langle C_\psi \rangle^{-1/2} T_\psi$ is an *isometry* from $L^2(\mathbb{R}^3; Cl_{3,0})$ to $L^2(\mathcal{G}; Cl_{3,0})$.

**Proof** By inserting (10.14) into the left side of (10.21), we obtain

$$
\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{3,0})} \\
&= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta})\,\{T_\psi g(a,\boldsymbol{b},\boldsymbol{\theta})\}^\sim\, d^3\boldsymbol{b}d\mu \\
&= \int_{\mathbb{R}^+}\int_{S0(3)} \frac{a^3}{(2\pi)^6}\Big(\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}^3}\hat{f}(\boldsymbol{\omega})\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^\sim e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}}d^3\boldsymbol{\omega} \\
&\qquad \int_{\mathbb{R}^3}\Big\{(\hat{g}(\boldsymbol{\omega}')\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}'))\}^\sim e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}'}\Big\}^\sim\, d^3\boldsymbol{\omega}'\Big]d^3\boldsymbol{b}\Big)d\mu. \qquad (10.22)
\end{aligned}
$$

For abbreviation, we use the notation

$$
F_a(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega})\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^\sim,\; G_a(\boldsymbol{\omega}') = \hat{g}(\boldsymbol{\omega}')\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}'))\}^\sim.
$$

Equation (10.22) can then be rewritten as

$$
\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{3,0})} \\
&= \frac{1}{(2\pi)^6}\int_{\mathbb{R}^+} a^3 \int_{S0(3)}\Big(\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}^3} F_a(\boldsymbol{\omega})e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}}\, d^3\boldsymbol{\omega} \\
&\qquad \int_{\mathbb{R}^3}\{G_a(\boldsymbol{\omega}')e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}'}\}^\sim\, d^3\boldsymbol{\omega}'\Big]d^3\boldsymbol{b}\Big)d\mu \\
&\overset{(3.7)}{=} \frac{1}{(2\pi)^6}\int_{\mathbb{R}^+} a^3 \int_{S0(3)}\left(\int_{\mathbb{R}^3}\hat{F}_a(-\boldsymbol{b})\,\{\hat{G}_a(-\boldsymbol{b})\}^\sim d^3\boldsymbol{b}\right)d\mu \\
&\overset{P.T.}{=} \int_{\mathbb{R}^+}\int_{S0(3)}\frac{a^3}{(2\pi)^3}\left(\int_{\mathbb{R}^3} F_a(\boldsymbol{\xi})\widetilde{G_a(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi}\right)d\mu \\
&= \int_{\mathbb{R}^3}\frac{1}{(2\pi)^3}\left(\int_{\mathbb{R}^+} a^3\int_{S0(3)}\hat{f}(\boldsymbol{\xi})\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))\}^\sim\,\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))\widetilde{\hat{g}(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi}\right)d\mu \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\hat{f}(\boldsymbol{\xi})\left(\int_{\mathbb{R}^+}\int_{S0(3)} a^3\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))\}^\sim\,\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))d\mu\right)\widetilde{\hat{g}(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi} \\
&\overset{10.5}{=} \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\hat{f}(\boldsymbol{\xi})C_\psi\widetilde{\hat{g}(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi} \\
&\overset{P.T.}{=} \int_{\mathbb{R}^3} f(\boldsymbol{x})C_\psi\widetilde{g(\boldsymbol{x})}\, d^3\boldsymbol{x} = (fC_\psi, g)_{L^2(\mathbb{R}^3;Cl_{3,0})},
\end{aligned}
$$

where P.T. denotes the Plancherel theorem of table 7.1. $\square$

As a consequence of theorem 10.6, we immediately obtain

**Corollary 10.7 (Norm relation)** *Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford mother wavelet that satisfies the admissibility condition (10.5). Then for any $f \in$*

$L^2(\mathbb{R}^3; Cl_{3,0})$ *we have*

$$\begin{aligned} \|T_\psi f\|^2_{L^2(\mathcal{G};Cl_{3,0})} &= \langle (fC_\psi, f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle \;=\; C_\psi * (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})} \\ &= \langle C_\psi\rangle \|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} + \langle (f\langle C_\psi\rangle_1, f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle \\ &= \langle C_\psi\rangle \|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} + \langle C_\psi\rangle_1 * \langle (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle_1 \end{aligned} \tag{10.23}$$

According to (2.21) we can rewrite the left hand side of (10.23) in the form

$$\|T_\psi f\|^2_{L^2(\mathcal{G};Cl_{3,0})} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^+}\int_{SO(3)} \sum_A \langle T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\rangle_A^2 \, d\mu d^3\boldsymbol{b}. \tag{10.24}$$

## 10.5 Inverse Clifford wavelet transform, reproducing kernel

In the following we will first derive the important inverse Clifford $Cl_{3,0}$ wavelet transform for multivector functions.

