To appear in: K. Adhav (ed.), Proceedings of the *International Conference on Relativity 2005 (ICR2005)*, University of Amravati, India, January 2005.

Relativistic Physics as Application of Geometric Algebra

By Eckhard S. M. Hitzer
University of Fukui, 3-9-1 Bunkyo, 910-8507 Fukui, Japan
http://sinai.mech.fukui-u.ac.jp/

Einstein's Background

In the beginning God created the heavens and the earth. Now the earth was formless and empty, darkness was over the surface of the deep, and the Spirit of God was hovering over the waters. And God said, "Let there be light," and there was light.
Thora, Genesis 1:1-3

Abstract

This review of relativistic physics integrates the works of Hamilton, Grassmann, Maxwell, Clifford, Einstein, Hestenes and lately the Cambridge (UK) Geometric Algebra Research Group. We start with the geometric algebra of spacetime (STA). We show how frames and trajectories are described and how Lorentz transformations acquire their fundamental rotor form. Spacetime dynamics deals with spacetime rotors, which have invariant and frame dependent splits. Spacetime rotor equations yield the proper acceleration (bivector) and the Fermi (vector) derivative.

A first application is given with the relativistic STA formulation of the Lorentz force law, leading to the description of spin precession in magnetic fields and Thomas precession. Now the stage is ready for introducing the STA Maxwell equation, which combines all 4 equations in one single STA equation. STA has procedures to extract from the electromagnetic field strength bivector F, electric and magnetic fields (also for relative motion observers) and field invariants, field momentum and stress-energy tensor. The Leonhard-Wiechert potential gives the retarded field of a point charge.

In addition we formulate the Dirac equation in STA, both massless and massive. From the Dirac equation we can derive STA expressions for Dirac observables. Plane wave states are described with the help of rotor decomposition. Finally we briefly review a STA gauge theory of gravity built on displacement and rotation gauge principles.

1 History

This paper is based on my ICR 2005 lecture in Amravati (India) on Wed. 12 Jan. 2005. I want to begin with a few historical remarks. Gauss, Rodgrigues and Hamilton invented quaternions in the first half of the 19th century [1]. Grassmann invented exterior algebra, the branch of mathematics, which according to him far surpasses all others [2]. In 1873 Maxwell's four partial differential equations were published in a fully developped form [3]. In 1878 W.K. Clifford applied Grassmann's algebra to create *geometric algebras* (GA), ultimately unifying Grassmann's and Hamilton's great ideas [4]. Clifford has speculated that physical space was curved (thus partly anticipating Einstein), and thought that "the ether and matter" were made of the "same stuff" [5].

In 1905 Einstein introduced the special theory of relativity (STR) and in 1915 he lectured before the Prussian Academy of Sciences on general relativity (GR), the curved space theory of gravity. In the 1960ies Hestenes reinvigorated the study of geometric algebra applied to physics in the form of spacetime algebra (STA) [6]. Amongst many followers, the Cambride (UK) Geometric Algebra Research Group (GARG) at the Cavendish institute systematically embraced Hestenes' approach, regarding GA as a general matematical framework for physics [7,8]. An excellent review of a gauge theory of gravity with geometric calculus (invented by the GARG group) was recently presented by Hestenes [9]. Geometric calculus simply means GA multivector calculus.

Nowadays the study and application of geometric algebras has spread to virtually all fields of science, including technological applications in image processing, robotics, speech analysis, etc. [16]. But we narrowly focus our attention to the description of STR, electromagnetism, relativistic quantum theory and general relativity within STA. We will mainly rely on [6,7,8,9].

It is wellknown that Einstein descended from a Jewish family. So one of the books he may have become familiar with already in early childhood may well have been the Jewish Thora, which begins with the famous account of the creation (Genesis) of the universe quoted above.

2 Introduction to Spacetime Algebra (STA)

2.1 Geometric Product

Two vectors *a*,*b* in Minkowski space are multiplied with the associative geometric product:

$$ab = |a||b|(\cos a + \mathbf{i} \sin a),$$

where **i** = $e_1 e_2$ is unit area element (bivector) of the *a*,*b* plane. The product has a symmetric scalar *inner part*

$$a \cdot b = (ab + ba)/2 = |a||b| \cos \alpha$$

and an antisymmetric bivector *outer part* (fully representing the parallelogram area spanned by the two vectors in space together with its direction and orientation)

$$a \wedge b = (ab - ba)/2 = |a||b| \, \mathrm{i} \sin a.$$

2.2 Reflections and Rotations

A very important use of the goemetric product is the elegant description of the reflection of a vector *x* at a plane (Fig. 1). The plane has the normal vector *a*. Reflection *x* → *x'* means to preserve the component of *x* parallel to the plane (perpendicular to *a*) and reverse the component of *x* perpendicular to the plane (parallel to *a*).

