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Geometric Algebra illustrated by Cinderella

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Abstract

Conventional illustrations of the rich elementary relations and physical applications of geometric algebra are helpful, but restricted in communicating full generality and time dependence. The main restrictions are one special perspective in each graph and the static character of such illustrations. Several attempts have been made to overcome such restrictions. But up till now very little animated and fully interactive, free, instant access, online material is available.

This report presents therefore a set of over 90 newly developed (freely online accessible[1]) JAVA applets. These applets range from the elementary concepts of vector, bivector, outer product and rotations to triangle relationships, oscillations and polarized waves. A special group of 21 applets illustrates three geometrically different approaches to the representation of conics; and even more ways to describe ellipses. Next Clifford's famous circle chain theorem is illustrated. Finally geometric applications important for crystallography and structural mechanics give a glimpse of the vast potential for applied mathematics. The interactive geometry software Cinderella[2] was used for creating these applets. The interactive features of many of the applets invite the user to freely explore by a few mouse clicks as many different special cases and perspectives as he likes. This is of great help in "visualizing" geometry encoded by geometric algebra.

1 Introduction

1.1 Geometric Algebra

About 150 years ago, in 1844, the German high school teacher Hermann Grassmann published an ambitious work entitled *The Linear Extension Theory, A New Branch of Mathematics.* For Grassmann this was indeed *The Branch* of mathematics, which in his own words "far surpasses" all others. His work won the prize of 45 gold ducats set out by the Princely Jablonowski Society for the recreation and further establishment of the geometric calculus invented by Leibniz. Leibnitz wrote[1] for example in 1679 to Huygens:

> Mais apres tous les progres que j'ay faits en ces matieres, je ne suis pas encore content de l'Algebre, en ce qu'elle ne donne ny les plus courtes voyes, ni les plus belles constructions de Geometrie. C'est pour quoy ... je croy qu'il nous faut encor une autre analyse proprement géométrique linéaire, qui nous exprime directement *situm*, comme l'Algebre exprime *magnitudinem* ... je croy qu'on pourroit manier par ce moyen la mécanique presque comme la Géometrie ... Enfin je n'espère pas qu'on puisse aller assez loin en Physique avant que d'avoir trouvé un tel abrégé pour soulager l'imagination. [3]

Leibnitz was also writing about his new ideas to L' Hospital and others. But in 1714 he writes[2] somewhat disappointed to Remond that L'Hospital and others think he is just dreaming:

> J'ay parlé de ma Specieuse generale à Mr.le Marquis de l'Hospital, et à d'autres; mais ils n'y ont point donné plus d'attention que si je leur avois conté *un songe*. [3]

Grassmann went on to prove the usefulness of his extension theory by applying it to the theory of tides and other phenomena in physics.

Grassmann's influence was far reaching. Under it the English mathematician W.K. Clifford published in 1878 his *Applications of Grassmann's extensive algebra*, describing "geometric algebra". Now this algebra is often simply referred to as "Clifford algebra." And the Italian G. Peano published in 1888 his *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann.* Four years later, in 1892 Felix Klein himself successfully began to push for a complete posthumous republication of Grassmann's works by the Royal Saxonian Society of Sciences.

Today, at the beginning of the 21st century, some people believe, that based on Grassmann's work soon more or less all of mathematics may be formulated as a single unified universal geometric calculus[4], with concrete geometrical foundations. The algebraic "grammar" such a geometric calculus uses is geometric algebra.

Yet why take *geometry* so important? We think the reason is that it is the kind of mathematics, which all of us can most easily imagine and visualize. It is

[1] But after all progress I have achieved in these matters, I am not yet satisfied with the algebra, inasmuch as it gives neither the shortest paths nor the most beautiful constructions in Geometry. This is why ... I believe we still need another analysis rightly geometric and linear that will allow us to express directly *situm* as the algebra expresses *magnitudinem*... I believe that in that way one will be able to deal with mechanics almost the same as with Geometry ... In conclusion, I do not hope one will be able to go ahead in physics, in a significant way, before having found such an abridgement to relieve the imagination. (Translated by J. Parra[3])

[2] I have told Mr.le Marquis de l'Hospital and others about my "Specieuse generale"; but they have paid no more attention as if I had told them a dream. (Translated by J. Parra[3])

what Wittgenstein believed to be the true source of understanding, the essential definition:

> What Wittgenstein believed is that our understanding comes in the area of our imagination. When we picture something in our mind we understand it. That, in essence, is the definition of that thing. [5]

Some of the modern engineering applications of geometric algebra are: computer vision, graphics and reconstruction, robotics, signal and image processing, structural dynamics and structural mechanics, control theory, quantum computing, bioengineering and molecular design, space dynamics, elasticity and solid mechanics, electromagnetism and wave propagation, geometric and Grassmann algebras, quaternions and screw theory, automated theorem proving, symbolic algebra and numerical algorithms.

1.2 Cinderella created JAVA applets

For readers interested in an introduction to geometric algebra, we recommend [6, 7, 8]. Our main purpose in the present paper is not a systematic introduction into the formulation of geometric algebra, but rather to *illustrate* geometric algebra. For this purpose we state in the following only the formulas relevant for the representative figures, which are included in the present work. More formulas and further descriptions are available online[1].

E.H. originally learned about the JAVA based interactive geometry software Cinderella[2] from the Amsterdam robotics researcher Leo Dorst[9]. Dorst also used it to do some geometric algebra illustrations. We elaborated this into a variety of interactive or animated online JAVA applets. Especially the interactive applets invite to visually explore the full meaning of geometric relationships. For part of the applets we have freely drawn with permission on [6]. Other important sources are the geometric algebra MATLAB tutorial GABLE[9], Sir R. Penrose's *afterword* to a recent Clifford biography[10] and new works on applications of geometric algebra to crystallography.[11]

Particularly in the section on conics we have compiled an instructive variety of ways to obtain conics (which include points, pairs of intersecting lines, circles, ellipses, parabolas and hyperbolas).

