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Vector Differential Calculus

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This paper treats the fundamentals of the *vector differential calculus* part of *universal geometric calculus.* Geometric calculus simplifies and unifies the structure and notation of mathematics for all of science and engineering, and for technological applications. In order to make the treatment self-contained, I first compile all important *geometric algebra* relationships, which are necessary for vector differential calculus. Then *differentiation by vectors* is introduced and a host of major vector differential and vector derivative relationships is proven explicitly in a very elementary step by step approach. The paper is thus intended to serve as reference material, giving details, which are usually skipped in more advanced discussions of the subject matter.

***Key Words* :** Geometric Calculus , Geometric Algebra, Clifford Algebra, Vector Derivative, Vector Differential Calculus

1. Introduction

"Now faith is being sure of what we hope for and certain of what we do not see. This is what the ancients were commended for. By faith we understand that the universe was formed at God's command, so that what is seen was not made out of what was visible." [7]

The German 19th century mathematician H. Grassmann had the clear vision, that his "extension theory (now developed to geometric calculus) … forms the keystone of the entire structure of mathematics."[6] The algebraic "grammar" of this universal form of calculus is *geometric algebra* (or Clifford algebra). That geometric calculus is a truly unifying approach to all of calculus will be demonstrated here by developing the *vector differential calculus* part of geometric calculus.

The basic geometric algebra necessary for this is compiled in section 2. Then section 3 develops vector differential calculus with the help of few simple definitions. This approach is generically coordinate free, and fully shows both the concrete and abstract geometric and algebraic beauty of the "keystone" of mathematics.

The underlying strategy of this paper is to demonstrate the proofs for all common formulas of vector differential calculus in an elementary step by step fashion. Thus enabling the interested reader to ultimately use this article as reference material, where other texts (e.g. [1],[2]) tend both to skip "elementary steps", and to presume, that the reader would be smart enough to fill in the gaps himself. I put the emphasis therefore on thorough proofs and not on comments, interpretations or application.

2. Basic Geometric Algebra

This section is a basic summary of important relationships in geometric algebra. For brevity they are stated without proof. This summary mainly serves as a reference section for the *vector differential calculus* to be developed in the following section. Most of the relationships listed here are to be found in the synopsis of geometric algebra and in chapters 1 and 2 of [1], as well as in chapter 1 of [2], together with relevant proofs. Beyond that [1] and [2] follow a much more didactic approach for newcomers to geometric algebra.

$\mathcal{G}(I)$ is the full *geometric algebra* over all vectors in the n-dimensional unit *pseudoscalar* $I = \vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n$. $\mathcal{A}_n \equiv \mathcal{G}^1(I)$ is the n-dimensional vector sub-space of grade-1 elements in $\mathcal{G}(I)$ spanned by $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. For

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vectors $\vec{a}, \vec{b}, \vec{c} \in \mathcal{A}_n \equiv \mathcal{G}^1(I)$ and scalars $\alpha, \beta, \lambda, \tau$; $\mathcal{G}(I)$ has the fundamental properties of

- associativity

$$\vec{a}(\vec{b}\vec{c}) = (\vec{a}\vec{b})\vec{c}, \; \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}, \qquad (1)$$

- commutativity

$$\alpha\vec{a} = \vec{a}\alpha, \quad \vec{a} + \vec{b} = \vec{b} + \vec{a}, \qquad (2)$$

- distributivity

$$\vec{a}(\vec{b} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}, \quad (\vec{b} + \vec{c})\vec{a} = \vec{b}\vec{a} + \vec{c}\vec{a}, \qquad (3)$$

- linearity

$$\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b} = (\vec{a} + \vec{b})\alpha, \qquad (4)$$

- scalar square (vector length $|\vec{a}|$)

$$\vec{a}^2 = \vec{a}\vec{a} = \vec{a} \cdot \vec{a} = |\vec{a}|^2. \qquad (5)$$

The *geometric product* $\vec{a}\vec{b}$ is related to the (scalar) *inner product* $\vec{a} \cdot \vec{b}$ and to the (bivector or 2-vector) *outer product* $\vec{a} \wedge \vec{b}$ by

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}, \qquad (6)$$

with

$$\vec{a} \cdot \vec{b} = \frac{1}{2}(\vec{a}\vec{b} + \vec{b}\vec{a}) = \vec{b} \cdot \vec{a} = \vec{a}\vec{b} - \vec{a} \wedge \vec{b} \overset{(13)}{=} \langle \vec{a}\vec{b} \rangle_0, \quad (7)$$

$$\vec{a} \wedge \vec{b} = \frac{1}{2}(\vec{a}\vec{b} - \vec{b}\vec{a}) = -\vec{b} \wedge \vec{a} = \vec{a}\vec{b} - \vec{a} \cdot \vec{b} \overset{(13)}{=} \langle \vec{a}\vec{b} \rangle_2 \quad (8)$$

The inner and the outer product are both linear and distributive

$$\vec{a} \cdot (\alpha\vec{b} + \beta\vec{c}) = \alpha\vec{a} \cdot \vec{b} + \beta\vec{a} \cdot \vec{c}, \qquad (9)$$

$$\vec{a} \wedge (\alpha\vec{b} + \beta\vec{c}) = \alpha\vec{a} \wedge \vec{b} + \beta\vec{a} \wedge \vec{c}. \qquad (10)$$

A unit vector $\hat{a}$ in the direction of $\vec{a}$ is

$$\hat{a} \equiv \frac{\vec{a}}{|\vec{a}|}, \text{ with } \hat{a}^2 = \hat{a}\hat{a} = 1, \; \vec{a} = \hat{a}|\vec{a}|. \qquad (11)$$

The inverse of a vector is

$$\vec{a}^{-1} = \frac{1}{\vec{a}} \equiv \frac{\vec{a}}{\vec{a}^2} = \frac{\vec{a}}{|\vec{a}|^2} = \frac{\hat{a}}{|\vec{a}|}. \qquad (12)$$

A multivector A can be uniquely decomposed into its homogeneous grade k parts ($\langle \; \rangle_k$ grade k selector):

$$A = \underbrace{\langle A \rangle_0}_{scalar} + \underbrace{\langle A \rangle_1}_{vector} + \underbrace{\langle A \rangle_2}_{bivector} + \ldots + \underbrace{\langle A \rangle_k}_{k-vector} + \ldots + \underbrace{\langle A \rangle_n}_{\substack{pseudo\\scalar}} \quad (13)$$

If A is homogeneous of grade k one often simply writes

$$A = \langle A \rangle_k = A_k. \qquad (14)$$

Grade selection is invariant under scalar multiplication

$$\lambda\langle A \rangle_k = \langle \lambda A \rangle_k. \qquad (15)$$

The consistent definition of inner and outer products of vectors $\vec{a}$ and r-vectors A_r is

$$\vec{a} \cdot A_r \equiv \langle \vec{a}A_r \rangle_{r-1} = \frac{1}{2}(\vec{a}A_r - (-1)^r A_r\vec{a}), \qquad (16)$$

$$\vec{a} \wedge A_r \equiv \langle \vec{a}A_r \rangle_{r+1} = \frac{1}{2}(\vec{a}A_r + (-1)^r A_r\vec{a}) \qquad (17)$$

By linearity the full geometric product of a vector and a multivector A is then

$$\vec{a}A = \vec{a} \cdot A + \vec{a} \wedge A. \qquad (18)$$

This extends to the distributive multiplication with arbitrary multivectors A, B

$$\vec{a}(A + B) = \vec{a}A + \vec{a}B. \qquad (19)$$

The inner and outer products of homogeneous multivectors A_r and B_s are defined ([2], p. 6, (1.21), (1.22)) as

$$A_r \cdot B_s \equiv \langle A_r B_s \rangle_{|r-s|} \text{ for } r, s > 0, \qquad (20)$$

$$A_r \cdot B_s \equiv 0 \text{ for } r = 0 \text{ or } s = 0, \qquad (21)$$

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{r+s}, \qquad (22)$$

$$A_r \wedge \lambda = \lambda \wedge A_r = \lambda A_r \text{ for scalar } \lambda. \qquad (23)$$

The inner (and outer) product is again linear and distributive

$$(\lambda A_r) \cdot B_s = A_r \cdot (\lambda B_s) = \lambda(A_r \cdot B_s) = \lambda A_r \cdot B_s, \quad (24)$$

$$A_r \cdot (B_s + C_t) = A_r \cdot B_s + A_r \cdot C_t, \qquad (25)$$

$$\lambda(B_s + C_t) = \lambda B_s + \lambda C_t. \qquad (26)$$

The *reverse* of a multivector is

$$\tilde{A} = \sum_{k=1}^{n} (-1)^{k(k-1)/2} \langle A \rangle_k. \qquad (27)$$

[2] uses a *dagger* instead of the *tilde*.
Special examples are

$$\tilde{\lambda} = \lambda, \; \tilde{\vec{a}} = \vec{a}, \; (\vec{a} \wedge \vec{b})^{\sim} = \vec{b} \wedge \vec{a} = -\vec{a} \wedge \vec{b}, \ldots \quad (28)$$

The scalar *magnitude* $|A|$ of a multivector A is

$$|A|^2 \equiv \underbrace{\tilde{A} * A}_{\text{scalar product}} \equiv \langle A \rangle_0^{\ 2} + \sum_{r=1}^{n} \langle \tilde{A} \rangle_r \cdot \langle A \rangle_r , \quad (29)$$

where the separate term $\langle A \rangle_0^{\ 2}$ is in particular due to the definition of the inner product in [2], p. 6, (1.21). The magnitude allows to define the inverse for simple k-blade vectors

$$A^{-1} \equiv \frac{\tilde{A}}{|A|^2}, \text{ with } A^{-1}A = AA^{-1} = 1. \quad (30)$$