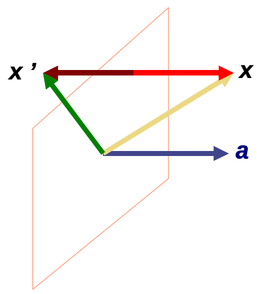

Fig. 1. Vector ***a*** perpendicular to plane of reflection *x* → *x'*.

This is easily done with the following geometric product

$$\mathbf{x'} = -a\,\mathbf{x}\,a^{-1}, \qquad a^{-1} = a/a^2 .$$

Two reflections at planes with dihedral angle $q_{a,b}$ are well known to produce a rotation (Fig. 2) by twice the angle $q_{a,b}$:

$$\theta_{x,x'} = 2\theta_{a,b}$$

We therefore get the rotation formula

$$x'' = ba\,x\,a^{-1} b^{-1} = ba\,x(ba)^{-1} = R\,x\,\tilde{R},$$

where the *rotor R* simply denotes the geometric product *ba* and $R^{\sim}$ its *reverse.*

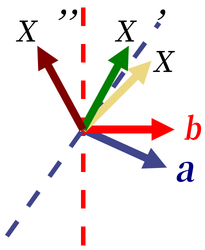

Fig. 2. Rotation by twice the angle between *a* and *b*.

Such rotations in the STA will not only comprise elements of *SO(3)*. They will include Lorentz transformations. For example a *boost* will simply be a rotation in a spacetime plane.

2.3 Geometric Algebra of Spacetime (STA)

The above definition of geometric algebra applies in fact to any space $R^{n,m}$, where *n,m* indicates the signature of positive and negative square vectors in a basis of $R^{n,m}$. Now let us turn our attention to the geometric algebra of flat spacetime (STA). The STA comprises 16 basis elements

- Real scalar multiples of 1,
- An orthonormal frame of the 4D Minkowski vector space $\{g_0, g_1, g_2, g_3\}$with metric h_{mn}
$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = diag(+1,-1,-1,-1).$$
- The unit oriented (pseudo-scalar) 4-volume (also used as *duality* operator)
$$I = \gamma_0 \gamma_1 \gamma_2 \gamma_3 , I^2 = -1$$
- six bivectors (three and three related by *duality*, i.e. multiplication with *I*)
$$\vec{\sigma}_i = \gamma_i \gamma_0 , \vec{\sigma}_i^{\,2} = +1 , , \quad I \vec{\sigma}_i , (I \vec{\sigma}_i)^2 = -1 , i = 1,2,3$$
- and four trivectors (*dual* to the four vectors)
$$I \gamma_\mu$$

3. Special Relativity in STA

3.1 Even STA Subalgebra

Let $x(l)$ be a spacetime *trajectory* and its tangent vector
$$x' = \frac{\partial x(\lambda)}{\partial \lambda}$$
The trajectory and its tangent vector are *timelike* if
$$x'^2 > 0.$$
This permits to define the *proper time* t and (unit vector) four velocity v
$$v = \partial_\tau x = \dot{x} , \qquad v^2 = 1$$
associated with the instantaneous rest frame. The trajectory and its tangent vector are called *null* (photons!) if
$$x'^2 = 0$$
A *relative* vector (actually a spacetime bivector) is defined by the outer product
$$x \wedge v = \vec{x} ,$$
which is the grade 2 part of the full geometric product
$$x v = x \cdot v + x \wedge v = t + \vec{x} .$$
The invariant distance therefore has the *built-in* form
$$x^2 = x v v x = t^2 - \vec{x}^{\,2}$$
The even STA subalgebra is *isomorphic* to the GA of 3D Euclidean space and has the 8-dimensional basis (*i*=1,2,3)
$$1 , \{\vec{\sigma}_i\} , \{I \vec{\sigma}_i\} , I$$

3.2 Velocity, Momentum and Wave Vectors

In rest space of *v* the proper velocity *u* of a particle can be measured relative to *v* in the form

$$u\,vv=(u\cdot v+u\wedge v)v=\gamma(1+\vec{u})v\,,\quad \gamma=u\cdot v\,,\quad \vec{u}=\frac{u\wedge v}{\gamma}$$

The *relative momentum* is defined by

$$\vec{p}:\,p\,v=p\cdot v+p\wedge v=E+\vec{p}\,,$$

with the *invariant m* of

$$m^2=p^2=E^2-\vec{p}^{\,2}$$

In the *null* case a photon wave vector can be measured relative to *v* in the form

$$k\,v=k\cdot v+k\wedge v=\omega+\vec{k}\,,0=k^2=\omega^2-\vec{k}^{\,2}$$

3.3 Lorentz Transformations

Let us assume for simplicity that two *frames* are related by the frame vector relations $e'_2=e_2$, $e'_3=e_3$, with scalar velocity