**Theorem 10.8 (Inverse Clifford $Cl_{3,0}$ wavelet transform)** *Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford mother wavelet that satisfies the admissibility condition (10.5). Then any $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ can be decomposed as*

$$\begin{aligned} f(\boldsymbol{x}) &= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta})\, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}, \\ &= \int_{\mathcal{G}} (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} C_\psi^{-1}\, d\mu d^3\boldsymbol{b}, \end{aligned} \tag{10.25}$$

*the integral converging in the weak sense.*

**Proof** Indeed, we have for every $g \in L^2(\mathbb{R}^3; Cl_{3,0})$

$$\begin{aligned} (T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{3,0})} &= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \{T_\psi g(a,\boldsymbol{\theta},\boldsymbol{b})\}^{\sim}\, d\mu d^3\boldsymbol{b} \\ &= \int_{\mathcal{G}}\int_{\mathbb{R}^3} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) \widetilde{g(\boldsymbol{x})}\, d^3\boldsymbol{x} d\mu d^3\boldsymbol{b} \\ &= \int_{\mathbb{R}^3}\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) \widetilde{g(\boldsymbol{x})}\, d\mu d^3\boldsymbol{b} d^3\boldsymbol{x} \\ &= \left(\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, d\mu d^3\boldsymbol{b}\ ,\ g\right)_{L^2(\mathbb{R}^3;Cl_{3,0})} \end{aligned} \tag{10.26}$$

Applying (10.21) of theorem 10.6 gives for every $g \in L^2(\mathbb{R}^3; Cl_{3,0})$

$$(fC_\psi, g)_{L^2(\mathbb{R}^3;Cl_{3,0})} = \left(\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, d\mu d^3\boldsymbol{b}\ ,\ g\right)_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{10.27}$$

Taking the scalar part of (10.27) we obtain

$$\langle (fC_\psi, g)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle = \langle \left( \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, d\mu d^3\boldsymbol{b}\ ,\ g \right)_{L^2(\mathbb{R}^3;Cl_{3,0})} \rangle. \tag{10.28}$$

Because the inner product identity (10.28) holds for every $g \in L^2(\mathbb{R}^3; Cl_{3,0})$ (and in particular for all basis elements of the Clifford module of def. 2.12) we conclude that

$$f(\boldsymbol{x})C_\psi = \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta})\psi_{a,\boldsymbol{b},\boldsymbol{\theta}}(\boldsymbol{x})\, d\mu d^3\boldsymbol{b}\,, \tag{10.29}$$

or equivalently, because of the assumed invertibility of $C_\psi$

$$\begin{aligned} f(\boldsymbol{x}) &= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta})\psi_{a,\boldsymbol{b},\boldsymbol{\theta}}(\boldsymbol{x})\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}. \\ &\overset{(10.12)}{=} \int_{\mathcal{G}} (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3;Cl_{3,0})}\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}. \end{aligned} \tag{10.30}$$

which completes the proof. $\square$

Weak convergence of (10.25) means that for all $g \in L^2(\mathbb{R}^3; Cl_{3,0})$ holds

$$(\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta})\, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} d\mu d^3\boldsymbol{b}\, C_\psi^{-1}\ ,\ g)_{L^2(\mathbb{R}^3;Cl_{3,0})} \to (f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{10.31}$$

Using the properties of the inner product (2.20), it is not difficult to show that (10.25) can alternatively be rewritten in the form ($C_\psi^{-1} = \widetilde{C_\psi^{-1}}$ because of (10.11))

$$f(\boldsymbol{x}) = C_\psi^{-1} \int_{\mathcal{G}} \{\psi_{a,\boldsymbol{b},\boldsymbol{\theta}}\}^\sim\, (\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, ,\, \tilde{f}\,)_{L^2(\mathbb{R}^3;Cl_{3,0})}\, d\mu d^3\boldsymbol{b}. \tag{10.32}$$

**Theorem 10.9 (Reproducing kernel)** *We define for an admissible Clifford mother wavelet* $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$

$$\mathbb{K}_\psi(a,\boldsymbol{\theta},\boldsymbol{b}; a',\boldsymbol{\theta}',\boldsymbol{b}') = (\psi_{a,\boldsymbol{\theta},\boldsymbol{b}} C_\psi^{-1}, \psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'})_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{10.33}$$

*Then* $\mathbb{K}_\psi(a,\boldsymbol{\theta},\boldsymbol{b}; a',\boldsymbol{\theta}',\boldsymbol{b}')$ *is a reproducing kernel in* $L^2(\mathcal{G}, d\lambda)$*, i.e,*

$$T_\psi f(a',\boldsymbol{\theta}',\boldsymbol{b}') = \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\mathbb{K}_\psi(a,\boldsymbol{\theta},\boldsymbol{b}; a',\boldsymbol{\theta}',\boldsymbol{b}')\, d\lambda. \tag{10.34}$$