Next W.K. Clifford's circle chain theorem in the ordinary Euclidean plane which refers to a "chain of theorems" of increasing complexity is visualized.[10] Every one of this infinite sequence of theorems must be true for the whole to be true. You will find the illustrations for $n = 2$ to $n = 8$ primary circles through one point O.

Finally geometric applications important for crystallography and structural mechanics give a glimpse of the vast potential for applied mathematics. It has recently been found that geometric algebra offers a natural way to describe both point and space groups using nothing but location vectors in the respective physical crystal.[11]

The applets work with Netscape 6.2 and 7, and Explorers 5 and 6, but not with Netscape 4.7. The first applet may take some time, because your browser has to load a 412k JAVA archive file. Later on some of the more involved applets may also take a few minutes to appear on your screen. One needs to take into account that Cinderella is inherently two-dimensional. That is three-dimensional projections may sometimes get disarranged. The easiest way to resolve this is to press the refresh button of the browser.

2 Vectors and bivectors

2.1 Vectors

This first group of applets illustrates how vectors can be represented by arrows, which are identical up to translations. The vector length and orientation (opposite directions) can be changed by dilations. The addition of vectors is defined as attaching one vector to the tip of another and drawing the result from the tail of the first to the tip of the second. The additive inverse of a vector brings it back to zero.

Special properties of vector addition are commutativity (Fig. 1),

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{c}, \tag{1}$$

associativity, and vector subtraction. The properly defined geometric multiplication of vectors

$$\mathbf{ab} = \mathbf{a} \rfloor \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{2}$$

also has a multiplicative inverse $\mathbf{a}^{-1} = \mathbf{a}/(\mathbf{aa})$, that is in geometric algebra one can naturally divide by vectors as well:

$$(\mathbf{xa})/\mathbf{a} = (\mathbf{x} \rfloor \mathbf{a})/\mathbf{a} + (\mathbf{x} \wedge \mathbf{a})/\mathbf{a} = \mathbf{x}(\mathbf{a}/\mathbf{a}) = \mathbf{x}. \tag{3}$$

One particular application of vector division (Fig. 2) is the calculation of projection $(\mathbf{x} \rfloor \mathbf{a})/\mathbf{a}$ and rejection $(\mathbf{x} \wedge \mathbf{a})/\mathbf{a}$ of one vector $\mathbf{x}$ with respect to another $\mathbf{a}$ by using the scalar[3] (here left contraction $\rfloor$) and outer product parts of the geometric product of vectors.

All these properties are illustrated in a set of interactive applets.

2.2 Bivectors

Area elements, the non scalar parts of the geometric product of vectors are simply called bivectors. Multiplying by a three-volume element results in a dual vector perpendicular to the bivector. The left contraction is naturally extended to include products of vectors and bivectors. This gives a new vector in the bivector plane perpendicular to the projection of the vector into the bivector plane.

[3]In much of the literature (e.g. [4, 6], etc.) the scalar part of the geometric product of vectors is denoted as inner product by $\mathbf{x} \cdot \mathbf{a}$ using definitions like the one given in [4]. But this definition has some drawbacks, as L. Dorst pointed out in [12]. We therefore use instead the *left contraction* with symbol '$\rfloor$' as e.g. defined in [7]. The left contraction applies not only to vectors, it has a general definition for the left contraction of multivectors.

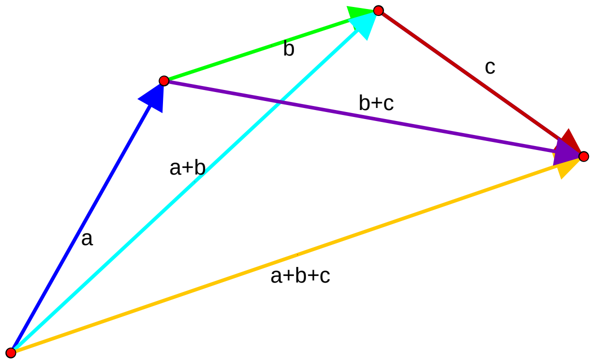

Figure 1: Commutativity of vector addition: $\mathbf{a}+(\mathbf{b}+\mathbf{c}) = (\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+\mathbf{b}+\mathbf{c}$.

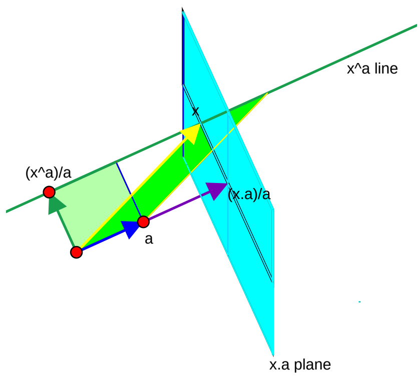

Figure 2: Vector division: $(\mathbf{xa})/\mathbf{a} = \mathbf{x}$.

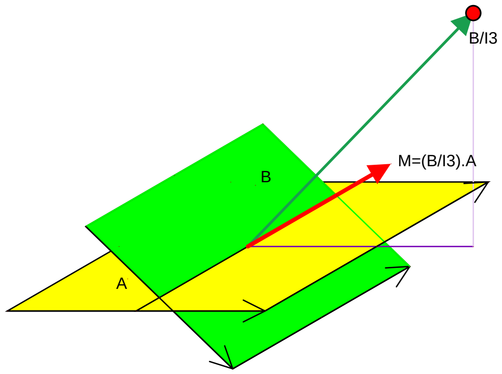

Figure 3: The intersection (meet M) of two planes A, B.

The projection and rejection of a vector with respect to a bivector use the same algebraic expressions as that of the vector vector case.

Geometric algebra easily allows to calculate the intersection line vector (meet M in Fig. 3) of two planes without ambiguity in one step.

$$M = (B/J) \rfloor A \tag{4}$$

Here A, B indicate the two planes by corresponding plane oriented bivectors and $J = I_3$ is the *join* of the two planes, i.e. a volume trivector I_3.