$$\gamma=e'_0\cdot e_0=ch(\alpha)\,,\quad \beta=th(\alpha)\,,$$

which allows to derive

$$e'_0=\gamma(e_0+\beta e_1)\,,\quad e'_1=\gamma(e_1+\beta e_0)\,.$$

Putting this in *rotor* form we obtain

$$\begin{aligned}e'_0&=ch(\alpha)e_0+sh(\alpha)e_1(e_0e_0)=\{ch(\alpha)+sh(\alpha)e_1e_0\}e_0\\&=\exp(\alpha e_1e_0)e_0=\exp(\alpha e_1e_0/2)e_0\exp(-\alpha e_1e_0/2)=R\,e_0\,\tilde{R}\end{aligned}$$

and in general we get Lorentz transformations in *rotor* form as

$$e'_\mu=R\,e_\mu\,\tilde{R}\,,\qquad R=\exp(\alpha e_1e_0/2)\,.$$

3.4 Addition of Velocities and Redshift

Two observers with velocities

$$v_1=e^{\alpha_1e_1e_0}e_0\,,\ \ v_2=e^{\alpha_2e_1e_0}e_0$$

have the relative velocity

$$\begin{aligned}\frac{v_1\wedge v_2}{v_1\cdot v_2}&=\frac{\langle\exp((\alpha_1+\alpha_2)e_1e_0)\rangle_2}{\langle\exp((\alpha_1+\alpha_2)e_1e_0)\rangle_0}=\frac{sh(\alpha_1+\alpha_2)}{ch(\alpha_1+\alpha_2)}e_1e_0\\&=th(\alpha_1+\alpha_2)e_1e_0\end{aligned}$$

The frequency redshift of a photon emitted from particle 1 towards particle 2 can then be calculated from the velocity vector of particle 1 in the rest system of particle 2, the photon *null* vector in the rest system of particle 2, and the velocity of particle 2 itself

$$v_1=ch(\alpha)e_0-sh(\alpha)e_1,\ \ k=\omega_2(e_0+e_1)\,,\ \ v_2=e_0\,.$$

The redshift result is simply

$$1+z=\frac{\omega_1}{\omega_2}=\frac{v_1\cdot k}{v_2\cdot k}=\frac{\omega_2(ch\alpha+sh\alpha)}{\omega_2}=e^{\alpha}=\left(\frac{1+th\alpha}{1-th\alpha}\right)^{1/2}$$

4. Spacetime Dynamics -- Spacetime Rotors

4.1 Invariant Decomposition

Every restricted Lorentz transformation (*B* bivector)

$$a\rightarrow Ra\,\tilde{R}\,,\ \ R\tilde{R}=1,\ \ R=\pm\exp(B/2)$$

has the following *invariant decomposition* into a *boost* and a *SO(3) spatial rotation* factor

$$\boxed{R=\exp(\alpha\,\hat{B}/2)\exp(\beta\,I\,\hat{B}/2)}$$

with

$$B^2=\rho\exp(I\phi),\quad \hat{B}=\rho^{-1/2}\exp(-I\phi/2)B,\quad \hat{B}^2=1,$$
$$\hat{B}(I\hat{B})=(I\hat{B})\hat{B},\quad B=\rho^{1/2}\exp(I\phi/2)\hat{B}=\alpha\hat{B}+\beta I\hat{B}.$$

The timelike bivector $\hat{B}$ has two invariant null vectors (seen as *fixed points* on the past celestial sphere)

$$\hat{B}^2=1,\quad 2\hat{B}=n_-\wedge n_+,\quad n_\pm^2=0,$$
$$Rn_\pm\tilde{R}=\exp(\alpha)n_\pm$$

4.2 Decomposition into Pure Boost and Relative Rotation

A *pure* boost L from velocity g_0 to velocity $v = L g_0 \tilde{L}$, is given by

$$L=\frac{1+v\gamma_0}{\{2(1+v\cdot\gamma_0)^{1/2}\}}=\exp\left(\frac{\alpha}{2}\frac{v\wedge\gamma_0}{|v\wedge\gamma_0|}\right),\quad ch(\alpha)=v\cdot\gamma_0$$

A general Lorentz transformation R can then be obtained with an additional γ_0 invariant rotation

$$U=\exp(I\vec{b}/2),\qquad U\gamma_0=\gamma_0 U.$$

Together the pure boost L and the γ_0 invariant rotation U provide a second *frame dependent* (!) decomposition of the general Lorentz rotor R

$$\boxed{R = LU\,.}$$

4.3 Spacetime Rotor Equations

We are now about to rewrite spacetime dynamics by universally encapsulating it in the form of *rotors*. This will both apply to classical and quantum physics and open up a new geometric understanding of quantum wave functions, expounded in a later section. With the proper time t we have for the four velocity of a particle

$$v=\dot{x}=R(\tau)\gamma_0\tilde{R}(\tau),\quad v^2=1$$

From this we compute the proper acceleration bivector (projected into the instantaneous rest frame)

$$v\dot{v}=v\wedge\dot{v}=2(\dot{R}\tilde{R})\cdot v\,v$$

In the case of *Fermi transport* (pure boosts from instant to instant, keeping the instantaneous spatial rest frame as constant as possible, e.g. for inertial gyroscopes) we have