**Proof** By inserting (10.25) into the definition of the Clifford wavelet transform (10.12) we obtain

$$\begin{aligned} T_\psi f(a',\boldsymbol{\theta}',\boldsymbol{b}') &= \int_{\mathbb{R}^3} \left\{ \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})\, d\lambda\, C_\psi^{-1} \right\} \widetilde{\psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'}}(\boldsymbol{x})\, d^3\boldsymbol{x} \\ &= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \left\{ \int_{\mathbb{R}^3} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) C_\psi^{-1} \{\psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'}(\boldsymbol{x})\}^\sim\, d^3\boldsymbol{x} \right\} d\lambda \\ &= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta}) \mathbb{K}_\psi(a,\boldsymbol{\theta},\boldsymbol{b};a',\boldsymbol{\theta}',\boldsymbol{b}')\, d\lambda, \qquad (10.35) \end{aligned}$$

which completes the proof. □

# 11 Extension of complex Gabor wavelets to multivector Clifford Gabor wavelets

In signal processing complex Gabor wavelets are used extensively for signal analysis.[26, 27, 28] Complex Gabor wavelets are well localized in both *space* and *frequency* domains which is very important in understanding signals. Two-dimensional complex Gabor wavelets are composed of a complex exponential function and a Gaussian function. They generally can be written as

$$h(\boldsymbol{x}) = \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}\right)} \left[ e^{\boldsymbol{i}(u_0x_1+v_0x_2)} - e^{-\frac{1}{2}(\sigma_1^2u_0^2+\sigma_2^2u_0^2)} \right], \qquad (11.1)$$

where $\sigma_1$ and $\sigma_1$ are the standard deviations of the Gaussian function.

Complex Gabor wavelets can be extended to multivectors. This extension is obtained by replacing the complex kernel $e^{\boldsymbol{i}(u_0x_1+v_0x_2)}$ in the 2D complex Gabor wavelets (11.1) by the Clifford Fourier kernel $e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}$. It then takes the form

$$\begin{aligned} \psi^c(\boldsymbol{x}) &= g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3) \left( e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}} - e^{-\frac{1}{2}(\sigma_1^2u_0^2+\sigma_2^2u_0^2+\sigma_3^2w_0^2)} \right) \\ &= g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)\, e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}} - \eta(\boldsymbol{x}), \qquad (11.2) \end{aligned}$$

where $\boldsymbol{\omega}_0 = u_0\boldsymbol{e}_1 + v_0\boldsymbol{e}_2 + w_0\boldsymbol{e}_3$ denotes a frequency vector. The 3D Gaussian function $g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)$ in (11.2) is defined by

$$g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_1\sigma_2\sigma_3}\, e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}+\frac{x_3^2}{\sigma_3^2}\right)},$$

and

$$\eta(\boldsymbol{x}) = g(\boldsymbol{x}; \sigma_1, \sigma_2, \sigma_3) e^{-\frac{1}{2}(\sigma_1^2 u_0^2 + \sigma_2^2 u_0^2 + \sigma_3^2 w_0^2)}$$

is a correction term in order for equation (10.6) to be satisfied (see [22]). Applying the shift and the scaling properties of table 7.1, we can rewrite the Clifford Gabor wavelets (11.2) in terms of the $Cl_{3,0}$ Clifford Fourier transform as follows

$$\begin{aligned} \mathcal{F}\{\psi^c\}(\boldsymbol{\omega}) &= e^{-\frac{1}{2}\left(\sigma_1^2(\omega_1 - u_0)^2 + \sigma_2^2(\omega_2 - v_0)^2 + \sigma_3^2(\omega_3 - w_0)^2\right)} - \\ &\quad e^{-\frac{1}{2}\left(\sigma_1^2(\omega_1^2 + u_0^2) + \sigma_2^2(\omega_2^2 + v_0^2) + \sigma_3^2(\omega_3^2 + w_0^2)\right)}. \end{aligned} \tag{11.3}$$

It is easy to see that $\mathcal{F}\{\psi^c\}(0) = 0$. The representation of the Clifford Gabor wavelets (11.2) shows that they are formally analogous to the 3D complex Gabor wavelets. We can apply the Euler formula to the trivector exponential which gives the Clifford Gabor wavelets (11.2) in the form

$$\psi^c(\boldsymbol{x}) = g(\boldsymbol{x}; \sigma_1, \sigma_2, \sigma_3) \cos(\boldsymbol{\omega}_0 \cdot \boldsymbol{x}) + i_3 g(\boldsymbol{x}; \sigma_1, \sigma_2, \sigma_3) \sin(\boldsymbol{\omega}_0 \cdot \boldsymbol{x}) - \eta(\boldsymbol{x}). \tag{11.4}$$

This shows that the resulting wavelets consist of a real scalar part and a trivector part. We note that (11.3) is a real-valued scalar function. As a consequence the admissibility constant (10.5) will also be real. It means that we have

$$0 < C_{\psi^c} = \int_{\mathbb{R}^+} \int_{SO(3)} a^3 \left[\widehat{\psi^c}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\right]^2 d\mu \overset{(10.7)}{=} \int_{\mathbb{R}^3} \frac{(\widehat{\psi^c}(\boldsymbol{\xi}))^2}{|\boldsymbol{\xi}|^3} \, d^3\boldsymbol{\xi} < \infty, \tag{11.5}$$

is a real positive scalar constant and finite at a.e. $\boldsymbol{\omega} \in \mathbb{R}^3$.