Alternative ways to express $\vec{a} \in \mathcal{A}_n \equiv \mathcal{G}^1(I)$ are

$$I \wedge \vec{a} = 0 \text{ or } I\vec{a} = I \cdot \vec{a} . \quad (31)$$

The projection of $\vec{a}$ into $\mathcal{A}_n \equiv \mathcal{G}^1(I)$ is

$$P_I(\vec{a}) = P(\vec{a}) \equiv \sum_{k=1}^{n} \vec{a}^k \vec{a}_k \cdot \vec{a} = \sum_{k=1}^{n} \vec{a}_k \vec{a}^k \cdot \vec{a}, \quad (32)$$

where $\vec{a}^k$ is the *reciprocal frame* defined by

$$\vec{a}^k \cdot \vec{a}_j = \delta_j^k = \text{Kronecker delta} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}. \quad (33)$$

A general convention is that inner products $\vec{a} \cdot \vec{b}$ and outer products $\vec{a} \wedge \vec{b}$ have priority over geometric products $\vec{a}\vec{b}$, e.g.

$$\vec{a} \cdot \vec{b}\vec{c} \wedge \vec{d}\vec{e} = (\vec{a} \cdot \vec{b})(\vec{c} \wedge \vec{d})\vec{e} . \quad (34)$$

The projection of a multivector B on a subspace described by a simple m-vector (m-blade) $A_m = \vec{a}_1 \wedge \vec{a}_2 \wedge ... \wedge \vec{a}_m, m \leq n$ is

$$P_A(B) \equiv \underbrace{(B \cdot A) \cdot A^{-1}}_{general} \overset{s \leq m}{=} \underbrace{A^{-1} \cdot (A \cdot B_{(s)})}_{\text{degree dependent}}, \quad (35)$$

$$P_A(\langle B \rangle_0) \equiv \langle B \rangle_0, \quad P_A(\langle B \rangle_n) \equiv \langle B \rangle_n \cdot AA^{-1}, \quad (36)$$

the exceptions for scalars $\langle B \rangle_0$ and pseudoscalars $\langle B \rangle_n$ being again due to the definition of the inner product in [2], p. 6, (1.21). A projection of one factor of an inner product has the effect

$$\vec{a} \cdot P(\vec{b}) = P(\vec{a}) \cdot P(\vec{b}) = P(\vec{a}) \cdot \vec{b}. \quad (37)$$

For a multivector $B \in \mathcal{G}(A_m)$, with $A = A_m$ we have

$$(\vec{a} \wedge B) \cdot A = (\vec{a} \wedge B)A = \vec{a} \cdot (BA) \text{ if } \vec{a} \wedge A = 0 . \quad (38)$$

Reordering rules for products of homogeneous multivector are

$$A_r \cdot B_s = (-1)^{r(s-r)} B_s \cdot A_r \text{ for } r \leq s , \quad (39)$$

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r . \quad (40)$$

Elementary combinations that occur often are

$$\vec{a} \cdot (\vec{b} \wedge \vec{c}) = (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b} = \vec{a} \cdot \vec{b}\vec{c} - \vec{a} \cdot \vec{c}\vec{b}, \quad (41)$$

$$\begin{aligned}&(\vec{a} \wedge \vec{b}) \cdot (\vec{c} \wedge \vec{d}) = \vec{a} \cdot (\vec{b} \cdot (\vec{c} \wedge \vec{d})) = \\ &(\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}),\end{aligned} \quad (42)$$

$$\begin{aligned}&(\vec{a} \wedge \vec{b})^2 = (\vec{a} \wedge \vec{b}) \cdot (\vec{a} \wedge \vec{b}) = (\vec{a} \cdot \vec{b})^2 - \vec{a}^2 \vec{b}^2 = \\ &-(\vec{b} \wedge \vec{a}) \cdot (\vec{a} \wedge \vec{b}) = -|\vec{a} \wedge \vec{b}|^2 ,\end{aligned} \quad (43)$$

and the *Jacobi identity*

$$\vec{a} \cdot (\vec{b} \wedge \vec{c}) + \vec{b} \cdot (\vec{c} \wedge \vec{a}) + \vec{c} \cdot (\vec{a} \wedge \vec{b}) = 0. \quad (44)$$

The *commutator product* of multivectors A,B is

$$A \times B \equiv \frac{1}{2}(AB - BA). \quad (45)$$

One useful identity using it is

$$(\vec{a} \wedge \vec{b}) \times A = \vec{a}\vec{b} \cdot A - A \cdot \vec{a}\vec{b} = \vec{a}\vec{b} \wedge A - A \wedge \vec{a}\vec{b}. \quad (46)$$

The commutator product is to be distinguished from the *cross product,* which is strictly limited to the three-dimensional Euclidean case with unit pseudoscalar I_3 :

$$\vec{a} \times \vec{b} \equiv (\vec{b} \wedge \vec{a})I_3 = -(\vec{a} \wedge \vec{b})I_3. \quad (47)$$

For more on basic geometric algebra I refer to [1], [2] and to section 3 of [3].

3. Vector Differential Calculus

This section shows how to differentiate functions on linear subspaces of the universal geometric algebra $\mathcal{G}$ by vectors. It has wide applications particularly to mechanics and physics in general [1]. Separate concepts of gradient, divergence and curl merge into a single concept of vector derivative, united by the geometric product.

The relationship of differential and derivative is clarified. The *Taylor expansion* (P. 12) is applied to important examples, yielding e.g. the *Legendre polynomials* (P. 36). The *adjoint* (Def. 57) and the *integrability* (P. 42, etc.) of multivector functions are defined and discussed. Throughout this section a number of basic differentials and derivations are performed explicitly illustrating ease and power of the calculus.

Since my emphasis here is on explicit step by step proofs, I refer the reader, who is interested in the philosophy, comments and interpretation to the literature ([1]-[5]).

As for the notation: P. 7 refers to proposition 7 of this section. Def. 13 refers to definition 13 of this section. (6) refers to equation number (6) in the previous section on basic geometric algebra.

Standard definitions of continuity and scalar differentiability apply to multivector-valued functions, because the scalar product determines a unique "distance" $|A-B|$ between two elements $A,B \in \mathcal{G}(I)$.

Definition 1 (*directional derivative*)

$F = F(\vec{x})$ multivector-valued function of a vector variable $\vec{x}$ defined on an *n*-dimensional vector space $\mathcal{A}_n = \mathcal{G}^1(I)$, *I* unit pseudoscalar. $\vec{a} \in \mathcal{A}_n$.

$$\vec{a}\cdot\vec{\partial}F \equiv \frac{dF(\vec{x}+\vec{a}\,\tau)}{d\tau} = \lim_{\tau\to 0}\frac{dF(\vec{x}+\vec{a}\,\tau)-dF(\vec{x})}{\tau}$$

Nomenclature: *derivative* of *F* in the *direction* $\vec{a}$, $\vec{a}$*-derivative* of *F*. ([1] uses $\nabla \equiv \vec{\partial}$, [2] uses $\partial \equiv \vec{\partial}$.)

Proposition 2 (*distributivity w.r.t. vector argument*)

$$(\vec{a}+\vec{b})\cdot\vec{\partial}F = \vec{a}\cdot\vec{\partial}F + \vec{b}\cdot\vec{\partial}F\,,\quad \vec{a},\vec{b} \in \mathcal{A}_n$$

Proof 2

$$(\vec{a}+\vec{b})\cdot\vec{\partial}F \stackrel{def1}{=} \lim_{\tau\to 0}\frac{F(\vec{x}+\tau\vec{b}+\tau\vec{a})-F(\vec{x})}{\tau} =$$

$$\lim_{\tau\to 0}\left\{\frac{F(\vec{x}+\tau\vec{b}+\tau\vec{a})-F(\vec{x}+\tau\vec{b})}{\tau}\right.$$

$$\left.+\frac{F(\vec{x}+\tau\vec{b})-F(\vec{x})}{\tau}\right\}$$

$$\stackrel{def1}{=} \lim_{\tau\to 0}\vec{a}\cdot\vec{\partial}F(\vec{x}+\tau\vec{b}) + \vec{b}\cdot\vec{\partial}F = \vec{a}\cdot\vec{\partial}F + \vec{b}\cdot\vec{\partial}F$$

Proposition 3

For scalar λ

$$(\lambda\vec{a})\cdot\vec{\partial}F = \lambda(\vec{a}\cdot\vec{\partial}F)$$

Proof 3

$$(\lambda\vec{a})\cdot\vec{\partial}F \stackrel{def1}{=} \lim_{\tau\to 0}\frac{F(\vec{x}+\lambda\vec{a}\,\tau)-F(\vec{x})}{\tau}$$

case 1: $\lambda \neq 0$

$$(\lambda\vec{a})\cdot\vec{\partial}F \stackrel{def1}{=} \lim_{\tau\to 0}\lambda\frac{F(\vec{x}+\vec{a}(\lambda\tau))-F(\vec{x})}{\lambda\tau}$$

$$\underset{rp}{\stackrel{\lambda\neq 0}{=}} \lambda\lim_{\tau'\to 0}\frac{F(\vec{x}+\vec{a}\,\tau')-F(\vec{x})}{\tau'} \stackrel{def1}{=} \lambda(\vec{a}\cdot\vec{\partial}F)$$

(rp: reparametrization: $\tau \to \tau' = \lambda\tau$)

case 2: $\lambda = 0$

$$(\lambda\vec{a})\cdot\vec{\partial}F \stackrel{def1}{=} \lim_{\tau\to 0}\frac{F(\vec{x})-F(\vec{x})}{\tau} = 0 = 0(\vec{a}\cdot\vec{\partial}F)\,.$$

Proposition 4 (*distributivity w.r.t. multivector-valued function*)

$$\vec{a}\cdot\vec{\partial}(F+G) = \vec{a}\cdot\vec{\partial}F + \vec{a}\cdot\vec{\partial}G$$