$$\dot{R}\tilde{R}=\frac{1}{2}v\dot{v}$$

It is useful to define the socalled *Fermi derivative*

$$\frac{Da}{D\tau}=\dot{a}+a\cdot(v\dot{v})$$

which vanishes for Fermi transport, preserving a^2 and $a\cdot v$. We especially have for Fermi transport with

$$a\cdot v=0,\qquad \frac{Da}{D\tau}=\dot{a}\wedge v\dot{v}\quad .$$

4.4 Lorentz Force Law

With the SRT electromagnetic *Faraday field bivector* defined as

$$F=\vec{E}+I\vec{B}$$

the *Lorentz force law* becomes (q = charge of particle with mass m)

$$m\dot{v}=q\,F\cdot v \quad .$$

This immediately leads to the acceleration bivector

$$\dot{v}\,v=\frac{q}{m}(F\cdot v)\wedge v=\frac{q}{m}\vec{E}_v \quad .$$

The expression on the right hand side of the second equality is in terms of the Electric field in the instantaneous v frame. Now we can express the dynamics in the computationally robust *rotor form*

$$\dot{R}=\frac{q}{2\mathrm{m}}F\,R\,, \quad \dot{x}=v=R\,v_0\,\tilde{R}.$$

In the special case of a constant field we get with the invariant decomposition of section 4.1 for the Faraday bivector

$$F=\alpha\,\hat{F}+I\,\beta\,\hat{F}$$

The particle trajectory composed of linear acceleration and periodic rotation terms is then

$$x-x_0=\frac{\exp(q\alpha\,\hat{F}\tau/m)-1}{q\alpha/m}\hat{F}\cdot v_0-\frac{\exp(q\beta\,I\,\hat{F}\tau/m)-1}{q\beta/m}(I\,\hat{F})\cdot v_0$$

4.5 Gyromagnetic Moment

The precession of a spin vector s interacting with a magnetic field yields the STA form of the Bargmann-Michel-Telegdi equation

$$s=\vec{s}\,\gamma_{0,} \quad \dot{s}=\frac{q}{m}F\cdot s+(g-2)\frac{q}{2\mathrm{m}}(F\cdot s)\wedge v\,v$$

For g= 2 (like for the Lorentz force) this reduces to

$$\dot{s}=\frac{q}{m}F\cdot s$$

For $s\cdot v=0$ therefore both *trajectory*

$$\dot{v}=2(\dot{R}\,\tilde{R})\cdot v\,,$$

and spin *precession* follow from the robust rotor dynamics expression of section 4.4.

$$\dot{s}=2(\dot{R}\,\tilde{R})\cdot s.$$

4.6 Thomas Precession

Let us consider a particle circling with radius a, and angular frequency ω

$$x(\tau)=t(\tau)+a\{\cos(\omega\,t)\gamma_1+\sin(\omega\,t)\gamma_2\}$$

Its velocity can be expressed in rotor factor form as

$$v=R_\omega\,R_\alpha\gamma_0\,\tilde{R}_\alpha\,\tilde{R}_\omega\,, \quad th\alpha=a\,\omega\,,\ \ ch\alpha=\dot{t}\,,$$
$$R_\omega=\exp(-\omega\,t\,I\,\vec{\sigma}_3/2)\,,\ \ R_\alpha=\exp(-\alpha\vec{\sigma}_2/2)$$

The total rotor (in terms of three rotor factors) is then

$$R=R_\omega\,R_\alpha\Phi\,, \quad \Phi\gamma_0=\gamma_0\Phi$$

Assuming Fermi transport yields for the third rotor factor

$$\Phi=\exp(ch(\alpha)\omega\,t\,I\,\vec{\sigma}_3/2)$$

The precession angle *q(T)* of the Fermi transported e_1 after one complete orbit (*t = 0 ... T = 2p/w*) will therefore be

$$e_1(T)\cdot e_1(0)=\cos(2\pi(ch\alpha-1))$$
$$\boxed{\theta(T)=2\pi(ch\alpha-1)}$$

5. Maxwell’s Electromagnetism

Let us now turn our attention to Maxwell’s four partial differential equations for electromagnetic fields [3]. The description of the relativistic motion of light by these equations was instrumental in shaping special relativity.