We summarize some important properties of Clifford Gabor wavelet transform in the following theorems corresponding to theorem 10.6, corollary 10.7 and theorem 10.8.

**Theorem 11.1 (Inner product relation)** *Let $\psi^c \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford Gabor wavelet and $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$ arbitrary. Then we have*

$$(T_{\psi^c} f, T_{\psi^c} g)_{L^2(\mathcal{G}; Cl_{3,0})} = C_{\psi^c} (f, g)_{L^2(\mathbb{R}^3; Cl_{3,0})}. \tag{11.6}$$

In other words the operator $C_{\psi^c}^{-\frac{1}{2}} T_{\psi^c}$ is an *isometry* from $L^2(\mathbb{R}^3; Cl_{3,0})$ to $L^2(\mathcal{G}; Cl_{3,0})$. An immediate consequence of (11.6) is

**Theorem 11.2 (Norm relation)** *Let $\psi^c \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford Gabor wavelet that satisfies the admissibility condition in the sense of (11.5). Then for any $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ we get*

$$\|T_{\psi^c} f\|_{L^2(\mathcal{G}; Cl_{3,0})}^2 = C_{\psi^c} \|f\|_{L^2(\mathbb{R}^3; Cl_{3,0})}^2 \tag{11.7}$$

**Theorem 11.3 (Reconstruction formula)** *Let $\psi^c \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford Gabor wavelets that satisfies the admissibility condition (11.5). Then any $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ can be decomposed as*

$$f(\boldsymbol{x}) = C_{\psi^c}^{-1} \int_{\mathcal{G}} (f, \psi^c_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \, \psi^c_{a,\boldsymbol{\theta},\boldsymbol{b}} d\mu d^3\boldsymbol{b}, \tag{11.8}$$

*the integral converging in the weak sense.*

This theorem shows that any multivector function $f$ can be reconstructed from the Clifford Gabor transform.

# 12 Precision limits - minimal variance products

## 12.1 The uncertainty principle for geometric algebra Fourier transforms

The uncertainty principle plays an important role in the development and understanding of quantum physics. It is also central for information processing [19]. In quantum physics it states e.g. that particle momentum and position cannot be simultaneously known. In Fourier analysis such conjugate entities correspond to a function and its Fourier transform which cannot both be simultaneously sharply localized. Futhermore much work (e.g. [19, 20]) has been devoted to extending the uncertainty principle to a function and its Fourier transform. From the view point of geometric algebra an uncertainty principle gives us information about how a multivector valued function and its Clifford Fourier transform are related.

**Theorem 12.1** *Let $f$ be a multivector valued function in $\mathcal{G}_3$ which has the Clifford Fourier transform $\mathcal{F}\{f\}(\boldsymbol{\omega})$. Assume $\int_{\mathbb{R}^3} \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} = F < \infty$, then the following inequality holds for arbitrary constant vectors $\boldsymbol{a}$, $\boldsymbol{b}$:*

$$\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \geq (\boldsymbol{a}\cdot\boldsymbol{b})^2 \, \frac{(2\pi)^3}{4} F^2 \tag{12.1}$$

**Proof** Applying previous results we have[10]

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \; \|f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x} \; \int_{\mathbb{R}^3} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \; \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \\
\overset{(5.1)}{=} \quad & \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \; \|f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x} \int_{\mathbb{R}^3} \; \|\mathcal{F}\{\boldsymbol{b}\cdot\nabla f\}(\boldsymbol{\omega})\|^2 \; d^3\boldsymbol{\omega} \\
\overset{(7.5)}{=} \quad & (2\pi)^3 \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \; \|f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x} \int_{\mathbb{R}^3} \; \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x} \\
\overset{\text{footnote 10}}{\geq} \quad & (2\pi)^3 \left( \int_{\mathbb{R}^3} \; \boldsymbol{a}\cdot\boldsymbol{x} \; \|f(\boldsymbol{x})\| \; \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\| \; d^3\boldsymbol{x} \right)^2 \\
\overset{(2.11)}{\geq} \quad & (2\pi)^3 \left( \int_{\mathbb{R}^3} \; \boldsymbol{a}\cdot\boldsymbol{x} |\langle \widetilde{f(\boldsymbol{x})} \; \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle| \; d^3\boldsymbol{x} \right)^2 \\
\geq \quad & (2\pi)^3 \left( \int_{\mathbb{R}^3} \; \boldsymbol{a}\cdot\boldsymbol{x} \langle \widetilde{f(\boldsymbol{x})} \; \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle \; d^3\boldsymbol{x} \right)^2 .
\end{aligned}
$$