$F = F(\vec{x}), G = G(\vec{x})$ multivector-valued functions of a vector variable $\vec{x}$. In the notation of Def. 13:

$$\underline{F+G} = \underline{F} + \underline{G}\,.$$

Proof 4

$$\vec{a}\cdot\vec{\partial}(F+G) \stackrel{def1}{=}$$

$$\lim_{\tau\to 0}\frac{F(\vec{x}+\vec{a}\,\tau)+G(\vec{x}+\vec{a}\,\tau)-F(\vec{x})-G(\vec{x})}{\tau} =$$

$$\lim_{\tau\to 0}\frac{F(\vec{x}+\vec{a}\,\tau)-F(\vec{x})}{\tau} +$$

$$\lim_{\tau\to 0}\frac{G(\vec{x}+\vec{a}\,\tau)-G(\vec{x})}{\tau} \stackrel{def1}{=} \vec{a}\cdot\vec{\partial}F + \vec{a}\cdot\vec{\partial}G.$$

Proposition 5 (*product rule*)

$$\vec{a}\cdot\vec{\partial}(FG) = (\vec{a}\cdot\vec{\partial}F)G + F(\vec{a}\cdot\vec{\partial}G)$$

In the notation of Def. 13:

$$\underline{FG} = \underline{F}G + F\underline{G}\,.$$

Proof 5

$$\vec{a}\cdot\vec{\partial}(FG) \stackrel{def1}{=}$$

$$\lim_{\tau\to 0}\frac{F(\vec{x}+\vec{a}\,\tau)G(\vec{x}+\vec{a}\,\tau)-F(\vec{x})G(\vec{x})}{\tau}$$

$$= \lim_{\tau\to 0}\left\{\frac{F(\vec{x}+\vec{a}\,\tau)G(\vec{x}+\vec{a}\,\tau)-F(\vec{x})G(\vec{x}+\vec{a}\,\tau)+}{\tau}\right.$$

$$+F(\vec{x})G(\vec{x}+\vec{a}\tau)-F(\vec{x})G(\vec{x})\Big\}$$

$$=\lim_{\tau\to 0}\left\{\frac{F(\vec{x}+\vec{a}\tau)-F(\vec{x})}{\tau}G(\vec{x}+\vec{a}\tau)\right.$$

$$\left.+F(\vec{x})\frac{G(\vec{x}+\vec{a}\tau)-G(\vec{x})}{\tau}\right\}$$

$$\overset{def1}{=}(\vec{a}\cdot\vec{\partial}F)\lim_{\tau\to 0}G(\vec{x}+\vec{a}\tau)+F(\vec{a}\cdot\vec{\partial}G)$$

$$=(\vec{a}\cdot\vec{\partial}F)G+F(\vec{a}\cdot\vec{\partial}G)$$

Proposition 6 (*grade invariance*)

$$\vec{a}\cdot\vec{\partial}\left\langle F\right\rangle_k=\left\langle\vec{a}\cdot\vec{\partial}F\right\rangle_k$$

$\vec{a}\cdot\vec{\partial}$ is therefore said to be a *scalar differential operator.*

Proof 6

$$\vec{a}\cdot\vec{\partial}\left\langle F\right\rangle_k\overset{def1}{=}\lim_{\tau\to 0}\frac{\left\langle F(\vec{x}+\vec{a}\tau)\right\rangle_k-\left\langle F(\vec{x})\right\rangle_k}{\tau}$$

$$\overset{\tau=scalar}{=}\lim_{\tau\to 0}\left\langle\frac{F(\vec{x}+\vec{a}\tau)-F(\vec{x})}{\tau}\right\rangle_k$$

$$\overset{def1}{=}\left\langle\vec{a}\cdot\vec{\partial}F\right\rangle_k.$$

Proposition 7 (*scalar chain rule*)

$$\vec{a}\cdot\vec{\partial}F=(\vec{a}\cdot\vec{\partial}\lambda)\frac{dF}{d\lambda}$$

$F=F(\lambda(\vec{x})),\ \ \lambda=\lambda(\vec{x})$ scalar valued function.

Proof 7

Using the Taylor expansions:

$$F(\lambda+\tau\Delta\lambda)=F(\lambda)+\tau\Delta\lambda\frac{dF}{d\lambda}+\frac{\tau^2(\Delta\lambda)^2}{2}\frac{d^2F}{d\lambda^2}+$$

$\ldots$

$$\lambda(\vec{x}+\tau\vec{a})\overset{P12}{=}\lambda(\vec{x})+\tau\vec{a}\cdot\vec{\partial}\lambda(\vec{x})+\frac{\tau}{2}(\vec{a}\cdot\vec{\partial})^2\lambda(\vec{x})+\ldots$$

we have

$$\vec{a}\cdot\vec{\partial}F(\lambda(\vec{x}))\overset{def1}{=}\lim_{\tau\to 0}\frac{F(\lambda(\vec{x}+\tau\vec{a}))-F(\lambda(\vec{x}))}{\tau}\overset{P12}{=}$$

$$\lim_{\tau\to 0}\frac{F(\lambda(\vec{x})+\tau\vec{a}\cdot\vec{\partial}\lambda(\vec{x}))-F(\lambda(\vec{x}))}{\tau}\overset{Taylor}{=}$$

$$\lim_{\tau\to 0}\frac{F(\lambda(\vec{x}))+\tau\left\{\vec{a}\cdot\vec{\partial}\lambda(\vec{x})\right\}\frac{dF}{d\lambda}-F(\lambda(\vec{x}))}{\tau}=$$

$$=(\vec{a}\cdot\vec{\partial}\lambda)\frac{dF}{d\lambda}.$$

Proposition 8 (*identity*)

$$\vec{a}\cdot\vec{\partial}\vec{x}=\vec{a}\,.$$

Proof 8

$$F(\vec{x})=\vec{x}\,,\ \ \vec{a}\cdot\vec{\partial}\vec{x}\overset{def1}{=}\lim_{\tau\to 0}\frac{\vec{x}+\tau\vec{a}-\vec{x}}{\tau}=\lim_{\tau\to 0}\vec{a}=\vec{a}.$$

Proposition 9 (*constant function*)

A independent of $\vec{x}$:

$$\vec{a}\cdot\vec{\partial}A=0\,.$$

Proof 9

$$F(\vec{x})=A,\ \ \vec{a}\cdot\vec{\partial}A\overset{def1}{=}\lim_{\tau\to 0}\frac{A-A}{\tau}=0.$$

Proposition 10 (*vector length*)

$$\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|=\frac{\vec{a}\cdot\vec{x}}{\left|\vec{x}\right|}=\vec{a}\cdot\hat{x}\,.$$

$\hat{x}\equiv\dfrac{\vec{x}}{\left|\vec{x}\right|}$ unit vector in the direction of $\vec{x}$ (11).

Proof 10

$$\vec{a}\cdot\vec{\partial}\vec{x}^2\overset{P5}{=}(\vec{a}\cdot\vec{\partial}\vec{x})\vec{x}+\vec{x}(\vec{a}\cdot\vec{\partial}\vec{x})\overset{P8}{=}\vec{a}\vec{x}+\vec{x}\vec{a}=2\vec{a}\cdot\vec{x},$$

$$\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|^2\overset{P5}{=}(\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|)\left|\vec{x}\right|+\left|\vec{x}\right|(\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|)=2\left|\vec{x}\right|(\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|),$$

$$\vec{x}^2=\left|\vec{x}\right|^2\Rightarrow\vec{a}\cdot\vec{\partial}\vec{x}^2=\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|^2$$

$$\Rightarrow\vec{a}\cdot\vec{x}=\left|\vec{x}\right|(\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|)\Rightarrow\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|=\frac{\vec{a}\cdot\vec{x}}{\left|\vec{x}\right|}=\vec{a}\cdot\hat{x}.$$

Proposition 11 (*direction function*)

$$\vec{a}\cdot\vec{\partial}\hat{x}=\frac{\vec{a}-\vec{a}\cdot\hat{x}\hat{x}}{\left|\vec{x}\right|}=\frac{\hat{x}\hat{x}\wedge\vec{a}}{\left|\vec{x}\right|}.$$

Proof 11

$$\vec{a}\cdot\vec{\partial}\hat{x}=\vec{a}\cdot\vec{\partial}\frac{\vec{x}}{\left|\vec{x}\right|}\overset{P5}{=}\frac{\vec{a}\cdot\vec{\partial}\vec{x}}{\left|\vec{x}\right|}+\vec{x}\vec{a}\cdot\vec{\partial}\frac{1}{\left|\vec{x}\right|}\overset{P7,8}{=}$$

$$\frac{\vec{a}}{\left|\vec{x}\right|}+\vec{x}(\vec{a}\cdot\vec{\partial}\left|\vec{x}\right|)\frac{d\frac{1}{\left|\vec{x}\right|}}{d\left|\vec{x}\right|}\overset{P10}{=}\frac{\vec{a}}{\left|\vec{x}\right|}+\vec{x}(\vec{a}\cdot\hat{x})\frac{-1}{\left|\vec{x}\right|^2}=$$

$$\frac{\vec{a}}{\left|\vec{x}\right|}-\hat{x}(\vec{a}\cdot\hat{x})\frac{1}{\left|\vec{x}\right|}=\frac{\vec{a}-\vec{a}\cdot\hat{x}\hat{x}}{\left|\vec{x}\right|}=\frac{\hat{x}\hat{x}\vec{a}-\hat{x}\hat{x}\cdot\vec{a}}{\left|\vec{x}\right|}=$$

$$= \frac{\hat{x}(\hat{x}\cdot\vec{a}+\hat{x}\wedge\vec{a})-\hat{x}\hat{x}\cdot\vec{a}}{|\vec{x}|} = \frac{\hat{x}\hat{x}\wedge\vec{a}}{|\vec{x}|}.$$

Proposition 12 (*Taylor expansion*)

$$F(\vec{x}+\vec{a}) = \exp(\vec{a}\cdot\vec{\partial})F(\vec{x}) = \sum_{k=0}^{\infty}\frac{(\vec{a}\cdot\vec{\partial})^k}{k!}F(\vec{x}).$$

Proof 12

This proof is done without referring to P7 to P11!

$$G(\tau) \equiv F(\vec{x}+\vec{a}\tau)$$

$$\Rightarrow \frac{dG(0)}{d\tau} = \left.\frac{dF(\vec{x}+\vec{a}\tau)}{d\tau}\right|_{\tau=0} \overset{def1}{=} \vec{a}\cdot\vec{\partial}F$$

$$\Rightarrow \frac{d^2G(0)}{d\tau^2} = \frac{d}{d\tau}\left.\frac{dF(\vec{x}+\vec{a}\tau)}{d\tau}\right|_{\tau=0} \overset{def1}{=}$$

$$\vec{a}\cdot\vec{\partial}\left.\frac{dF(\vec{x}+\vec{a}\tau)}{d\tau}\right|_{\tau=0} \overset{def1}{=} \vec{a}\cdot\vec{\partial}\left(\vec{a}\cdot\vec{\partial}F(\vec{x})\right) =$$

$$(\vec{a}\cdot\vec{\partial})^2 F(\vec{x}).$$

General: $\dfrac{d^kG(0)}{d\tau^k} = (\vec{a}\cdot\vec{\partial})^k F(\vec{x})$.