3 Outer product and triangle

The outer product of two vectors (Fig. 4) is the non-scalar, antisymmetric part of their geometric product

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \tag{5}$$

resulting in an oriented area element, which can be defined by one vector sweeping along the other. JAVA applets make it possible to animate this motion. But the outer product is not restricted to two dimensions. Sweeping an area element bivector along a third vector gives an oriented volume element trivector. It does not matter which side face of a parallelepiped is used for sweeping, we always get the same trivector with positive or negative orientation.

The outer product is distributive (Fig. 5),

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}. \tag{6}$$

The outer products of the side vectors of a triangle immediately lead to the law of (oriented) sines. Both facts are illustrated by applets in the triangle category.

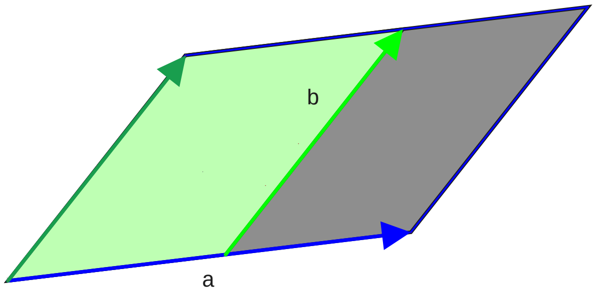

Figure 4: Outer product $\mathbf{a} \wedge \mathbf{b}$ of two vectors.

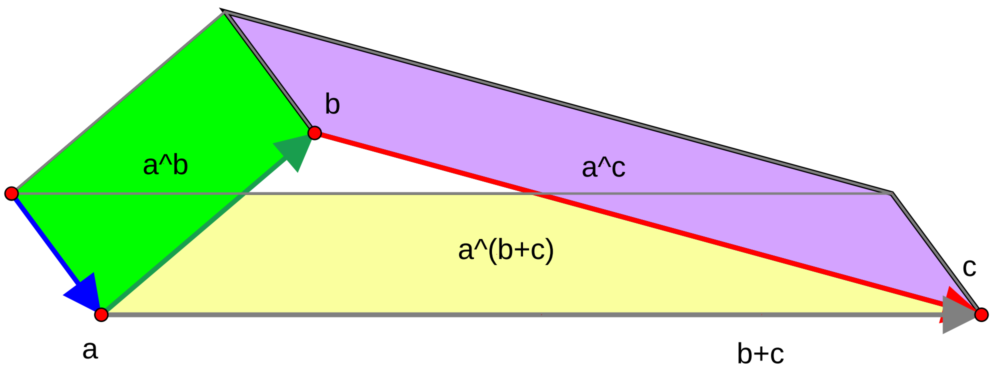

Figure 5: The outer product is distributive: $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$.

4 Rotations

Geometric algebra offers elegant ways to describe rotations without introducing coordinates or matrices. E.g. the full geometric multiplication of a vector with a (unit area) bivector $I_2 = \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$ gives a 90 degree rotation (Fig. 6):

$$\begin{aligned} \mathbf{x} &\to \mathbf{x}I_2 = \mathbf{x} \rfloor I_2, \\ \text{e.g. } \mathbf{e}_1 &\to \mathbf{e}_2 = \mathbf{e}_1 I_2 = \mathbf{e}_1 \rfloor I_2. \end{aligned} \tag{7}$$

Capitalizing on this one can easily demonstrate how the exponential of a bivector yields arbitrary plane rotations (Fig. 7).

$$\begin{aligned} \mathbf{x} &\to \mathbf{x}\cos\phi + \mathbf{x}I_2 \sin\phi \\ &= \mathbf{x}(\cos\phi + I_2\sin\phi) \\ &= \mathbf{x}\exp(I_2\phi) \\ &= \exp(-I_2\phi/2)\mathbf{x}\exp(I_2\phi/2) \\ &= R\,\mathbf{x}\,\widetilde{R}. \end{aligned} \tag{8}$$

Another simple way to describe rotations is through two successive reflections at planes which comprise half of the angle of the resulting rotation (Fig. 8). These planes of reflection can be denoted by unit vectors $\mathbf{n}, \mathbf{m}$ perpendicular to them. The geometric product $\mathbf{nm}$ of these two unit bivectors is the desired rotation operator (rotor) R, to be applied in the orders $R = \mathbf{nm}$ from the left and $\widetilde{R} = \mathbf{mn}$ from the right.

$$\mathbf{x} \to \mathbf{x}'' = \mathbf{nm}\,\mathbf{x}\,\mathbf{mn} = R\,\mathbf{x}\,\widetilde{R}, \;\; \mathbf{m}^2 = \mathbf{n}^2 = 1. \tag{9}$$

Rotors can be written in terms of the angle and the oriented plane area element (e.g. I_2) defined by the two vectors $\mathbf{n}, \mathbf{m}$. This brings us full swing back to the exponential description of rotations already mentioned in (8), now in the proper general form for the rotation of objects in higher dimensions. The same coordinate free rotor R rotation can be applied to vectors and bivectors (Fig. 9).

$$\begin{aligned} R = \exp(-I_2\phi/2), \widetilde{R} &= \exp(I_2\phi/2) \\ \mathbf{x} &\to R\,\mathbf{x}\,\widetilde{R} \\ B &\to RB\widetilde{R} \end{aligned} \tag{10}$$

The exponential rotor form of rotations gives us a coordinate and matrix free understanding of the well known Euler angles (Fig. 10, $\mathbf{n}$ =line of nodes).

$$R = \exp(-\frac{1}{2}I_3\boldsymbol{\sigma}_3\psi)\exp(-\frac{1}{2}I_3\mathbf{n}\,\theta)\exp(-\frac{1}{2}I_3\mathbf{e}_3\varphi) \tag{11}$$

And it leads on to an incredibly simple algorithm for interpolating between arbitrary orientations (rotors R_A, R_B) (Fig. 11), e.g. of plane area elements.

$$\begin{aligned} R_A &= \exp(-I_3\mathbf{v}_A\,\Phi_A/2), \\ R_B &= \exp(-I_3\mathbf{v}_B\,\Phi_B/2), \\ R_{inc} &= \exp(\log(R_A/R_B)/n), \end{aligned} \tag{12}$$

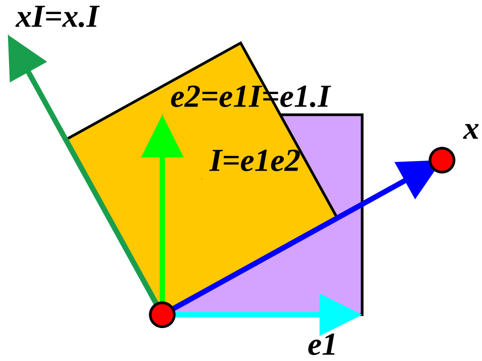

Figure 6: Right angle rotation with unit area element.