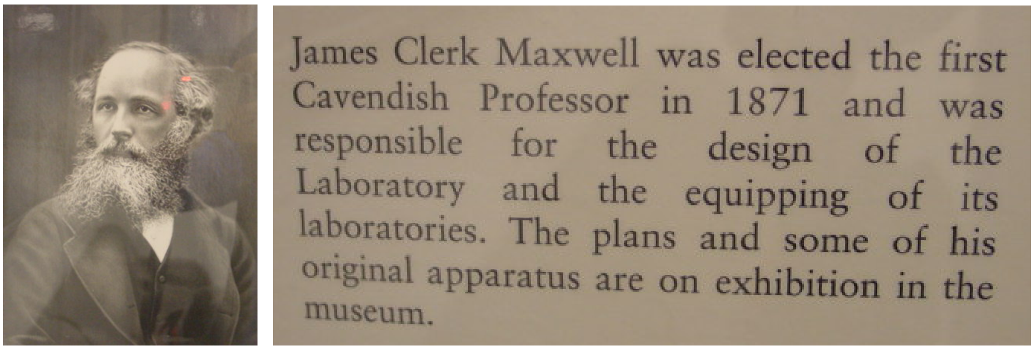

Fig. 3. J.C. Maxwell wrote Psalm 111:2 in golden letters at the front gate of his Cambridge (UK) Cavendish Laboratory:
Great are the works of the Lord; They are studied by all who delight in them.

5.1 The Single Maxwell Equation in STA

The spacetime (Dirac) vector derivative can be expressed in terms of familiar partial derivatives as
$$\nabla=\gamma^{\mu}\partial_{\mu},\ \ \partial_{\mu}=\frac{\partial}{\partial x^{\mu}},$$
$$\nabla\gamma_0=\partial_t-\vec{\sigma}_i\partial_i=\partial_t-\vec{\nabla}$$
This surprisingly allows a *complete unification* of all four Maxwell equations into the unique *single* multivector Maxwell equation
$$\boxed{\nabla F=J}$$

with electromagnetic Faraday field bivector *F* and current vector *J* given by

$$F=\vec{E}+I\,\vec{B}=\nabla\wedge A$$
$$J=(\rho+\vec{J})\gamma_{0.}$$

Grade selection neatly separates the vector and trivector parts (equivalent differential form expressions in brackets) into
$$\nabla\cdot F=J\ ,\quad (\partial_{\mu}F^{\mu\nu}=J^{\nu})$$
$$\nabla\wedge F=\nabla\wedge\nabla\wedge A=0,\quad (\epsilon^{\mu\nu\rho\sigma}\partial_{\nu}F_{\rho\sigma}=0)$$
The totally natural unification achieved in STA *cannot* be obtained by means of differential forms! Further imposing the Lorentz gauge condition we get the electromagnetic wave equation as

$$\nabla\cdot A=0\ \ (F=\nabla A),\quad \nabla^2 A=J$$

5.2 Field Strength and Relative Motion

The (observer γ_0–dependent) electric and magnetic relative bivector components of the Faraday field bivector can easily be computed by

$$\vec{E}=\frac{1}{2}(F-\gamma_0 F\gamma_0), \quad I\vec{B}=\frac{1}{2}(F+\gamma_0 F\gamma_0)$$

The Faraday field bivector is subject to Lorentz transformations

$$F\rightarrow \tilde{R}\,F\,R$$

For a stationary (γ_0–frame) charge the field bivector will be

$$F=E=E_x\vec{\sigma}_1+E_y\vec{\sigma}_2$$

Another observer moving in the γ_1 direction is boosted with

$$R=\exp(\alpha\vec{\sigma}_1/2)$$

resulting in

$$\tilde{R}\,F\,R=E_x\vec{\sigma}_1+E_y\exp(-\alpha\vec{\sigma}_1)\vec{\sigma}_2 \quad \Rightarrow$$

$$E'_x=E_x, \quad E'_y=ch(\alpha)E_y, B'_z=-sh(\alpha)E_y$$

The scalar (~ 1) and pseudoscalar (~ I) Lorentz invariants are simply given by

$$F^2=\langle F^2\rangle_0+\langle F^2\rangle_4=(\vec{E}^2-\vec{B}^2) \quad + \quad I(2\vec{E}\cdot\vec{B})$$

Lagrange density *fully Lorentz invariant*

5.3 Fields of a Point Charge

Charge q moving along trajectory $x_0(t)$ radiates to an observer positioned at x along

$$X=x-x_0(\tau), \qquad X^2=0,$$

therefore we get surfaces of constant t. That means the observer receives electromagnetic radiation from the world-point of intersection of his past light cone with $x_0(t)$:

$$\forall x \ \ \exists_1 \tau(x): \ \ X^2=0.$$

The Lienard-Wiechert potential in the rest frame of the charge at the retarded position (described above) is

$$A=\frac{q}{4\pi}\frac{v}{X\cdot v},$$

resulting in the Faraday field bivector with velocity and long range radiation terms

$$\Rightarrow F=\nabla A=\frac{q}{4\pi}\left(\frac{X\wedge v}{(X\cdot v)^2}+\frac{X\,\Omega_v\,X}{2(X\cdot v)^3}\right).$$

The accelaration bivector in the Faraday bivector is

$$\Omega_v=\dot{v}\wedge v.$$

For constant v we get with the above procedure

$$x_0(\tau)=v\tau$$

and hence

$$\Rightarrow F=\frac{q}{4\pi}\frac{x\wedge v}{(x\cdot v)^3}.$$

F splits in the γ_0-observer rest frame into

$$\vec{E}=\frac{q\gamma}{4\pi d^3}(\vec{x}-\vec{v}\,t), \quad \vec{B}=\frac{q\gamma}{4\pi d^3}I\,\vec{x}\wedge\vec{v}, \quad \vec{x}=\vec{x}(t).$$