The Taylor series for *G* is:

$$G(1) = G(0+1) = G(0) + \frac{dG(0)}{d\tau} + \frac{1}{2}\frac{d^2G(0)}{d\tau^2} + \ldots$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^kG(0)}{d\tau^k}$$

$$\Rightarrow G(1) = F(\vec{x}+\vec{a}) =$$

$$\sum_{k=0}^{\infty}\frac{1}{k!}(\vec{a}\cdot\vec{\partial})^k F(\vec{x}) = \exp(\vec{a}\cdot\vec{\partial})F(\vec{x}).$$

Definition 13 (*continuously differentiable, differential*)

F is *continuously differentiable* at $\vec{x}$ if for each fixed $\vec{a}$ $\vec{a}\cdot\vec{\partial}F(\vec{y})$ exists and is a continuous function of $\vec{y}$ for each $\vec{y}$ in a neighborhood of $\vec{x}$.

If *F* is defined and continuously differentiable at $\vec{x}$, then, for fixed $\vec{x}$, $\vec{a}\cdot\vec{\partial}F(\vec{x})$ is a linear function of $\vec{a}$, the (*first*) *differential* of *F*.

$$\underline{F}(\vec{a},\vec{x}) = F_{\vec{a}}(\vec{x}) \equiv \vec{a}\cdot\vec{\partial}F(\vec{x}).$$

([1], p. 107 uses $F' \equiv \underline{F}$.)

Suppressing $\vec{x}$, or for fixed $\vec{x}$:

$$\underline{F} = \underline{F}(\vec{a}) = F_{\vec{a}} \equiv \vec{a}\cdot\vec{\partial}F.$$

Proposition 14 (*linearity*)

$$\underline{F}(\vec{a}+\vec{b}) = \underline{F}(\vec{a}) + \underline{F}(\vec{b})$$

λ scalar: $\underline{F}(\lambda\vec{a}) = \lambda\underline{F}(\vec{a})$

Proof 14

Propositions 2 and 3.

Proposition 15 (*linear approximation*)

For $|\vec{r}| = |\vec{x}-\vec{x}_0|$ sufficiently small:

$$F(\vec{x}) - F(\vec{x}_0) \approx \underline{F}(\vec{x}-\vec{x}_0) = \underline{F}(\vec{x}) - \underline{F}(\vec{x}_0).$$

Proof 15

$$F(\vec{x}) = F(\vec{x}+\vec{r}) \overset{P12}{=}$$

$$F(\vec{x}_0) + \vec{r}\cdot\vec{\partial}F(\vec{x}_0) + \frac{1}{2}(\vec{r}\cdot\vec{\partial})^2F(\vec{x}_0) + \ldots =$$

$$F(\vec{x}_0) + |\vec{r}|\hat{r}\cdot\vec{\partial}F(\vec{x}_0) + \frac{|\vec{r}|^2}{2}(\hat{r}\cdot\vec{\partial})^2F(\vec{x}_0) + \ldots$$

$$+\frac{|\vec{r}|^k}{2}(\hat{r}\cdot\vec{\partial})^kF(\vec{x}_0) + \ldots,$$

with $\hat{r} \equiv \dfrac{\vec{r}}{|\vec{r}|}$. For sufficiently small $|\vec{r}|$:

$$F(\vec{x}) = F(\vec{x}_0) + \vec{r}\cdot\vec{\partial}F(\vec{x}_0) =$$

$$F(\vec{x}_0) + (\vec{x}-\vec{x}_0)\cdot\vec{\partial}F(\vec{x}_0) \overset{def13}{=}$$

$$F(\vec{x}_0) + \underline{F}(\vec{x}-\vec{x}_0,\vec{x}_0) \overset{def13}{=} F(\vec{x}_0) + \underline{F}(\vec{x}-\vec{x}_0) \overset{P14}{=}$$

$$F(\vec{x}_0) + \underline{F}(\vec{x}) - \underline{F}(\vec{x}_0)$$

$$\Rightarrow F(\vec{x}) - F(\vec{x}_0) \approx \underline{F}(\vec{x}-\vec{x}_0) = \underline{F}(\vec{x}) - \underline{F}(\vec{x}_0).$$

Proposition 16 (*chain rule*)

$$\frac{dF}{dt}\left(\vec{x}(t)\right) = \left(\frac{d}{dt}\vec{x}(t)\right)\cdot\vec{\partial}F(\vec{x})\Big|_{\vec{x}=\vec{x}(t)}$$

Proof 16

Using the Taylor expansion

$$\vec{x}(t+\tau) = \vec{x}(t) + \tau\frac{d}{dt}\vec{x}(t) + \frac{\tau^2}{2}\frac{d^2}{dt^2}\vec{x}(t) + \ldots,$$

$$\frac{dF}{dt}\left(\vec{x}(t)\right) = \lim_{\tau\to 0}\frac{F(\vec{x}(t+\tau)) - F(\vec{x}(t))}{\tau} \overset{Taylor}{=}$$

$$\lim_{\tau\to 0}\frac{F(\vec{x}(t) + \tau\frac{d}{dt}\vec{x}(t)) - F(\vec{x}(t))}{\tau} \overset{def1}{=}$$

$$= \left(\frac{d}{dt} \vec{x}(t) \right) \cdot \vec{\partial} F(\vec{x}) \Big|_{\vec{x}=\vec{x}(t)} .$$

Definition 17 (*vector derivative*)

Differentiation of F by its argument $\vec{x}$

$$\vec{\partial}_{\vec{x}} F(\vec{x}) = \vec{\partial} F ,$$

with the differential operator $\vec{\partial}_{\vec{x}}$, assumed to

(i) have the algebraic properties of a vector in $\mathcal{A}_n \equiv \mathcal{G}^1(I)$, I unit pseudoscalar; and

(ii) that $\vec{a} \cdot \vec{\partial}_{\vec{x}}$ with $\vec{a} \in \mathcal{A}_n$ is $\vec{a} \cdot \vec{\partial}_{\vec{x}} F$ as in Def. 1.

Proposition 18 (*algebraic properties of* $\vec{\partial}_{\vec{x}}$)

$$I \wedge \vec{\partial}_{\vec{x}} \overset{(31)}{=} 0$$

$$I \vec{\partial}_{\vec{x}} \overset{(31)}{=} I \cdot \vec{\partial}_{\vec{x}}$$

$$\vec{\partial}_{\vec{x}} = P_I(\vec{\partial}_{\vec{x}}) \overset{(32)}{=} \sum_{k=1}^{n} \vec{a}^k \vec{a}_k \cdot \vec{\partial}_{\vec{x}} ,$$

where the $\vec{a}^k$ express the algebraic vector properties and the $\vec{a}_k \cdot \vec{\partial}_{\vec{x}}$ the scalar differential properties.

Definition 19 (*gradient*)

The vector field $\vec{f} = \vec{f}(\vec{x}) \equiv \vec{\partial}_{\vec{x}} \Phi(\vec{x}) = \vec{\partial}\Phi$ for a scalar function $\Phi = \Phi(\vec{x})$ is called the gradient of Φ.

Propostion 20 (*3-dimensional cross product*)

For $\vec{b}$ independent of $\vec{x} \in \mathcal{A}_3 \equiv \mathcal{G}^1(I_3)$:

$$\vec{a} \cdot \vec{\partial}(\vec{x} \times \vec{b}) = \vec{a} \times \vec{b} .$$

Only here $\times$ means the 3-dimensional cross product (47), not the commutator product in P. 81.

Proof 20

$$\vec{a} \cdot \vec{\partial}(\vec{x} \times \vec{b}) \overset{(47)}{=} -\vec{a} \cdot \vec{\partial}\left((\vec{x} \wedge \vec{b}) I_3\right) \overset{P5}{=}$$
$$\left[-\vec{a} \cdot \vec{\partial}(\vec{x} \wedge \vec{b})\right] I_3 - (\vec{x} \wedge \vec{b}) \vec{a} \cdot \vec{\partial} I_3 \underset{P9}{\overset{I_3 = const}{=}}$$
$$\left[-\vec{a} \cdot \vec{\partial}(\vec{x} \wedge \vec{b})\right] I_3 \overset{(8)}{=} \left[-\vec{a} \cdot \vec{\partial} \left\langle \vec{x}\vec{b} \right\rangle_2 \right] I_3 \overset{P6}{=}$$
$$-\left\langle \vec{a} \cdot \vec{\partial}\vec{x}\vec{b} \right\rangle_2 I_3 \overset{P5}{=} -\left\langle (\vec{a} \cdot \vec{\partial}\vec{x})\vec{b} \right\rangle_2 I_3 - \left\langle \vec{x}(\vec{a} \cdot \vec{\partial}\vec{b}) \right\rangle_2 I_3$$
$$\overset{P8,9}{=} -\left\langle \vec{a}\vec{b} \right\rangle_2 I_3 \overset{(8)}{=} -(\vec{a} \wedge \vec{b}) I_3 \overset{(47)}{=} \vec{a} \times \vec{b}.$$

Proposition 21

$$\vec{a} \cdot \vec{\partial}\left(\vec{x} \cdot \langle A \rangle_r\right) = \vec{a} \cdot \langle A \rangle_r ,$$

A independent of $\vec{x}$.