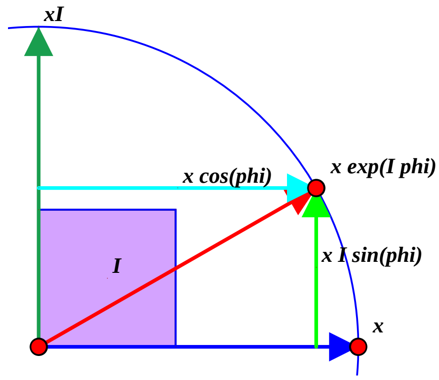

Figure 7: Rotation with exponential of unit area element.

where R_{inc} is the incremental rotor and n the desired number of increments. This is of great use for computer graphics, virtual reality, robot manipulations, orbits and aerospace.

5 Osciallations and waves

5.1 Oscillations

All basic forms of symmetric and antisymmetric oscillations in longitudinal (Fig. 12), transverse (Fig. 13) and circular (Fig. 14) directions are easily formulated by geometric algebra and visualized by animated applets. For the symmetric circular case we have

$$\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{a}\exp(I_2\omega t), \tag{13}$$

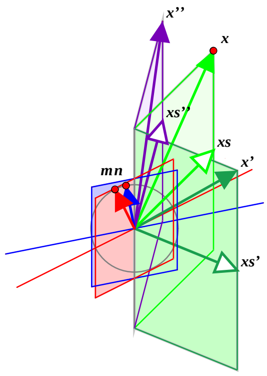

Figure 8: Rotation by two reflections: $\mathbf{x}'' = \mathbf{nm}\,\mathbf{x}\,\mathbf{mn}$.

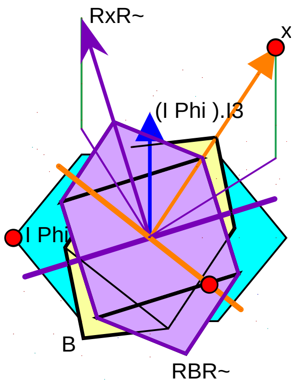

Figure 9: General rotation by rotor R operation.

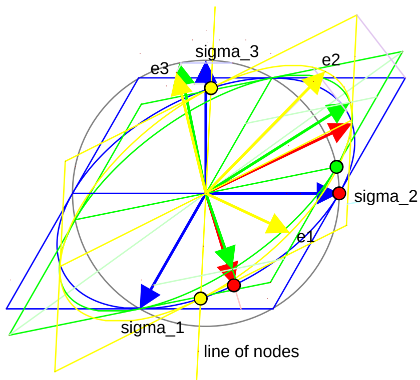

Figure 10: Euler angles of rotation $R(\psi, \theta, \varphi)$.

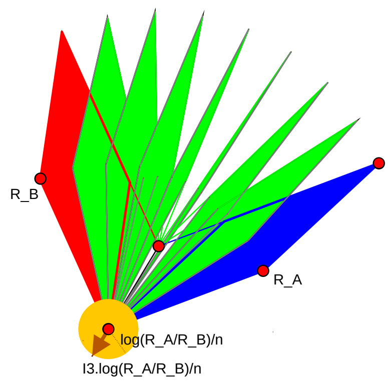

Figure 11: Rotation interpolation $R_{inc} = \exp(\log(R_A/R_B)/n)$.

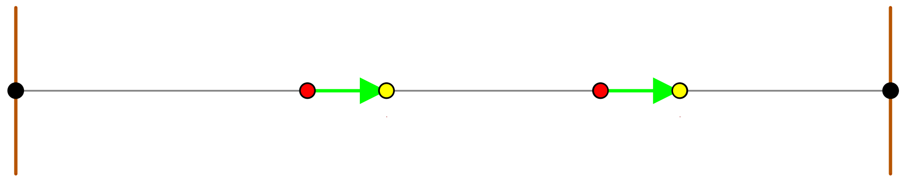

Figure 12: Symmetric longitudinal oscillations.

and for the antisymmetric circular case

$$\mathbf{q}_1 = \mathbf{a}\exp(I_2\omega t), \quad \mathbf{q}_2 = \mathbf{a}\exp(-I_2\omega t). \tag{14}$$

Longitudinal and transverse modes are simply projections into the corresponding directions as explained in section 2.1.

As one may expect by now, geometric algebra can again do it without coordinates. It really brings out the full geometric nature of oscillations without artificially resorting to a Cartesian coordinate system.

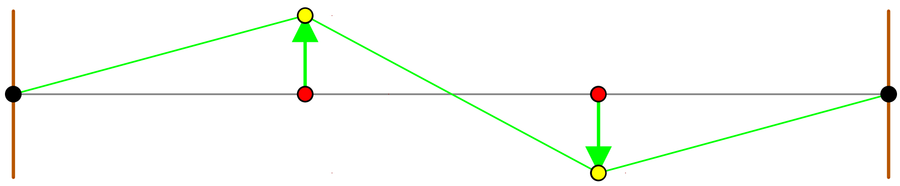

Figure 13: Antisymmetric transverse oscillations.