5.3.1 Point Charge -- Circlular Orbit

Let us consider a point charge moving on a circle in the Is_3 plane with speed

$$|\vec{v}|=th\,\alpha$$

$$x_0(\tau)=ch(\alpha)\gamma_0+a\{\cos(\omega\tau)\gamma_1+\sin(\omega\tau)\gamma_2\}$$

$$\Rightarrow v(\tau)\Rightarrow\Omega_v(\tau)=\dot{v}\wedge v.$$

Numerically the past x light cone intersection with $x_0(t)$ leads step by step to the point charge Faraday field bivector $F(t)$ itself.

$$(x-x_0(\tau))^2=0 \ \Rightarrow \ t(\tau) \quad \Rightarrow \quad \tau(t)\Rightarrow v(t), \quad \Omega(t)\Rightarrow F(t)$$

Numerically the fieldlines of F in the x,y plane can then be explicitly pictured for varying speed parameter values (see e.g. the fieldline diagrams for α = 0.1, 0.4 and 1.0 in [8]).

5.4 Stress-Energy Tensor

The electromagnetic flux vector of the relativistic 4-momentum across a hypersurface perpendicular to a vector *a* in STA is

$$T(a) = -FaF/2.$$

This compact expression needs to be compared with the less elegant conventional tensor formula

$$T^{\mu}{}_{\nu} = \frac{1}{4}\delta^{\mu}_{\nu} F^{\alpha\beta} F_{\alpha\beta} + F^{\mu\alpha} F_{\alpha\nu}$$

The nonlocal (covariant) field momentum (inegration over spacelike hypersurfaces*)* is given in STA by

$$P = \frac{-1}{2} \int dA\, F\, n\, F$$

The flux vector *T* has the following properties:

1. Symmetry $\quad a \cdot T(b) = T(a) \cdot b$
2. Positivity (for timelike velocities v) $\quad v \cdot T(v) > 0$
3. Conserved current

$$\nabla \cdot T(\gamma_{\mu}) = 0, \quad \mu = 0 \ldots 3$$

6. Quantum Theory

6.1 Dirac, Geometry and STA

The well-known complex four-by-four *Dirac matrices* represent spacetime basis vectors, their algebra is in fact *isomorphic* to the STA. This leads to a most natural complete STA expression of Dirac theory. In his 1970 Boston lecture Dirac showed *"... how physical ideas can be pictured immediately in geometrical terms ..."* [10]. STA beautifully unfolds the geometry underlying Dirac's algebra.

6.2 Relativistic Dirac Spin

Pauli and Dirac spinors can be mapped 1:1 to the even subalgebra of STA. The Pauli spinor matrices (isomorphic to Hamilton's quaternions) can be mapped 1:1 to a real linear combination of scalars and relative bivectors

$$\Psi = \begin{pmatrix} a^0 + i\,a^3 \\ -a^2 + i\,a^1 \end{pmatrix} \leftarrow\rightarrow \psi = a^0 + a^k\, I\, \vec{\sigma}_k$$

The relativistic Dirac spinor matrices (Φ and Λ are Pauli spinors) can also be mapped 1:1 to even STA multivectors using

$$\Psi = \begin{pmatrix} \Phi \\ \Lambda \end{pmatrix} \leftarrow\rightarrow \psi = \varphi + \lambda\, \vec{\sigma}_3$$

Dirac matrix operators (distinguishing by a hat)

$$\hat{\gamma}_{\mu}(\mu = 0, \ldots, 3),\, i\,, \hat{\gamma}_5$$

are then mapped to

$$\hat{\gamma}_{\mu} \Psi \leftarrow\rightarrow \gamma_{\mu} \psi \gamma_0\,, \quad i\Psi \leftarrow\rightarrow \psi\, I\, \vec{\sigma}_3\,, \quad \hat{\gamma}_5 \Psi \leftarrow\rightarrow \psi\, \vec{\sigma}_3\,.$$

6.3 Massless Dirac Equation

The *massless Dirac equation* is indeed the simplest linear STA vector derivative equation

$$\nabla \psi = 0.$$

From y we obtain the right and left-handed helicity eigenstates as

$$\psi = \psi \frac{1}{2}(1+\vec{\sigma}_3) + \psi \frac{1}{2}(1-\vec{\sigma}_3) = \psi_+ + \psi_-$$
$$\nabla \psi_\pm = 0$$
Taking the massless Dirac equation in two dimensions in fact simply constitutes a set of Cauchy-Riemann equations
$$\vec{\nabla} = \vec{e}_1 \partial_x + \vec{e}_2 \partial_y, \qquad \psi = u + \vec{e}_1 \vec{e}_2 v$$
$$\vec{\nabla} \psi = (\partial_x u - \partial_y v)\vec{e}_1 + (\partial_y u + \partial_x v)\vec{e}_2 = 0$$

This helps to understand how the STA vector derivative in the massless Dirac equation $\nabla y = 0$ generalizes analytic functions to higher dimensions [11].