Proof 21

$$\vec{a} \cdot \vec{\partial}\left(\vec{x} \cdot \langle A \rangle_r\right) \overset{(16)}{=} \vec{a} \cdot \vec{\partial} \left\langle \vec{x} \langle A \rangle_r \right\rangle_{r-1} \overset{P6}{=}$$
$$\left\langle \vec{a} \cdot \vec{\partial}\left(\vec{x} \langle A \rangle_r\right) \right\rangle_{r-1} \overset{P5}{=} \left\langle (\vec{a} \cdot \vec{\partial}\vec{x}) \langle A \rangle_r + \vec{x}\vec{a} \cdot \vec{\partial} \langle A \rangle_r \right\rangle_{r-1} \overset{P8,9}{=}$$
$$\left\langle \vec{a} \langle A \rangle_r \right\rangle_{r-1} \overset{(16)}{=} \vec{a} \cdot \langle A \rangle_r .$$

Proposition 22

$$\vec{a} \cdot \vec{\partial}[\vec{x} \cdot (\vec{x} \wedge \vec{b})] = \vec{a} \cdot (\vec{x} \wedge \vec{b}) + \vec{x} \cdot (\vec{a} \wedge \vec{b})$$

Proof 22

$$\vec{a} \cdot \vec{\partial}\left[\vec{x} \cdot (\vec{x} \wedge \vec{b})\right] \overset{(41)}{=} \vec{a} \cdot \vec{\partial}\left[\vec{x}^2 \vec{b} + \vec{x} \cdot \vec{b}\vec{x}\right] \overset{P4,5}{=}$$
$$\left(\vec{a} \cdot \vec{\partial}\vec{x}^2\right)\vec{b} + \vec{x}^2 \vec{a} \cdot \vec{\partial}\vec{b} + \vec{a} \cdot \vec{\partial}(\vec{x} \cdot \vec{b})\vec{x} + \vec{x} \cdot \vec{b}\left(\vec{a} \cdot \vec{\partial}\vec{x}\right)$$
$$\underset{P9,21,8}{\overset{Proof\,10}{=}} \vec{a} \cdot \vec{x}\vec{b} - \vec{a} \cdot \vec{b}\vec{x} + \vec{x} \cdot \vec{a}\vec{b} + \vec{x} \cdot \vec{b}\vec{a} \overset{(41)}{=}$$
$$\vec{a} \cdot (\vec{x} \wedge \vec{b}) + \vec{x} \cdot (\vec{a} \wedge \vec{b}).$$

Proposition 23

For $\vec{x}'$ independent of $\vec{x}$ and $r \equiv |\vec{r}| = |\vec{x} - \vec{x}'|$:

$$\vec{a} \cdot \vec{\partial} r = \vec{a} \cdot \frac{\vec{r}}{r} = \vec{a} \cdot \hat{r} ,$$

where $\hat{r} = \frac{\vec{r}}{r}$.

Proof 23

Compare [1], p. 681. $r^2 = (\vec{x} - \vec{x}')(\vec{x} - \vec{x}')$, then

$$\vec{a} \cdot \vec{\partial} r^2 \overset{P5}{=}$$
$$\left[\vec{a} \cdot \vec{\partial}(\vec{x} - \vec{x}')\right](\vec{x} - \vec{x}') + (\vec{x} - \vec{x}')\left[\vec{a} \cdot \vec{\partial}(\vec{x} - \vec{x}')\right]$$
$$\overset{P4,9}{=} \vec{a}(\vec{x} - \vec{x}') + (\vec{x} - \vec{x}')\vec{a} = 2\vec{a} \cdot \vec{r} ,$$

$$\vec{a} \cdot \vec{\partial} r^2 \overset{P5}{=} 2r(\vec{a} \cdot \vec{\partial} r) \Rightarrow 2\vec{a} \cdot \vec{r} = 2r(\vec{a} \cdot \vec{\partial} r)$$
$$\Rightarrow \vec{a} \cdot \vec{\partial} r = \vec{a} \cdot \frac{\vec{r}}{r} \overset{(11)}{=} \vec{a} \cdot \hat{r}.$$

Proposition 24

$$\vec a\cdot\vec\partial\hat r=\frac{\hat r\hat r\wedge\vec a}{r}.$$

Proof 24

Compare [1], p. 681.

$$\vec a\cdot\vec\partial\hat r=\vec a\cdot\frac{\vec r}{r}\overset{P5}{=}\frac{1}{r}\vec a\cdot\vec\partial\vec r+\vec r\vec a\cdot\vec\partial\frac{1}{r}\overset{P7}{=}$$

$$\frac{1}{r}\vec a\cdot\vec\partial(\vec x-\vec x')-\vec r\frac{1}{r^2}\vec a\cdot\vec\partial r\overset{P8,23}{=}\frac{1}{r}\vec a-\frac{\hat r}{r}\vec a\cdot\hat r\overset{P8}{=}$$

$$\frac{\hat r\hat r\vec a-\hat r(\hat r\cdot\vec a)}{r}\overset{(8)}{=}\frac{\hat r\hat r\wedge\vec a}{r}.$$

Proposition 25

$$\vec a\cdot\vec\partial(\hat r\cdot\vec a)=\frac{\left|\hat r\wedge\vec a\right|^2}{r}.$$

Proof 25

$$\vec a\cdot\vec\partial(\hat r\cdot\vec a)=\vec a\cdot\vec\partial\left(\frac{\vec r\cdot\vec a}{r}\right)\overset{P5}{=}$$

$$\frac{1}{r}\vec a\cdot\vec\partial(\vec r\cdot\vec a)+\vec r\cdot\vec a(\vec a\cdot\vec\partial)\frac{1}{r}\overset{P7,23}{=}$$

$$\frac{1}{r}\vec a\cdot\vec\partial(\vec x\cdot\vec a-\vec x'\cdot\vec a)-\frac{\vec r\cdot\vec a}{r^2}\vec a\cdot\hat r\overset{P21}{=}$$

$$\frac{1}{r}\vec a\cdot\vec a-\frac{(\hat r\cdot\vec a)^2}{r}\overset{(11)}{=}\frac{\hat r^2\vec a^2-(\hat r\cdot\vec a)^2}{r}\overset{(43)}{=}$$

$$\frac{(\hat r\wedge\vec a)\cdot(\vec a\wedge\hat r)}{r}\overset{(43)}{=}\frac{\left|\hat r\wedge\vec a\right|^2}{r}.$$

Proposition 26

$$\vec a\cdot\vec\partial(\hat r\wedge\vec a)=\frac{\hat r\cdot\vec a\vec a\wedge\hat r}{r}.$$

Proof 26

$$\vec a\cdot\vec\partial(\hat r\wedge\vec a)\overset{(8)}{=}\vec a\cdot\vec\partial(\hat r\vec a-\hat r\cdot\vec a)\overset{P4}{=}$$

$$\vec a\cdot\vec\partial(\hat r\vec a)-\vec a\cdot\vec\partial(\hat r\cdot\vec a)\overset{P5,25}{=}$$

$$\vec a\cdot\vec\partial(\hat r)\vec a+\hat r\vec a\cdot\vec\partial\vec a-\frac{\left|\hat r\wedge\vec a\right|^2}{r}\underset{(43)}{\overset{P9,24}{=}}$$

$$\frac{\hat r\hat r\wedge\vec a}{r}\vec a-\frac{(\vec a\wedge\hat r)(\hat r\wedge\vec a)}{r}\underset{(40)}{\overset{(39)}{=}}$$

$$\frac{\vec a\wedge\hat r\hat r\vec a-(\vec a\wedge\hat r)(\hat r\wedge\vec a)}{r}\overset{(3)}{=}\frac{\vec a\wedge\hat r\big(\hat r\vec a-(\hat r\wedge\vec a)\big)}{r}$$

$$\overset{(7)}{=}\frac{\vec a\wedge\hat r\hat r\cdot\vec a}{r}=\frac{\hat r\cdot\vec a\vec a\wedge\hat r}{r}.$$

Proposition 27

$$\vec a\cdot\vec\partial\left|\hat r\wedge\vec a\right|=-\frac{\hat r\cdot\vec a\left|\hat r\wedge\vec a\right|}{r}.$$

Proof 27

$$\vec a\cdot\vec\partial\left|\hat r\wedge\vec a\right|^2\overset{P5}{=}2\left|\hat r\wedge\vec a\right|\vec a\cdot\vec\partial\left|\hat r\wedge\vec a\right|,$$

$$\vec a\cdot\vec\partial\left|\hat r\wedge\vec a\right|^2\overset{(43)}{=}\vec a\cdot\vec\partial\big[(\hat r\wedge\vec a)\cdot(\vec a\wedge\hat r)\big]\overset{(43)}{=}$$

$$\vec a\cdot\vec\partial\big[\hat r^2\vec a^2-(\hat r\cdot\vec a)^2\big]\overset{P5}{=}\vec a\cdot\vec\partial\vec a^2-2(\hat r\cdot\vec a)\vec a\cdot\vec\partial(\hat r\cdot\vec a)$$

$$\overset{P9,25}{=}-2(\hat r\cdot\vec a)\frac{\left|\hat r\wedge\vec a\right|^2}{r}$$

$$\Rightarrow 2\left|\hat r\wedge\vec a\right|\vec a\cdot\vec\partial\left|\hat r\wedge\vec a\right|=-2\frac{(\hat r\cdot\vec a)\left|\hat r\wedge\vec a\right|^2}{r}$$

$$\Rightarrow \vec a\cdot\vec\partial\left|\hat r\wedge\vec a\right|=-\frac{(\hat r\cdot\vec a)\left|\hat r\wedge\vec a\right|}{r}.$$

Proposition 28

$$\vec a\cdot\vec\partial\frac{1}{\vec r}=-\frac{1}{\vec r}\vec a\frac{1}{\vec r}.$$