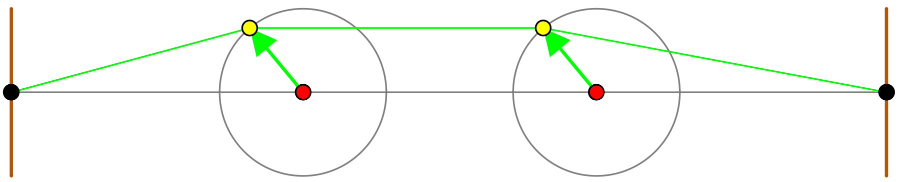

Figure 14: Symmetric circular oscillations $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{a}\exp(I_2\omega t)$.

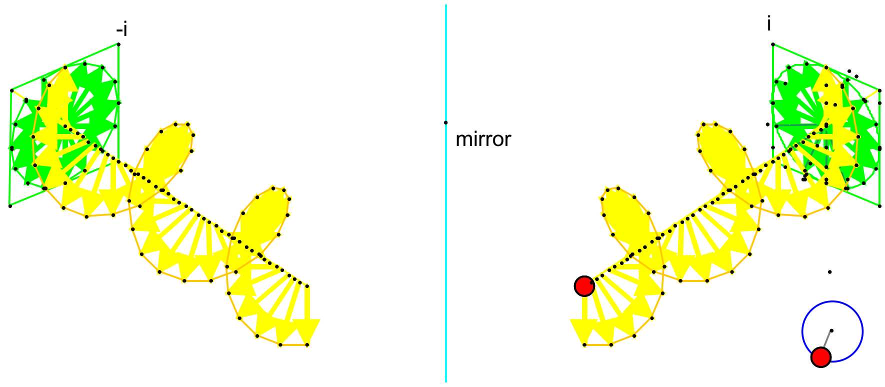

Figure 15: Right and left circular polarized waves $\mathbf{q} = \mathbf{a}\exp(\mp I(\omega t - kx))$.

5.2 Circular polarized waves

Circular polarized waves standing, rotating or traveling in arbitrary directions of any wavelengths receive an elegant fully geometric description in geometric algebra.

$$\mathbf{q} = \mathbf{a}\exp(I(\omega t - kx)), \tag{15}$$

where $I = \hat{\mathbf{k}} \rfloor I_3$ is the bivector of the plane rotation for fixed distance x, and $\mathbf{k}$ is the wave vector with length $k = |\mathbf{k}|$. $I \to -I$ produces negative helicity (right-circularly polarized).

The applets in this category show right and left polarized circular waves (Fig. 15) which can be turned interactively, or observed traveling by animation. The user can continuously choose the wavelength $2\pi/k$ interactively.

6 Conic intersections

Conic intersections are a vast and immensely important topic in geometry, mathematics and its applications. The geometric and analytic properties of these particular curves are most fascinating, but the incredible variety of equivalent descriptions can be very confusing and rather elusive at first. To address this

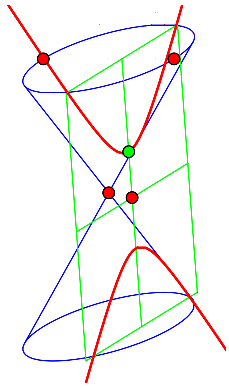

Figure 16: Conic intersection.

subject a comprehensive study has been carried out to first show the "conic" nature of circles, ellipses, parabolas, hyperbolas, points and intersecting pairs of lines, and to fully illustrate with interactive "hands on" the variety of equivalent descriptions. Cinderella even allows to capture the infinite properties of parabolas and hyperbolas by visualizing the spatial infinity in its spherical view.

6.1 Intersecting the cone

This is now meant to be taken literally as the procedure for obtaining all conics. In both finite Euclidean view and in the (infinite) spherical view the position and orientation of a plane intersecting a cone can be freely manipulated interactively (Fig. 16). This creates before the eyes of the user all known varieties of conics. The spherical view (Fig. 17) shows how finite ellipses become parabolas closed at infinity and then continuously open up to become hyperbolas. By just dragging the mouse pointer across the screen the user can freely explore, what many pages of text books and many conventional illustrations just cannot show due to their limitations to selective static perspectives.

6.2 Semi-latus rectum formula

A famous formula for all conics is

$$r = \frac{l}{1 + \boldsymbol{\varepsilon} \cdot \hat{\mathbf{r}}}, \tag{16}$$

with l the semi-latus rectum, r the length of the radius vector pointing in the unit vector direction $\hat{\mathbf{r}}$, and $\boldsymbol{\varepsilon}$ the excentricity vector.

How formula (16) achieves to include circles, ellipses, parabolas (Fig. 18) and hyperbolas is made visible by a variety of applets. In the interactive version, the user can continuously create each conic by dragging the mouse along the directrix line. Special animated versions show how all values of the formula

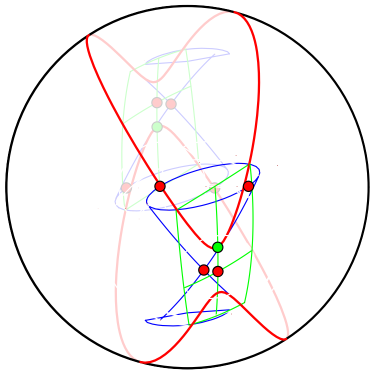

Figure 17: Spherical projection.

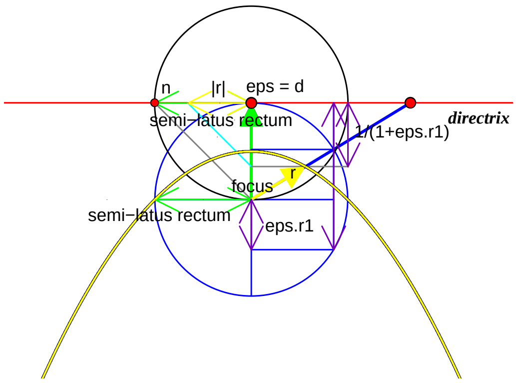

Figure 18: Semi-latus rectum generation of parabola: $r = l/(1 + \boldsymbol{\varepsilon} \cdot \hat{\mathbf{r}})$.

sweep out complete conics. At the same time the infinite behavior is captured again with the help of the spherical view (Fig. 19).