Because of

$$
(\boldsymbol{b}\cdot\nabla)\|f\|^2 = 2\langle \widetilde{f} \; (\boldsymbol{b}\cdot\nabla) f\rangle, \tag{12.2}
$$

we furthermore obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \; \|f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x} \; \int_{\mathbb{R}^3} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \; \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \\
\geq \quad & (2\pi)^3 \left( \int_{\mathbb{R}^3} \; \boldsymbol{a}\cdot\boldsymbol{x} \; \frac{1}{2} \; (\boldsymbol{b}\cdot\nabla\|f\|^2) \; d^3\boldsymbol{x} \right)^2 \\
\overset{\text{Prop. 2.11}}{=} \quad & \frac{(2\pi)^3}{4} \left( \left[ \int_{\mathbb{R}^2} \boldsymbol{a}\cdot\boldsymbol{x} \|f(\boldsymbol{x})\|^2 d^2\boldsymbol{x} \right]_{b\cdot x=-\infty}^{b\cdot x=\infty} \right. \\
& \qquad \left. - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)(\boldsymbol{a}\cdot\boldsymbol{x})] \; \|f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x} \right)^2 \\
= \quad & \frac{(2\pi)^3}{4} \left( 0 - \boldsymbol{a}\cdot\boldsymbol{b} \int_{\mathbb{R}^3} \; \|f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x}) \right)^2 \\
= \quad & (\boldsymbol{a}\cdot\boldsymbol{b})^2 \; \frac{(2\pi)^3}{4} \; F^2 .
\end{aligned}
$$

Choosing $\boldsymbol{b} = \pm\boldsymbol{a}$, with $\boldsymbol{a}^2 = 1$ we get the following **uncertainty principle**, i.e.

$$
\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \; \|f(\boldsymbol{x})\|^2 \; d^3\boldsymbol{x} \; \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{\omega})^2 \; \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \geq \frac{(2\pi)^3}{4} \; F^2 . \tag{12.3}
$$

[10] $\phi, \psi : \mathbb{R}^n \to \mathbb{C}, \quad \int_{\mathbb{R}^n} |\phi(x)|^2 d^n x \int_{\mathbb{R}^n} |\psi(x)|^2 d^n x \geq (\int_{\mathbb{R}^n} \phi(x)\bar{\psi(x)} \, d^n x)^2$

In (12.3) equality holds for Gaussian multivector valued functions (See appendix B)

$$f(\boldsymbol{x}) = C_0 \, e^{-k \, \boldsymbol{x}^2} \tag{12.4}$$

where $C_0 \in \mathcal{G}_3$ is a constant multivector, $0 < k \in \mathbb{R}$.

**Theorem 12.2** *For $\boldsymbol{a} \cdot \boldsymbol{b}$ =0, we get*

$$\int_{\mathbb{R}^3} (\boldsymbol{a} \cdot \boldsymbol{x})^2 \, \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{b} \cdot \boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \, \geq 0. \tag{12.5}$$

**Proof** The right side of equation (12.1) is 0 for $(\boldsymbol{a} \cdot \boldsymbol{b}) = 0$.
Note that with

$$\boldsymbol{x}^2 = \sum_{k=1}^{3} x_k^2 = \sum_{k=1}^{3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2, \qquad \boldsymbol{\omega}^2 = \sum_{l=1}^{3} \omega_l^2 = \sum_{l=1}^{3} (\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \tag{12.6}$$

we can extend the formula of the uncertainty principle to

**Theorem 12.3** *Under the same assumptions as in theorem 12.1, we obtain*

$$\int_{\mathbb{R}^3} \boldsymbol{x}^2 \, \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \boldsymbol{\omega}^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \geq 3 \, \frac{(2\pi)^3}{4} F^2. \tag{12.7}$$

**Proof** Direct calculation gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \boldsymbol{x}^2 \, \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \boldsymbol{\omega}^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \\
&\overset{(12.6)}{=} \sum_{k,\, l=1}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 \, d^3\boldsymbol{\omega} \\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 \, d^3\boldsymbol{\omega} \\
&\quad \underbrace{+ \sum_{k \neq l}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 \, d^3\boldsymbol{\omega}}_{\geq 0} \\
&\overset{\text{Theor. 12.2}}{\geq} \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 \, d^3\boldsymbol{\omega} \\
&\overset{\text{Theor. 12.1}}{=} 3 \, \frac{(2\pi)^3}{4} \, F^2.
\end{aligned}$$

In the last step we used theorem 12.1 with $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{e}_k$, $k = 1, 2, 3$.

## 12.2 Uncertainty principles for Clifford algebra $Cl_{3,0}$ wavelets

The uncertainty principle for the continuous wavelet transforms establishes a lower bound of the product of the variances of the continuous wavelet transform of a function and its Fourier transform (see e.g. [21]).