6.4 Massive Dirac Equation and Observables

Geometrically the (even multivector) spinor y acts like a Lorentz *rotor*, which rotates g_0 to the relativistic 4-momentum *p/m.* Expressing the momentum by the Dirac vector derivative we naturally get for the massive Dirac equation
$$p\psi = m\psi\gamma_0 \;\longleftrightarrow\; \nabla \psi I \vec{\sigma}_3 = m\psi\gamma_0$$
The Dirac current vector
$$J = \psi\gamma_0\tilde{\psi}, \qquad \nabla \cdot J = 0$$
geometrically is an instruction to rotate the vector g_0 into the direction of the current and dilate it with the real scalar probability density r to produce the (timelike and future-pointing) current vector *J.*
$$\psi\tilde{\psi} = \rho \exp(I\beta), \quad \rho \geq 0, \beta \in \mathbb{R}$$
The *duality* rotation exp(*Ib*) is related to particle and antiparticle wavefunctions, with b the famous Yvon-Takabayashi angle [12]. The full relation of the Lorentz rotor R to the Dirac spinor y is explicitly given by
$$(\rho \neq 0) \quad R = \psi \rho^{-1/2} \exp(-I\beta/2), \quad R\tilde{R} = 1$$
$$J = \rho R \gamma_0 \tilde{R}$$
$J_0 = J \cdot g_0 \geqq 0$ then gives the positive definite probability density for locating the electron. The spin vector (see also section 4.5) for the *intrinsic spin* of a particle is yet another observable obtained analogously with the Dirac spinor (rotor)
$$s = \rho R \gamma_3 \tilde{R}$$

6.5 Plane Wave States

Plane waves are nothing but the positive and negative energy solutions of the Dirac equation
$$\pm p \psi^{(\pm)} \tilde{\psi}^{(\pm)} = m J$$
Using the decomposition of section 4.2 we can analyze plane waves as pure boosts L: $mg_0 \rightarrow p$, combined with pure spatial (g_0 invariant) rotations U
$$L(p) = \frac{E + m + \vec{p}}{\{2m(E+m)\}^{1/2}}$$

Therefore Lorentz rotors *LU* multiplied by phase factors of *positive* and *negative* energies constitute the plane waves
$$\psi^{(+)}(x) = L(p)U \exp(-I\vec{\sigma}_3\, p \cdot x)$$
$$\psi^{(-)}(x) = L(p)U\, I \exp(I\vec{\sigma}_3\, p \cdot x)$$

7. STA gauge theory of gravity

In this final section I will give a very brief review of a fascinating STA gauge theory of gravity developped by the GARG group in Cambridge referring to the introductory form given to it recently by D. Hestenes [9].

Again as in section 4.3 (proper time) parametrized rotors R(t) serve to express particle dynamics by a generalized rotational bivector velocity and to define comoving frames. A *flat space* model is employed, to identify the spacetime manifold with the Minkowski vector space. A spacetime map then describes the spaciotemporal partial ordering of physical events as points in the spacetime manifold. Fields on spacetime take values in the flat tangent space algebra.

Two fundamental gauge principles, the *displacement gauge principle* (DGP) that *equations of physics must be invariant under arbitrary smooth remappings of events on spacetime*; and the *rotation gauge principle* (RGP) of *covariance under (active) local Lorentz rotations* correspond to globally homogeneous (DGP) and locally isotropic (RGP) spacetime.

The DGP is realized with the help of a new physical (gravitational) field, an invertible gauge field (or tensor), equivalent to a *tetrad field*. This leads to position gauge *invariant* forms of (vector) derivative, velocity, and the line element with symmetric metric tensor; and completely decouples the remapping of events in spacetime from coordinate changes. The DGP is a true symmetry principle, because the displacement *symmetry* group leaves the flat spacetime background invariant.

The RGP leads to a gauge covariant derivative (vector coderivative) including a bivector valued connexion tensor. Torsion free Riemannian geometry is assumed. The commutator of the coderivatives defines the curvature, which can be expressed as a covariant bivector-valued function of a bivector variable. Derivations in completely coordinate free form are given for curvature, curvature contractions, coderivative identities and Bianchi identities.

This foundation permits to fully formulate Einstein's equation, electrodynamics with gravity, the derivation of equations of motion from Einstein's equation, particle motion, parallel transfer, gravitational precession and the real Dirac equation with (full electromagnetic and) gravitational interaction(s). Einstein's tensor is given a new unitary form. It becomes further possible in various gauges to establish *very compact multivector solutions* for the curvature tensors $\boldsymbol{R}$ of static black holes

$$\boldsymbol{R}(B) = -\, M\,(B+3\,\mathbf{x_0}\,B\,\mathbf{x_0})/2r3\ ,$$

with B a bivector variable, M the mass of the black hole, $r = |\mathbf{x}| = |x\wedge g_0|$, and $\mathbf{x_0}=e_r g_0$ a relative unit radial bivector. The same is possible for rotating (Kerr) black holes and can e.g. be applied to elementary black hole physics and (gravitational) orbital precession.