6.3 Ellipse in various disguises

One conic, the ellipse has a particular variety of equivalent ways to describe it:

$$\begin{aligned}
\mathbf{r} &= \mathbf{a}\cos\varphi + \mathbf{b}\, sin\varphi, \\
\mathbf{r} &= \mathbf{a}_+ \exp(I_2\Phi) + \mathbf{a}_- \exp(-I_2\Phi), \quad I_2 = \mathbf{e}_1\mathbf{e}_2, \\
\mathbf{r} &= \tfrac{1}{2}\mathbf{a}\left(\exp(I_+\Phi) + \exp(-I_-\Phi)\right),
\end{aligned} \tag{17}$$

with I_+, I_- two unit bivectors characterizing two planes. There are besides the literal conic intersection and the semi-latus rectum formula three other interesting descriptions. Yet another way to describe it as second order curve will be treated in the next section.

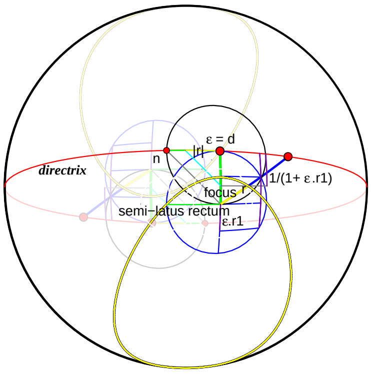

Figure 19: Parabola in spherical view closing at infinity.

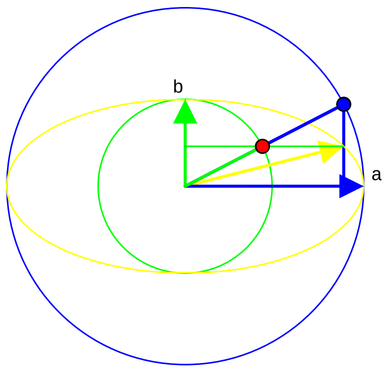

Figure 20: Description of ellipse by polar angle: $\mathbf{r} = \mathbf{a}\cos\varphi + \mathbf{b}\, sin\varphi$.

The three descriptions mentioned above are illustrated in both interactive and animated ways by remarkably simple and aesthetic Cinderella applets (Figs. 20,21,22). Needless to say that the elegant descriptions of planes as bivectors and of rotations by rotors make all this in the framework of geometric algebra quite easy.

6.4 Second order curves

All curves of the form

$$\mathbf{r} = \frac{\mathbf{a}_0 + \mathbf{a}_1\lambda + \mathbf{a}_2\lambda^2}{\alpha + \lambda^2}, \tag{18}$$

with any three vectors $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and scalars λ and α describe conics. Taking $\mathbf{r}$ in (18) as functions of λ yields a specific conic. For $\alpha > 0$ ellipses, for $\alpha = 0$

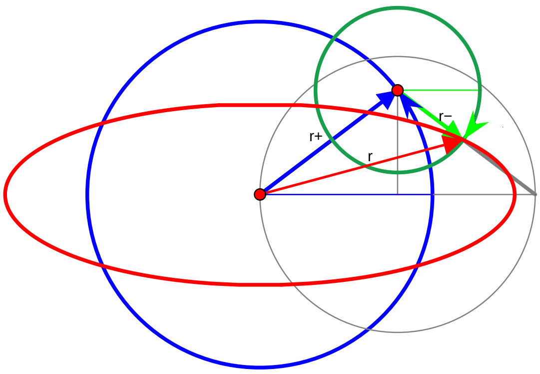

Figure 21: Description of ellipse by two coplanar circles: $\mathbf{r} = \mathbf{a}_+ \exp(I_2\Phi) + \mathbf{a}_- \exp(-I_2\Phi)$.

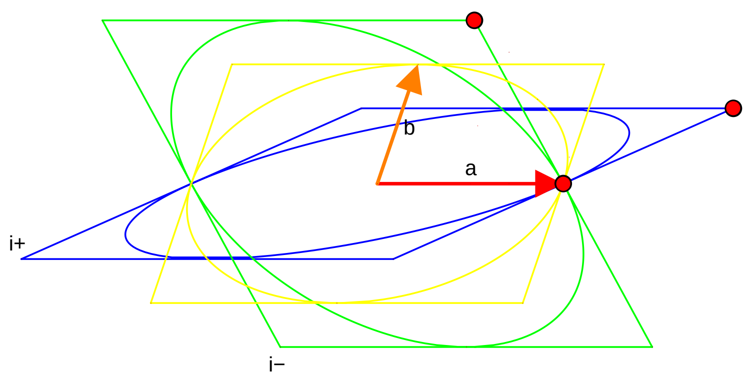

Figure 22: Description of ellipse by and two non-coplanar planes $\mathbf{r} = \frac{1}{2}\mathbf{a}\left(\exp(I_+\Phi) + \exp(-I_-\Phi)\right)$.

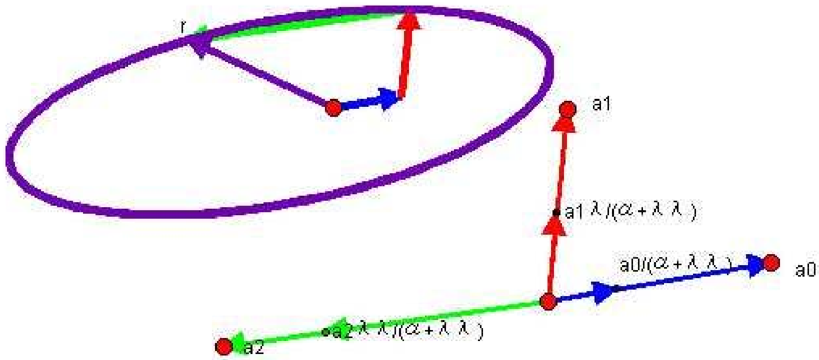

Figure 23: Ellipse as second order curve: $\mathbf{r} = (\mathbf{a}_0 + \mathbf{a}_1\lambda + \mathbf{a}_2\lambda^2)/(\alpha + \lambda^2), \ \ \alpha > 0$.

parabolas and for $\alpha < 0$ hyperbolas. It sounds incredible, but the truth of (18) can be immediately visualized using interactive and animated JAVA applets created with Cinderella (Fig. 23). In the interactive version all three vectors and the values of the scalar parameters can be freely varied to provide full intuition on how the formula works. The spherical view captures again what happens at infinity.