We extend this idea to the Clifford algebra $Cl_{3,0}$ wavelet transform, i.e. we show how the Clifford algebra $Cl_{3,0}$ wavelet transform and the Clifford Fourier transform of a multivector function are related.

### 12.2.1 Uncertainty principles for general admissibility constant

Let us first formulate a general statement in the following theorem. That this is indeed the generalized form of an uncertainty principle will be seen in the special case of scalar $C_\psi$ in corollary 12.7, which follows in section 12.2.2.

**Theorem 12.4 (Generalized Clifford wavelet uncertainty principle)** *Let $\psi$ be a Clifford algebra wavelet that satisfies the admissibility condition (10.7). Then for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, the following inequality holds*

$$\begin{aligned} &\|\boldsymbol{b} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})}\ C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \\ &\qquad \geq \frac{3(2\pi)^3}{4}\ \left[C_\psi * (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^2. \end{aligned} \tag{12.8}$$

Before we attempt the proof of theorem 12.4 we derive the following two useful lemmas.

**Lemma 12.5 (Integrated variance of CFT of Cliff. wavelet transf.)**

$$\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\ d\mu = C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{12.9}$$

**Proof** We observe that

$$\begin{aligned} &\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\ d\mu \qquad (12.10)\\ &\overset{(2.22)}{=} \int_{\mathbb{R}^3}\int_{\mathbb{R}^+}\int_{SO(3)} \boldsymbol{\omega}^2[\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}(\boldsymbol{\omega})] * \widetilde{\mathcal{F}}\{T_\psi f(a,\boldsymbol{\theta},.)\}(\boldsymbol{\omega})\ d\mu\, d^3\boldsymbol{\omega} \\ &\overset{(10.16)}{=} \int_{\mathbb{R}^3}\underbrace{\int_{\mathbb{R}^+}\int_{SO(3)} a^3[\widetilde{\widehat{\psi}}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))]}\, *[\tilde{\hat{f}}(\boldsymbol{\omega})\hat{f}(\boldsymbol{\omega})]\ \boldsymbol{\omega}^2 d\mu d^3\boldsymbol{\omega} \\ &= \quad C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})}\,. \end{aligned}$$

□

In some cases only the scalar part of the admissibility constant matters on the right hand side of (12.9), as shown in

**Lemma 12.6 (With scalar admissibility constant)** *If either one of the factors is scalar, or the two vector parts are perpendicular, i.e.* $\langle C_\psi \rangle_1 \perp \langle (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle_1$ *we get instead*

$$\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\ d\mu = \langle C_\psi \rangle_0\ \|\boldsymbol{\omega}\hat{f}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{12.11}$$

Now we begin with the proof of theorem 12.4.

**Proof** We apply to $T_{\psi^c} f(a,\boldsymbol{\theta},\boldsymbol{b})$, where $\boldsymbol{b} \in \mathbb{R}^3$ is regarded as the main independent variable and $a, \boldsymbol{\theta}$ as function parameters, the uncertainty principle for multivector functions of theor. 12.3 in order to get with (2.21)

$$\begin{aligned} &\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \times \|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \\ &\qquad \geq \frac{3(2\pi)^3}{4}\|T_\psi f(a,\boldsymbol{\theta},.)\|^4_{L^2(\mathbb{R}^3;Cl_{3,0})} \end{aligned} \tag{12.12}$$

Taking the square root on both sides of (12.12) we obtain

$$\begin{aligned} &\left[\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^{\frac{1}{2}} \times \left[\|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^{\frac{1}{2}} \\ &\qquad \geq \frac{\sqrt{3}(2\pi)^{3/2}}{2}\|T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \end{aligned} \tag{12.13}$$

Integrating both sides of (12.13) with respect to $d\mu$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}^+}\int_{SO(3)} \left(\left[\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^{\frac{1}{2}}\right. \\ &\qquad\qquad \left.\times \left[\|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^{\frac{1}{2}}\right) d\mu \\ &\geq \frac{\sqrt{3}(2\pi)^{3/2}}{2}\int_{\mathbb{R}^+}\int_{SO(3)} \|T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\ d\mu. \end{aligned} \tag{12.14}$$

Applying the multivector Cauchy-Schwartz inequality to the left hand side of (12.14) gives

$$\begin{aligned}
&\left(\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},\boldsymbol{.})\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu\right)^{\frac{1}{2}} \\
&\qquad\times\left(\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\;\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu\right)^{\frac{1}{2}} \\
&\qquad\geq \frac{\sqrt{3}(2\pi)^{3/2}}{2}\int_{\mathbb{R}^+}\int_{SO(3)} \|T_\psi f(a,\boldsymbol{\theta},\boldsymbol{.})\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu. \qquad (12.15)
\end{aligned}$$

Taking the square on both sides of (12.15) and inserting the definitions of the norms of lines 1 and 3 of (12.15) we get with (2.22)