A position gauge invariant *canonical energy-momentum split* of Einstein's equations allows to identify *total* and *gravitational field* energy-momentum tensor densities. The total density is the divergence of a bivector valued *superpotential* (function of the vector connexion). We therefore have a coordinate independent energy-momentum tensor! There is a rotation gauge dependence, but superpotential and energy-momentum tensor still preserve local gauge equivalence.

8. Conclusion

I very much hope you enjoyed this *brief* tour de force of vast realms of physics cast into the single unified framework of the STA algebra. This algebra can systematically replace differential forms and proves infact to be even more general [13]. If you want to inform youself further [14,15] may be a good idea to begin with.

9. Acknowledgements

First of all I want to thank God my creator and saviour: *Soli Deo Gloria*. (J.S. Bach). I want to round this contribution off with two quotations: one from a reference to Genesis in the New Testament and the second from Einstein himself.

> *In the beginning was the Word and the Word was with God, and the Word was God. And he was with God in the beginning. Through him all things were made; without him nothing was made that has been made. In him was life, and that life was the light of man. The light shines in the darkness, but the darkness has not understood it.* (John in the Bible)

> *Raffiniert ist der Herrgott, aber boshaft ist er nicht.* (A. Einstein)

I further thank my family for total loving support and finally the great organizers of this memorable conference.

10. References

[1] See p. 44 of T. Needham, *Visual Complex Analysis*, OUP, 1997. The following homepage has many important *papers of W.R. Hamilton* online: http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/
[2] H. Grassmann, (F. Engel, editor) *Die Ausdehnungslehre von 1844 und die Geometrische Analyse*, vol. 1, part 1, Teubner, Leipzig, 1894. English translation by L. C. Kannenberg, *A New Branch of Mathematics. The Ausdehnungslehre of 1844, and Others Works. Hermann Grassmann.* Open Court, Ilinois, 1995.
[3] J.C. Maxwell, *A Treatise on Electricity and Magnetism*. Clarendon Press, Oxford. 1873.
[4] W.K. Clifford, *Applications of Grassmann's Extensive Algebra*, American Journal of Mathematics Pure and Applied 1, 350-358 (1878). Online: http://sinai.mech.fukui-u.ac.jp/gcj/wkconline.html
[5] J.S.R. Chisholm, M. Chisholm, *Darwin, V. Woolf, and Curved Space: The Story of W. and L. Clifford,* http://wwwmaths.anu.edu.au/events/AustMS2001/publect.html
[6] D. Hestenes, *Spacetime Algebra*, Gordon and Breach, New York 1966. See also Hestenes' homepage for many online papers: http://modelingnts.la.asu.edu/GC_R&D.html
[7] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, CUP, 2003. See also many *GARG online papers* at http://www.mrao.cam.ac.uk/~clifford/
[8] *Physical Applications of Geometric Algebra* Lecture Course Material http://www.mrao.cam.ac.uk/~clifford/ptIIIcourse/ and
[9] D. Hestenes, *Gauge Theory Gravity with Geometric Calculus*, Found. of Physics, Vol. 35, No. 6, 2005.
[10] P.A.M. Dirac, *Talk: Projective Geometry*, *Origin of Quantum Equations*, (Audio recording made by John B. Hart), Boston University, October 30, 1972. PDF transcript: http://www.aias.us/Comments/Paul Dirac Talk - Projective Geometry (2).pdf
[11] F.Brackx, R. Delanghe, F.Sommen, *Clifford Analysis*, Pitman, 1982.
[12] G. Casanova, *L`algebra de Clifford et ses applications*, Adv. in App. Cliff. Alg., **12**, # S1 (2002).
[13] R. Delanghe, *On primitives of monogenic functions*, preprint submitted to Complex Variables.
[14] *GA-Net Updates* (http://sinai.mech.fukui-u.ac.jp/GAblog/), we*blog* for everyone interested in Clifford Algebra and Geometric algebra. *GA-Net* itself is a monthly electronic newsletter for the same interest group (http://sinai.mech.fukui-u.ac.jp/GA-Net/index.html).
[15] E. Hitzer, *Geometric Calculus International* website: http://sinai.mech.fukui-u.ac.jp/gcj/gc_int.html
[16] R. Ablamowicz, G. Sobczyk (eds.), *Lect. On Cliff. (Geom.) Algebras and Applications*, Birkhaeuser, Basel, 2004. R. Ablamowicz (ed.), *Cliff. Algebras – Applications to Math., Phys., and Engin.*, Birkhaeuser, Basel, 2004.