Proof 28

$$\vec a\cdot\vec\partial\frac{1}{\vec r}\overset{(12)}{=}\vec a\cdot\vec\partial\frac{\vec r}{r^2}\overset{P5}{=}\frac{1}{r^2}\vec a\cdot\vec\partial\vec r+\vec r\vec a\cdot\vec\partial\frac{1}{r^2}\overset{P7}{=}$$

$$\frac{1}{r^2}\vec a+\vec r\left(-\frac{2}{r^3}\right)\vec a\cdot\vec\partial r\overset{P23}{=}\frac{1}{r^2}\vec a-2\frac{\vec r}{r^3}\vec a\cdot\hat r\overset{(12),(11)}{=}$$

$$\frac{1}{\vec r}\frac{1}{\vec r}\vec a-2\frac{1}{\vec r}\vec a\cdot\frac{1}{\vec r}\overset{(7)}{=}\frac{1}{\vec r}\frac{1}{\vec r}\vec a-\frac{1}{\vec r}\vec a\frac{1}{\vec r}-\frac{1}{\vec r}\frac{1}{\vec r}\vec a=$$

$$-\frac{1}{\vec r}\vec a\frac{1}{\vec r}.$$

Proposition 29

$$\vec a\cdot\vec\partial\frac{1}{r^2}=-2\frac{\vec a\cdot\hat r}{r^3}.$$

Proof 29

$$\vec a\cdot\vec\partial\frac{1}{r^2}\overset{P7}{=}-\frac{2}{r^3}\vec a\cdot\vec\partial r\overset{P23}{=}-\frac{2}{r^3}\vec a\cdot\vec\partial\hat r.$$

Proposition 30

$$\frac{1}{2}(\vec a\cdot\vec\partial)^2\frac{1}{r^2}=\frac{3(\vec a\cdot\hat r)^2-\left|\hat r\wedge\vec a\right|^2}{r^4}.$$

Proof 30

$$\frac{1}{2}(\vec a\cdot\vec\partial)^2\frac{1}{r^2}\overset{P29}{=}\frac{1}{2}\vec a\cdot\vec\partial\left(-2\frac{\vec a\cdot\hat r}{r^3}\right)\overset{P5}{=}$$

$$-\left(\vec a\cdot\vec\partial\frac{1}{r^3}\right)\vec a\cdot\hat r-\frac{1}{r^3}\vec a\cdot\vec\partial(\vec a\cdot\hat r)\overset{P7,23,25}{=}$$

$$\frac{3\vec a\cdot\hat r}{r^4}\vec a\cdot\hat r-\frac{1}{r^3}\frac{\left|\hat r\wedge\vec a\right|^2}{r}=\frac{3(\vec a\cdot\hat r)^2-\left|\hat r\wedge\vec a\right|^2}{r^4}.$$

Proposition 31

$$\frac{1}{6}(\vec a\cdot\vec\partial)^3\frac{1}{r^2}=\frac{-4(\vec a\cdot\hat r)^3+4\left|\hat r\wedge\vec a\right|^2\vec a\cdot\hat r}{r^5}.$$

Proof 31

$$\frac{1}{6}(\vec a\cdot\vec\partial)^3\frac{1}{r^2}=\frac{1}{3}(\vec a\cdot\vec\partial)\frac{1}{2}(\vec a\cdot\vec\partial)^2\frac{1}{r^2}\overset{P30}{=}$$

$$\frac{1}{3}(\vec a\cdot\vec\partial)\frac{3(\vec a\cdot\hat r)^2-\left|\hat r\wedge\vec a\right|^2}{r^4}\overset{P5,7}{=}$$

$$\frac{1}{3}\frac{3\ 2(\vec a\cdot\hat r)\vec a\cdot\vec\partial(\vec a\cdot\hat r)-2\left|\hat r\wedge\vec a\right|\vec a\cdot\vec\partial\left|\hat r\wedge\vec a\right|}{r^4}+$$

$$\frac{1}{3}\left[3(\vec a\cdot\hat r)^2-\left|\hat r\wedge\vec a\right|^2\right](-4)\frac{1}{r^5}\vec a\cdot\vec\partial r\overset{P23,25,27}{=}$$

$$\frac{2(\vec a\cdot\hat r)\dfrac{\left|\hat r\wedge\vec a\right|^2}{r}+\dfrac{2}{3}\left|\hat r\wedge\vec a\right|\hat r\cdot\vec a\dfrac{\left|\hat r\wedge\vec a\right|}{r}}{r^4}-$$

$$\frac{4}{3}\left[3(\vec a\cdot\hat r)^2-\left|\hat r\wedge\vec a\right|^2\right]\frac{1}{r^5}\vec a\cdot\hat r=$$

$$\frac{2(\vec a\cdot\hat r)\left|\hat r\wedge\vec a\right|^2+\dfrac{2}{3}\left|\hat r\wedge\vec a\right|^2\hat r\cdot\vec a-4(\vec a\cdot\hat r)^3+}{r^5}$$

$$\frac{\dfrac{4}{3}\left|\hat r\wedge\vec a\right|^2\hat r\cdot\vec a}{}=\frac{-4(\vec a\cdot\hat r)^3+4\left|\hat r\wedge\vec a\right|^2\vec a\cdot\hat r}{r^5}.$$

Proposition 32

$$\vec a\cdot\vec\partial\log r=\frac{\vec a\cdot\vec r}{r^2}.$$

Proof 32

$$\vec a\cdot\vec\partial\log r\overset{P7}{=}\frac{1}{r}\vec a\cdot\vec\partial r\overset{P23}{=}\frac{\vec a\cdot\hat r}{r}\overset{(11)}{=}\frac{\vec a\cdot\vec r}{r^2}.$$

Proposition 33

For integer k and $\vec r\neq 0$ if k<0 :

$$\vec a\cdot\vec\partial\vec r^{\,2k}=2k\vec a\cdot\vec r\vec r^{\,2(k-1)}.$$

Proof 33

$$\vec a\cdot\vec\partial\vec r^{\,2k}=\vec a\cdot\vec\partial r^{2k}\overset{P7}{=}2kr^{2k-1}\vec a\cdot\vec\partial r\overset{P23}{=}2kr^{2k-1}\vec a\cdot\hat r$$
$$=2kr^{2k-2}\vec a\cdot\vec r=2k\vec a\cdot\vec r\vec r^{\,2(k-1)}.$$

Proposition 34

For integer k and $\vec r\neq 0$ if 2k+1<0 :

$$\vec a\cdot\vec\partial\vec r^{\,2k+1}=\vec r^{\,2k}(\vec a+2k\vec a\cdot\hat r\hat r).$$

Proof 34

$$\vec a\cdot\vec\partial\vec r^{\,2k+1}=\vec a\cdot\vec\partial(\vec r^{\,2k}\vec r)\overset{P5}{=}(\vec a\cdot\vec\partial\vec r^{\,2k})\vec r+\vec r^{\,2k}\vec a\cdot\vec\partial\vec r$$
$$\overset{P8,33}{=}2k\vec a\cdot\vec r\vec r^{\,2(k-1)}\vec r+\vec r^{\,2k}\vec a=\vec r^{\,2k}(\vec a+2k\vec a\cdot\hat r\hat r).$$

Proposition 35 (*Taylor expansion of* $\frac{1}{\vec x-\vec a}$)

$$\frac{1}{\vec x-\vec a}=\frac{1}{\vec x}+\frac{1}{\vec x}\vec a\frac{1}{\vec x}+\frac{1}{\vec x}\vec a\frac{1}{\vec x}\vec a\frac{1}{\vec x}+\ldots\ .$$

Proof 35

$$\frac{1}{\vec x-\vec a}\overset{P12}{=}\exp(-\vec a\cdot\vec\partial)\frac{1}{\vec x}=\sum_{k=0}^{\infty}\frac{(-\vec a\cdot\vec\partial)^k}{k!}\frac{1}{\vec x},$$

$$(-\vec a\cdot\vec\partial)^k\frac{1}{\vec x}=(-\vec a\cdot\vec\partial)^{k-1}(-\vec a\cdot\vec\partial)\frac{1}{\vec x}\overset{P28}{=}$$

$$(-\vec a\cdot\vec\partial)^{k-1}\frac{1}{\vec x}\vec a\frac{1}{\vec x}=(-\vec a\cdot\vec\partial)^{k-2}(-\vec a\cdot\vec\partial)\left[\frac{1}{\vec x}\vec a\frac{1}{\vec x}\right]\overset{P5}{=}$$

$$(-\vec a\cdot\vec\partial)^{k-2}\left[(-\vec a\cdot\vec\partial\frac{1}{\vec x})\vec a\frac{1}{\vec x}+\frac{1}{\vec x}\vec a(-\vec a\cdot\vec\partial\frac{1}{\vec x})\right]\overset{P28}{=}$$

$$2(-\vec a\cdot\vec\partial)^{k-2}\frac{1}{\vec x}\vec a\frac{1}{\vec x}\vec a\frac{1}{\vec x}\overset{P5,28}{=}$$

$$2\ 3(-\vec a\cdot\vec\partial)^{k-3}\frac{1}{\vec x}\vec a\frac{1}{\vec x}\vec a\frac{1}{\vec x}\vec a\frac{1}{\vec x}\overset{P5,28}{=}\ldots=$$

$$k!(-\vec a\cdot\vec\partial)^{k-k}\frac{1}{\vec x}\left(\vec a\frac{1}{\vec x}\right)^k=k!\frac{1}{\vec x}\left(\vec a\frac{1}{\vec x}\right)^k$$

$$\Rightarrow\frac{1}{k!}(-\vec a\cdot\vec\partial)\frac{1}{\vec x}=\frac{1}{\vec x}\left(\vec a\frac{1}{\vec x}\right)^k$$

$$\Rightarrow\frac{1}{\vec x-\vec a}=\sum_{k=0}^{\infty}\frac{1}{\vec x}\left(\vec a\frac{1}{\vec x}\right)^k.$$

Proposition 36 (*Legendre Polynomials*)