7 Clifford's circle chain theorem

Clifford's circle chain theorem in the ordinary Euclidean plane refers to a "chain of theorems" of increasing complexity.[10] Every one of this infinite sequence of theorems must be true for the whole to be true. It begins with two circles passing through a common point O ($n = 2$). The next theorem in the chain is for the case of three circles through a common point O ($n = 3$) and so forth for $n = 4, 5, 6, \ldots$ If one takes the point O to infinity the n circles become n straight lines (circles with infinite radii.) The interactive JAVA applets show illustrations for the cases of $n = 2, \ldots, 8$ circles through O (Figs. 24, 25). These illustrations are each accompanied by a detailed description. Each illustration shows the magnificent intersection properties that lead to a final point (even n) or circle (odd n).

It is most interesting to note that a purely algebraic proof of the case $n = 5$, based on the *conformal* geometric algebra model of the Euclidean plane has been found[13]. With the help of the geometric product of vectors one can construct an algebra of Clifford brackets, with which the proof becomes possible. This illustrates how even Clifford's own inventions submit over time to the descriptive power of his geometric algebra.

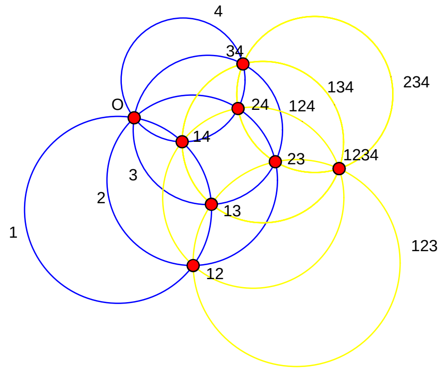

Figure 24: Clifford's circle chain theorem ($n = 4$).

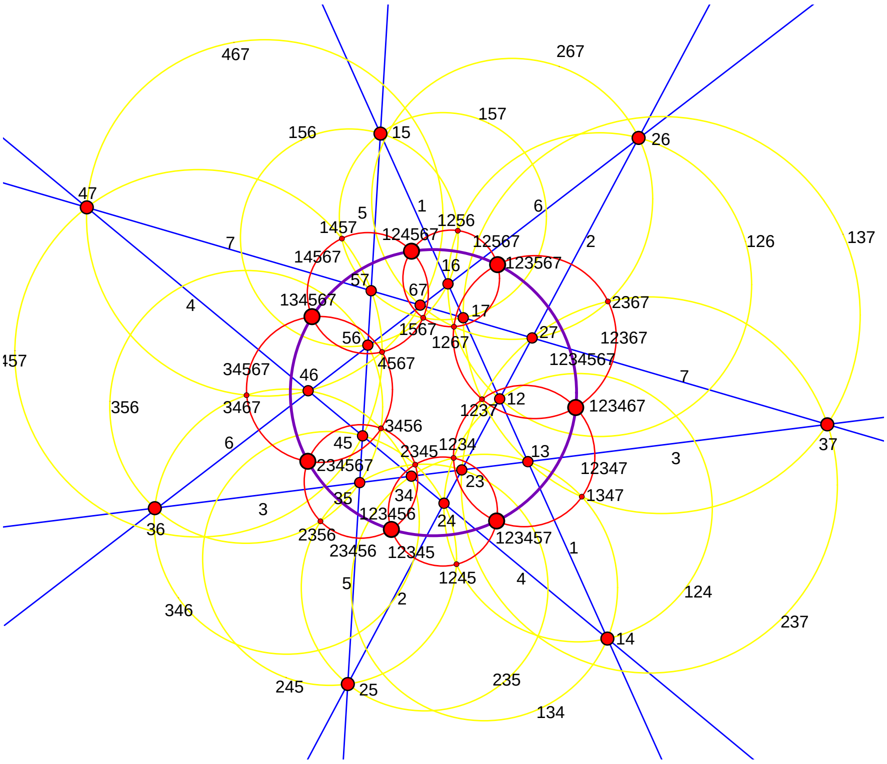

Figure 25: Clifford's circle chain theorem ($n = 7$).

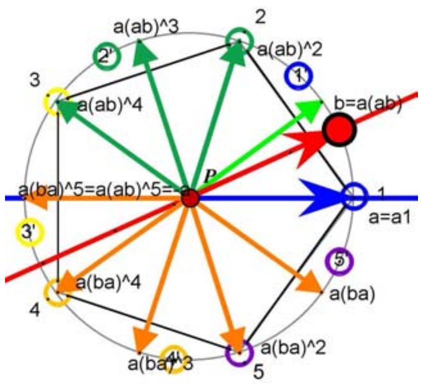

Figure 26: Oriented rotations of a pentagon.

8 Point groups

Point groups are essential for the study of symmetries of molecules and crystals. Geometric algebra naturally provides for a geometric treatment in terms of location vectors in the crystal.[11] To illustrate this the complete set of two dimensional point groups of regular polygons with $k = 1, ..., 6$ sides (corners) are implemented as applets. All rotations are represented as compositions of reflections: $\mathbf{x} \to \mathbf{rxr}$. The unit vectors $\mathbf{r}$ mark the direction of the line of reflection either directed to a corner, or the middle of a side of the regular polygons.