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{SO(3)}\int_{\mathbb{R}^3} \boldsymbol{b}^2 T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\;*[T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})]^\sim\,d\mu d^3\boldsymbol{b} \\
&\quad\times\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\;\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu \\
&\geq \frac{3(2\pi)^3}{4}\left(\int_{\mathbb{R}^+}\int_{SO(3)}\int_{\mathbb{R}^3} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\;*[T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})]^\sim\,d\mu d^3\boldsymbol{b}\right)^2 \qquad (12.16)
\end{aligned}$$

We now recognize the $L^2(\mathcal{G};Cl_{3,0})$-norms in lines 1 and 3 of (12.16) and with lemma 12.5 we replace the second line of (12.16) to become

$$\begin{aligned}
&\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})}\;\; C_\psi *(\widetilde{\boldsymbol{\omega}\hat{f}},\widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \\
&\qquad\geq\;\; \frac{3(2\pi)^3}{4}\|T_\psi f\|^4_{L^2(\mathcal{G};Cl_{3,0})}. \qquad (12.17)
\end{aligned}$$

Substituting for the right hand side (10.23) we finally get

$$\begin{aligned}
&\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})}\; C_\psi *(\widetilde{\boldsymbol{\omega}\hat{f}},\widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \\
&\qquad\geq\;\; \frac{3(2\pi)^3}{4}\;\left[C_\psi *(f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^2, \qquad (12.18)
\end{aligned}$$

which concludes the proof of theorem 12.4. □

### 12.2.2 Uncertainty principle for scalar admissibility constant

For scalar $C_\psi$ we get due to (12.11) and a similar identity for the right hand side of (12.18) the following corollary

**Corollary 12.7 (Uncertainty principle for Clifford wavelet)** *Let $\psi$ be a Clifford algebra wavelet that satisfies the admissibility constant (10.7). Then for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, the following inequality holds*

$$\|\boldsymbol{b}\, T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})} \, \|\boldsymbol{\omega} \hat{f}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \geq 3C_\psi \frac{(2\pi)^3}{4} \|f\|^4_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{12.19}$$

This shows indeed, that theorem 12.4 represents a multivector generalization of the uncertainty principle of corollary 12.7 for Clifford wavelets with scalar admissibility constant.

In the field of information theory and image processing corollary 12.7 establishes bounds for the effective width times frequency extension of processed signals or images.

### 12.2.3 Uncertainty principle for Clifford geometric algebra Gabor wavelets

As a consequence of the general uncertainty principle for Clifford wavelets with scalar admissibility constant of corollary 12.7 we have

**Theorem 12.8 (Uncertainty principle for Clifford Gabor wavelet)** *Let $\psi^c$ be a Clifford Gabor wavelet that satisfies the admissibility condition (11.5). Assume $\|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} = F < \infty$ for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, then the following inequality holds*

$$\|\boldsymbol{b}\, T_{\psi^c} f(a, \boldsymbol{\theta}, \boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})} \, \|\boldsymbol{\omega} \hat{f}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \geq 3C_{\psi^c} \frac{(2\pi)^3}{4} F^2. \tag{12.20}$$

## 13 Conclusions

We showed how the (real) Clifford geometric algebra Fourier transform (FT) extends the traditional Fourier transform on scalar functions to multivector functions. Basic properties and rules for differentiation, convolution, the Plancherel and Parseval theorems were demonstrated.

It is known that the Fourier transform is successfully applied to solving physical equations such as the heat equation, wave equations, etc. Therefore in the future, we can apply geometric algebra and the Clifford Fourier transform to solve such problems involving scalar, vector, bivector and pseudoscalar fields. A closely related relative of the geometric algebra FT is the quaternion FT (QFT), since quaternions are nothing but the bivector subalgebra of the geometric algebra of 3D Euclidean space $\mathbb{R}^3$. Based on the

QFT far reaching generalizations to high dimensional geometric algebra are possible, including the geometric algebra of spacetime [18].

As for the geometric algebra treatment of wavelets, we showed how Clifford algebra $Cl_{3,0}$-valued wavelets extend the classical wavelets on scalar functions to multivector functions. Important properties such as translation, dilation and rotation covariances, a reproducing kernel, and a reproduction formula for multivector functions were demonstrated.

We then applied our formalism by extending complex Gabor wavelets to Gabor multivector wavelets, and looked at some of their important properties.

We finally studied uncertainty principles in the geometric algebra $\mathcal{G}_3$ which describe how a multivector-valued function and its Clifford geometric algebra Fourier transform or geometric algebra wavelet transform relate.

For Clifford geometric algebra wavelets we infact established the general form of a new uncertainty principle, which becomes analogous to the usual scalar formulation (corollary 12.7) when the admissibility constant itself is scalar. We also established a new uncertainty principle for the Clifford Gabor wavelets.

In the field of information theory and image processing these geometric algebra uncertainty principles establish bounds or limits for the effective width times frequency extension of processed signals or images.