The Legendre Polynomials P_n are defined by:

$$\frac{1}{\left|\vec x-\vec a\right|}\equiv\sum_{n=0}^{\infty}\frac{P_n(\hat x\vec a)}{\left|\vec x\right|^{n+1}}=\sum_{n=0}^{\infty}\frac{P_n(\vec x\vec a)}{\left|\vec x\right|^{2n+1}}.$$

The explicit first four polynomials are:

$$P_0(\vec x\vec a)=1$$
$$P_1(\vec x\vec a)=\vec x\cdot\vec a$$
$$P_2(\vec x\vec a)=\frac{1}{2}\left[3(\vec x\cdot\vec a)^2-\vec a^2\vec x^2\right]$$
$$=(\vec x\cdot\vec a)^2+\frac{1}{2}(\vec x\wedge\vec a)^2$$
$$P_3(\vec x\vec a)=\frac{1}{2}\left[5(\vec x\cdot\vec a)^3-3\vec a^2\vec x^2\vec x\cdot\vec a\right]=$$
$$(\vec x\cdot\vec a)^3+\frac{3}{2}\vec x\cdot\vec a(\vec x\wedge\vec a)^2$$

$$P_n(\vec{x}\vec{a}) = |\vec{x}|^n P_n(\hat{x}\vec{a}) = |\vec{x}|^n |\vec{a}|^n P_n(\hat{x}\hat{a}).$$

Proof 36

$$F(\vec{x}-\vec{a}) = \frac{1}{|\vec{x}-\vec{a}|} \overset{P12}{=} \exp(-\vec{a}\cdot\vec{\partial})F(\vec{x}) =$$

$$\sum_{n=0}^{\infty} \frac{(-\vec{a}\cdot\vec{\partial})}{n!}\frac{1}{|\vec{x}|} =$$

$$\frac{1}{|\vec{x}|} - \vec{a}\cdot\vec{\partial}\frac{1}{|\vec{x}|} + \frac{1}{2}(\vec{a}\cdot\vec{\partial})^2\frac{1}{|\vec{x}|} - \frac{1}{6}(\vec{a}\cdot\vec{\partial})^3\frac{1}{|\vec{x}|} + \ldots$$

$$\Rightarrow P_0(\vec{x}\vec{a}) = 1,$$

$$-\vec{a}\cdot\vec{\partial}\frac{1}{|\vec{x}|} \overset{P7}{=} \frac{1}{|\vec{x}|^2}\vec{a}\cdot\vec{\partial}|\vec{x}| \overset{P10}{=} \frac{1}{|\vec{x}|^2}\vec{a}\cdot\hat{x} \overset{(11)}{=} \frac{\vec{a}\cdot\vec{x}}{|\vec{x}|^3}$$

$$\Rightarrow P_1(\vec{x}\vec{a}) = \vec{x}\cdot\vec{a},$$

$$\frac{1}{2}(\vec{a}\cdot\vec{\partial})^2\frac{1}{|\vec{x}|} = -\frac{1}{2}(\vec{a}\cdot\vec{\partial})(-\vec{a}\cdot\vec{\partial})\frac{1}{|\vec{x}|} =$$

$$-\frac{1}{2}(\vec{a}\cdot\vec{\partial})\frac{\vec{a}\cdot\vec{x}}{|\vec{x}|^3} \overset{P5}{=}$$

$$-\frac{1}{2}\left[\vec{a}\cdot\vec{\partial}\frac{1}{|\vec{x}|^3}\right]\vec{a}\cdot\vec{x} - \frac{1}{2}\frac{1}{|\vec{x}|^3}\vec{a}\cdot\vec{\partial}(\vec{a}\cdot\vec{x}) \overset{P7,21}{=}$$

$$-\frac{1}{2}\left[\frac{-3}{|\vec{x}|^4}\vec{a}\cdot\vec{\partial}|\vec{x}|\right]\vec{a}\cdot\vec{x} - \frac{1}{2}\frac{1}{|\vec{x}|^3}\vec{a}\cdot\vec{a} \overset{P10}{=}$$

$$\frac{1}{2}\frac{3}{|\vec{x}|^4}\vec{a}\cdot\hat{x}(\vec{a}\cdot\vec{x}) - \frac{1}{2}\frac{1}{|\vec{x}|^3}\vec{a}^2 \underset{(5)}{\overset{(11)}{=}}$$

$$\frac{1}{2}\frac{3(\vec{a}\cdot\vec{x})^2 - \vec{x}^2\vec{a}^2}{|\vec{x}|^5} = \frac{(\vec{a}\cdot\vec{x})^2 + \frac{1}{2}\{(\vec{a}\cdot\vec{x})^2 - \vec{x}^2\vec{a}^2\}}{|\vec{x}|^5}$$

$$\overset{(43)}{=} \frac{(\vec{a}\cdot\vec{x})^2 + \frac{1}{2}(\vec{a}\wedge\vec{x})^2}{|\vec{x}|^5}$$

$$\Rightarrow P_2(\vec{x}\vec{a}) = \frac{1}{2}\left[3(\vec{x}\cdot\vec{a})^2 - \vec{a}^2\vec{x}^2\right]$$

$$= (\vec{x}\cdot\vec{a})^2 + \frac{1}{2}(\vec{x}\wedge\vec{a})^2,$$

$$-\frac{1}{6}(\vec{a}\cdot\vec{\partial})^3\frac{1}{|\vec{x}|} = -\frac{1}{3}(\vec{a}\cdot\vec{\partial})\left\{\frac{1}{2}(\vec{a}\cdot\vec{\partial})^2\frac{1}{|\vec{x}|}\right\} =$$

$$-\frac{1}{3}(\vec{a}\cdot\vec{\partial})\left\{\frac{1}{2}\frac{3(\vec{a}\cdot\vec{x})^2 - \vec{x}^2\vec{a}^2}{|\vec{x}|^5}\right\} \overset{P4,5}{=}$$

$$= -\frac{1}{6}\left(\vec{a}\cdot\vec{\partial}\frac{1}{|\vec{x}|^5}\right)\{3(\vec{a}\cdot\vec{x})^2 - \vec{x}^2\vec{a}^2\}$$

$$-\frac{1}{6|\vec{x}|^5}\{3\ 2(\vec{a}\cdot\vec{x})\vec{a}\cdot\vec{\partial}(\vec{a}\cdot\vec{x}) - \vec{a}^2(\vec{a}\cdot\vec{\partial})\vec{x}^2\} \overset{P5,7,21,23}{=}$$

$$-\frac{-5}{6|\vec{x}|^6}(\vec{a}\cdot\vec{\partial}|\vec{x}|)\{3(\vec{a}\cdot\vec{x})^2 - \vec{x}^2\vec{a}^2\}$$

$$-\frac{1}{6|\vec{x}|^5}\{6(\vec{a}\cdot\vec{x})\vec{a}\cdot\vec{a} - \vec{a}^2 2(\vec{a}\cdot\vec{x})\} \overset{P10}{=}$$

$$\frac{5}{6|\vec{x}|^6}\frac{\vec{a}\cdot\vec{x}}{|\vec{x}|}\{3(\vec{a}\cdot\vec{x})^2 - \vec{x}^2\vec{a}^2\} - \frac{4\vec{a}^2(\vec{a}\cdot\vec{x})}{6|\vec{x}|^5} =$$

$$\frac{15(\vec{a}\cdot\vec{x})^3 - 5\vec{x}^2\vec{a}^2\vec{a}\cdot\vec{x} - 4\vec{x}^2\vec{a}^2(\vec{a}\cdot\vec{x})}{6|\vec{x}|^7} =$$

$$\frac{1}{2|\vec{x}|^7}\left[5(\vec{a}\cdot\vec{x})^3 - 3\vec{x}^2\vec{a}^2\vec{a}\cdot\vec{x}\right] =$$

$$\frac{1}{|\vec{x}|^7}\left[(\vec{a}\cdot\vec{x})^3 + \frac{3}{2}\vec{a}\cdot\vec{x}\{(\vec{a}\cdot\vec{x})^2 - \vec{x}^2\vec{a}^2\}\right] \overset{(43)}{=}$$

$$\frac{1}{|\vec{x}|^7}\left[(\vec{a}\cdot\vec{x})^3 + \frac{3}{2}\vec{a}\cdot\vec{x}(\vec{a}\wedge\vec{x})^2\right]$$

$$\Rightarrow P_3(\vec{x}\vec{a}) = \frac{1}{2}\left[5(\vec{x}\cdot\vec{a})^3 - 3\vec{a}^2\vec{x}^2\vec{x}\cdot\vec{a}\right] =$$

$$(\vec{x}\cdot\vec{a})^3 + \frac{3}{2}\vec{x}\cdot\vec{a}(\vec{x}\wedge\vec{a})^2$$

Homogeneity of degree *n* of the P_n:

$$\frac{(-\vec{a}\cdot\vec{\partial})^n}{n!}\frac{1}{|\vec{x}|} \overset{def}{=} \frac{P_n(\vec{x}\vec{a})}{|x|^{2n+1}}$$

$$\Rightarrow (-\vec{a}\cdot\vec{\partial})^n\frac{1}{|\vec{x}|} = n!\frac{P_n(\vec{x}\vec{a})}{|x|^{2n+1}}$$ and

$$(-\vec{a}\cdot\vec{\partial})^{n+1}\frac{1}{|\vec{x}|} = (n+1)!\frac{P_{n+1}(\vec{x}\vec{a})}{|x|^{2(n+1)+1}}$$

$$\Rightarrow \frac{(n+1)!}{n!}\frac{P_{n+1}(\vec{x}\vec{a})}{|\vec{x}|^{2(n+1)+1}} = (-\vec{a}\cdot\vec{\partial})\frac{P_n(\vec{x}\vec{a})}{|\vec{x}|^{2n+1}} \overset{P5}{=}$$

$$\left(-\vec{a}\cdot\vec{\partial}\frac{1}{|\vec{x}|^{2n+1}}\right)P_n(\vec{x}\vec{a}) - \frac{1}{|\vec{x}|^{2n+1}}\vec{a}\cdot\vec{\partial}P_n(\vec{x}\vec{a}) \overset{P7}{=}$$

$$\frac{2n+1}{|\vec{x}|^{2n+2}}P_n(\vec{x}\vec{a})\vec{a}\cdot\vec{\partial}|\vec{x}| - \frac{1}{|\vec{x}|^{2n+1}}\vec{a}\cdot\vec{\partial}P_n(\vec{x}\vec{a}) \overset{P10}{=}$$

$$\frac{2n+1}{|\vec{x}|^{2n+2}}P_n(\vec{x}\vec{a})\frac{\vec{a}\cdot\vec{x}}{|\vec{x}|} - \frac{1}{|\vec{x}|^{2n+1}}\vec{a}\cdot\vec{\partial}P_n(\vec{x}\vec{a}) =$$

$$= \frac{2n+1}{|\vec{x}|^{2n+3}} P_n(\vec{x}\vec{a})\vec{a}\cdot\vec{x} - \frac{1}{|\vec{x}|^{2n+1}} \vec{a}\cdot\vec{\partial}P_n(\vec{x}\vec{a}).$$