For each regular polygon the applets first illustrate the oriented reflections in the symmetry group $2H_k$, and second the oriented rotations (Fig. 26 for $k = 5$) in the dicyclic point symmetry group $2C_k$. In $2C_5$ we have the following ten symmetry rotations:

$$
\begin{aligned}
&\mathbf{x} \to \mathbf{x}' = R\mathbf{x}\widetilde{R}, \;\; \mathbf{a}^2 = \mathbf{b}^2 = 1, \\
&R = 1 = (\mathbf{ab})^0, \mathbf{ab}, (\mathbf{ab})^2, (\mathbf{ab})^3, (\mathbf{ab})^4, \\
&R = -1 = (\mathbf{ab})^5 = (\mathbf{ba})^5, \mathbf{ba} = -\mathbf{ba}(\mathbf{ab})^5 = -(\mathbf{ab})^4, (\mathbf{ba})^2 = -(\mathbf{ab})^3, \\
&\qquad (\mathbf{ba})^3 = -(\mathbf{ab})^2, (\mathbf{ba})^4 = -\mathbf{ab},
\end{aligned} \tag{19}
$$

where $\mathbf{a}$ points to a corner of the pentagon and $\mathbf{b}$ to an adjacent middle of a side. The first five rotors R describe rotations of positive sense, and the latter five rotors describe corresponding rotations of negative sense.

9 Structural mechanics

A basic problem of structural mechanics is the description of beams, their forces and momenta. The geometric nature of such problems lends itself naturally to the application of geometric algebra. To exemplify this we deal with the

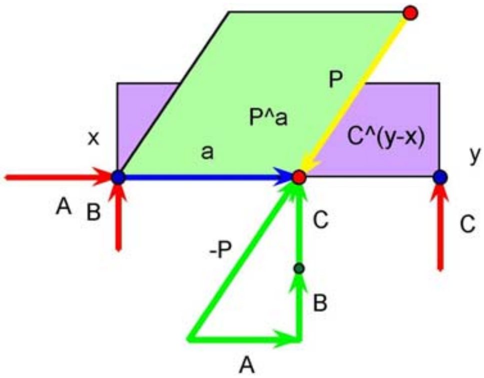

Figure 27: Beam with concentrated load.

equilibrium conditions for a plane 1-dimensional beam, represented by a vector in a plane. A plane beam has three degrees of freedom, e.g. two displacements of the barycenter and the rotation about the barycenter. Equilibrium means to make sure that the beam will not move or rotate. Newton's law leads us to the zero equilibrium conditions for the sums of all forces and of all force induced momenta, respectively.

Let $\mathbf{x}$ be the position of a hinge on the left end and $\mathbf{y}$ the position of a cart on the right end of the beam. The (concentrated) load force P is acting at position $\mathbf{a}$, somewhere between $\mathbf{x}$ and $\mathbf{y}$. The reaction forces at $\mathbf{x}$ and $\mathbf{y}$ are then simply (compare Fig. 27)

$$\begin{aligned} A &= -P \rfloor (\mathbf{y}-\mathbf{x})/(\mathbf{y}-\mathbf{x}), \\ B &= -P \wedge (\mathbf{y}-\mathbf{x})/(\mathbf{y}-\mathbf{x}), \\ C &= -P \wedge \mathbf{a}/(\mathbf{y}-\mathbf{x}), \end{aligned} \tag{20}$$

where it is a convention to perform the left contraction and the outer product before dividing by $\mathbf{y}-\mathbf{x}$.

In textbook algebra, the zero momentum condition is also a vector equation, because each momentum is a cross product of two vectors, and thus a vector that sticks out of the plane at 90 degree, i.e. is not part of the algebra of the plane. If we interpret the momentum condition in the framework of geometric algebra, all momenta are expressed by the outer product of force vector, and distance from the pivot. Thus the momentum naturally becomes a bivector in the geometric algebra of the plane. The two momenta of P and C relative to the hinge $\mathbf{x}$ are in equilibrium:

$$P \wedge \mathbf{a} + C \wedge (\mathbf{y}-\mathbf{x}) = 0. \tag{21}$$

Two applets show the behavior of a simply supported beam with a concentrated (Fig. 27) and a distributed load. The direction and magnitude of the load, the point where it is applied, the bending moment and the equilibrium conditions are all visualized interactively.

10 Cinderella and Geometric Algebra

Cinderella uses complex projective coordinates.[14] The geometric algebra of a two dimensional Euclidean space is four dimensional with basis: $\{1, \mathbf{e}_1, \mathbf{e}_2, I_2 = \mathbf{e}_1\mathbf{e}_2\}$. The even subalgebra with basis $\{1, I_2\}$ is isomorphic to the complex numbers, because $I_2^2 = -1$. But I_2 *anticommutes* with every vector in the $\mathbf{e}_1, \mathbf{e}_2$-plane:

$$(a_1\mathbf{e}_1 + a_2\mathbf{e}_2)I_2 = -I_2(a_1\mathbf{e}_1 + a_2\mathbf{e}_2). \tag{22}$$

The geometric algebra of the Euclidean plane comprises the Euclidean $\mathbf{e}_1, \mathbf{e}_2$-plane and has the complex numbers as its even subalgebra. But the geometric algebra of the Euclidean plane differs from a complex linear space, or complex algebra, where scalars are complex numbers, and where the imaginary unit *commutes* with all vectors.[7] Geometric algebra allows to easily incorporate projective geometry as well.[15]

Beyond that exists a double conformal model of Euclidean space in the geometric algebra $\mathcal{R}_{4,1}$. There the Euclidean space is modelled as a three dimensional section of a higher dimensional null cone. This description unfies the description of rotations and translations to monomials of multivectors alongside with a number of other notable benefits.[16] A first attempt at an interactive JAVA implementation of this double conformal model of three dimensional Euclidean space has already been made with the program KamiWaAi.[17]

11 Conclusion

We hope that this free, colorful, interactive way of mathematical exploration will bring fresh motivation and insight for anybody interested in geometry. For people who want to make their own applets: Cinderella is reasonably prized commercial software. Within three to five years a 3D version is expected. As for geometric algebra, its applications in science and engineering currently undergo rapid development.

E.H. thanks God for the joy of doing research in the wonders of creation:

> The works of the LORD are great, sought out of all them that have pleasure therein. [18]

He thanks his wife, Fukui University, K. Shinoda and H. Ishi. *Soli deo Gloria.*

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