The formulas of the derived uncertainty principles in $\mathcal{G}_3$ can be extended to $\mathcal{G}_n$ using properties of the Clifford Fourier transform for geometric algebras with unit pseudoscalars squaring to -1. Similar extensions are possible for the uncertainty principle of the geometric algebra wavelet transform.

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**Ordering of the references**

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# A Multivector Cauchy-Schwartz inequality

We will show that

$$|\langle M\tilde{N}\rangle| \leq \|M\|\,\|N\| \tag{A1}$$

**Proof** Note that for any $t \in \mathbb{R}$ holds

$$\begin{aligned} 0 &\leq \|M+tN\|^2 = (M+tN) * (\widetilde{M+tN}) \\ &= M*\tilde{M} + t(M*\tilde{N} + N*\tilde{M}) + t^2 N*\tilde{N} \\ &= \|M\|^2 + 2t\langle M\tilde{N}\rangle + t^2\|N\|^2. \end{aligned} \tag{A2}$$

The negative discriminant of this quadratic polynomial implies

$$\langle M\tilde{N}\rangle^2 - \|M\|^2\|N\|^2 \leq 0. \tag{A3}$$

This proves (A1) and (2.11):

$$\langle M\widetilde{N}\rangle = M * \widetilde{N} \leq |\langle M\widetilde{N}\rangle| \leq \|M\|\,\|N\|. \tag{A4}$$

Inserting into (2.6) and (2.9) into the multivector Cauchy-Schwartz inequality (A4) we can express it in a basis (2.1) of the geometric algebra as

$$|\sum_A \alpha_A\beta_A| \leq \left(\sum_A \alpha_A^2\right)^{\frac{1}{2}} \left(\sum_B \beta_B^2\right)^{\frac{1}{2}}. \tag{A5}$$

## B Uncertainty equality for Gaussian multivector functions

Note that according to line 3 of the proof for theorem 12.1 the uncertainty principle (12.3) can be rewritten as

$$(2\pi)^3 \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \|\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})\|^2 d^3\boldsymbol{x} \geq \frac{(2\pi)^3}{4} F^2. \tag{B1}$$

Now we have for Gaussian multivector functions (12.4)

$$\begin{aligned} \boldsymbol{a}\cdot\nabla f &= \boldsymbol{a}\cdot\nabla\, C_0\, e^{-k\boldsymbol{x}^2} \\ &= -2k\, \boldsymbol{a}\cdot\boldsymbol{x}\, C_0\, e^{-k\,\boldsymbol{x}^2} \\ &= -2k\, \boldsymbol{a}\cdot\boldsymbol{x}\, f. \end{aligned} \tag{B2}$$

so, we get

$$\boldsymbol{a}\cdot\boldsymbol{x}\, f = \frac{-1}{2k}\, \boldsymbol{a}\cdot\nabla f, \tag{B3}$$

and

$$\|\boldsymbol{a}\cdot\nabla f\|^2 = 4k^2\|(\boldsymbol{a}\cdot\boldsymbol{x})f\|^2 = 4k^2\, (\boldsymbol{a}\cdot\boldsymbol{x})^2\|f\|^2 \tag{B4}$$

Substituting (B4) and (B3) in the left side of (B1) we get for $\boldsymbol{a}^2 = 1$

$$\begin{aligned} &(2\pi)^3 \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \|\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})\|^2 d^3\boldsymbol{x} \\ &\overset{(B4)}{=} 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x}\; \boldsymbol{a}\cdot\boldsymbol{x} \|f\|^2 d^3\boldsymbol{x}\right)^2 \\ &= 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x}\boldsymbol{a}\cdot\boldsymbol{x}\langle f\tilde{f}\rangle d^3\boldsymbol{x}\right)^2 \\ &= 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x}\langle \boldsymbol{a}\cdot\boldsymbol{x} f\tilde{f}\rangle d^3\boldsymbol{x}\right)^2 \\ &\overset{(B3)}{=} 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \frac{\boldsymbol{a}\cdot\boldsymbol{x}}{-2k}\langle (\boldsymbol{a}\cdot\nabla f)\tilde{f}\rangle d^3\boldsymbol{x}\right)^2 \\ &\overset{(12.2)}{=} (2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x}\frac{1}{2}\boldsymbol{a}\cdot\boldsymbol{\nabla}\|f\|^2 d^3\boldsymbol{x}\right)^2 \\ &\overset{\text{P. 2.11}}{=} \frac{(2\pi)^3}{4} \left(\int_{\mathbb{R}^3} \underbrace{(\boldsymbol{a}\cdot\boldsymbol{\nabla}\; \boldsymbol{a}\cdot\boldsymbol{x})}_{=\boldsymbol{a}^2=1} \|f\|^2 d^3\boldsymbol{x}\right)^2 \\ &= \frac{(2\pi)^3}{4} F^2. \end{aligned}$$



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