$P_n(\vec{x}\vec{a})\vec{a}\cdot\vec{x}$ is a homogeneous function of degree *n+1*, if we assume P_n to be homogeneous of degree *n*:

$$P_n(\vec{x}\vec{a}) \overset{assume}{=} \sum_{k=0}^{n} \alpha_k (\vec{x}\cdot\vec{a})^k |\vec{x}|^{n-k} \vec{a}^{n-k} \quad ,$$

$\alpha_k = const.$ which is especially true for *n*=0,1,2,3. The right term

$$\vec{a}\cdot\vec{\partial}P_n(\vec{x}\vec{a}) = \vec{a}\cdot\vec{\partial}\sum_{k=0}^{n}\alpha_k(\vec{x}\cdot\vec{a})^k|\vec{x}|^{n-k}\vec{a}^{n-k} \overset{P4,5,9}{=}$$

$$\sum_{k=0}^{n}\alpha_k\left\{\vec{a}\cdot\vec{\partial}(\vec{x}\cdot\vec{a})^k\right\}|\vec{x}|^{n-k}\vec{a}^{n-k} +$$

$$\alpha_k(\vec{x}\cdot\vec{a})^k\left\{\vec{a}\cdot\vec{\partial}|\vec{x}|^{n-k}\right\}\vec{a}^{n-k} \overset{P5,7}{=}$$

$$\sum_{k=0}^{n}\alpha_k\left\{k(\vec{x}\cdot\vec{a})^{k-1}\vec{a}\cdot\vec{\partial}(\vec{x}\cdot\vec{a})\right\}|\vec{x}|^{n-k}\vec{a}^{n-k} +$$

$$\alpha_k(\vec{x}\cdot\vec{a})^k\left\{(n-k)|\vec{x}|^{n-k-1}\vec{a}\cdot\vec{\partial}|\vec{x}|\right\}\vec{a}^{n-k} \overset{P10,21}{=}$$

$$\sum_{k=0}^{n}\alpha_k\left\{k(\vec{x}\cdot\vec{a})^{k-1}\vec{a}\cdot\vec{a})\right\}|\vec{x}|^{n-k}\vec{a}^{n-k} +$$

$$\alpha_k(\vec{x}\cdot\vec{a})^k\left\{(n-k)|\vec{x}|^{n-k-1}\frac{\vec{a}\cdot\vec{x}}{|\vec{x}|}\right\}\vec{a}^{n-k} =$$

$$\sum_{k=0}^{n}\alpha_k k(\vec{x}\cdot\vec{a})^{k-1}\frac{|\vec{x}|^2}{|\vec{x}|^2}|\vec{x}|^{n-k}\vec{a}^2\vec{a}^{n-k} +$$

$$\alpha_k(n-k)(\vec{x}\cdot\vec{a})^{k+1}\frac{|\vec{x}|^{n-k-1}}{|\vec{x}|}\frac{|\vec{x}|}{|\vec{x}|}\vec{a}^{n-k} =$$

$$\frac{1}{|\vec{x}|^2}\sum_{k=0}^{n}\alpha_k k(\vec{x}\cdot\vec{a})^{k-1}|\vec{x}|^{n-k+2}\vec{a}^{n-k+2} +$$

$$\alpha_k(n-k)(\vec{x}\cdot\vec{a})^{k+1}|\vec{x}|^{n-k}\vec{a}^{n-k}$$

yields $|\vec{x}|^2\vec{a}\cdot\vec{\partial}P_n(\vec{x}\vec{a})$, to be homogeneous of degree *n+1*.

Hence $\dfrac{(n+1)!}{n!}\dfrac{P_{n+1}(\vec{x}\vec{a})}{|\vec{x}|^{2(n+1)+1}} =$

$\dfrac{(\text{Polynomial homogeneous of degree } n+1)}{|\vec{x}|^{2(n+1)+1}}$. By induction every P_n will therefore be homogeneous of degree *n*. This and the explicit expressions above for $\dfrac{P_{n+1}(\vec{x}\vec{a})}{|\vec{x}|^{2(n+1)+1}}$ fully prove for all *n*: $\quad P_n(\vec{x}\vec{a}) = |\vec{x}|^n P_n(\hat{x}\vec{a}) = |\vec{x}|^n|\vec{a}|^n P_n(\hat{x}\hat{a}).$

Definition 37 (*redefinition of differential, over-dots*)

$$\underline{F}(\vec{a}) = \vec{a}\cdot\vec{\partial}F \overset{(7)}{=} \frac{1}{2}\left(\vec{a}\vec{\partial}F + \dot{\vec{\partial}}\vec{a}\dot{F}\right),$$

where the *over-dots* indicate, that only *F* is to be differentiated and not $\vec{a}$.

Proposition 38

For $\vec{a} \notin \mathcal{A}_n \equiv \mathcal{G}^1(I)$, $P = P_I$:

$$\vec{a}\cdot\vec{\partial}_{\vec{x}} = \vec{a}\cdot P(\vec{\partial}_{\vec{x}}) = P(\vec{a})\cdot\vec{\partial}_{\vec{x}},\ P(\vec{a}) \in \mathcal{A}_n.$$

Proof 38

$$\vec{a}\cdot\vec{\partial}_{\vec{x}} \overset{P18}{=} \vec{a}\cdot P(\vec{\partial}_{\vec{x}}) \overset{P18}{=} \sum_{k=1}^{n}\vec{a}\cdot\vec{a}^k(\vec{a}_k\cdot\vec{\partial}_{\vec{x}}) =$$

$$\sum_{k=1}^{n}P(\vec{a})\cdot\vec{a}^k(\vec{a}_k\cdot\vec{\partial}_{\vec{x}}) \overset{P18}{=} P(\vec{a})\cdot\vec{\partial}_{\vec{x}}.$$

Proposition 39

$$\underline{F}(\vec{a}) = \underline{F}(P(\vec{a})) = P(\vec{a})\cdot\vec{\partial}F\ .$$

$$\underline{F}(\vec{a}) = 0,\ \text{if}\ \ P(\vec{a}) = 0\,.$$

Proof 39

$$\underline{F}(\vec{a}) \overset{def13,P38}{=} \underline{F}(P(\vec{a})) \overset{def13}{=} P(\vec{a})\cdot\vec{\partial}F\,,$$

$$\Rightarrow \underline{F}(\vec{a}) = 0,\ \text{if}\ \ P(\vec{a}) = 0\,.$$

Proposition 40 (*differential of composite functions*)

For $F(\vec{x}) = G(f(\vec{x}))$ and

$$f: \vec{x}\in\mathcal{A}_n = \mathcal{G}^1(I) \to f(\vec{x})\in\mathcal{A}'_n = \mathcal{G}^1(I')$$

$$\vec{a}\cdot\vec{\partial}F = \underline{f}(\vec{a})\cdot\vec{\partial}G$$

$$\underline{F}(\vec{a}) = \underline{G}(\underline{f}(\vec{a}))\quad \text{(Def. 13)}$$

The differential of composite functions is the composite of differentials.

$$\underline{F}(\vec{x},\vec{a}) = \underline{G}(f(\vec{x}),\underline{f}(\vec{x},\vec{a}))\quad \text{(explicit)}$$

Proof 40

Taylor expansion (P12):

$$f(\vec{x}+\tau\vec{a}) = f(\vec{x}) + \tau\vec{a}\cdot\vec{\partial}f(\vec{x}) + \frac{1}{2}\tau^2(\vec{a}\cdot\vec{\partial})^2 f(\vec{x}) + \ldots$$

$$\vec{a}\cdot\vec{\partial}G(f(\vec{x})) \overset{def1}{=} \frac{d}{d\tau}G(f(\vec{x}+\tau\vec{a}))\Big|_{\tau=0} \underset{def13}{\overset{Taylor(P12)}{=}}$$

$$= \frac{d}{d\tau} G(f(\vec{x}) + \tau \underline{f}(\vec{a}))\Big|_{\tau=0} \overset{def1}{=} \underline{f}(\vec{a}) \cdot \vec{\partial}_{\vec{x}'} G(\vec{x}')\Big|_{\vec{x}'=f(\vec{x})}$$

$= \underline{f}(\vec{a}) \cdot \vec{\partial} G$ (evaluation at corresponding points.)

Definition 41 (*second differential*)

$$F_{\vec{a}\vec{b}}(\vec{x}) \equiv \vec{b} \cdot \dot{\vec{\partial}} \vec{a} \cdot \vec{\partial} \dot{F}(\vec{x}) .$$

Suppressing $\vec{x}$: $F_{\vec{a}\vec{b}} \equiv \vec{b} \cdot \dot{\vec{\partial}} \vec{a} \cdot \vec{\partial} \dot{F}$.

Proposition 42 (*integrability condition*)

$$F_{\vec{a}\vec{b}} = F_{\vec{b}\vec{a}} .$$

The second differential is a *symmetric bilinear* function of its differential arguments $\vec{a}$, $\vec{b}$.

Proof 42

$$F_{\vec{a}\vec{b}}(\vec{x}) \equiv \vec{b} \cdot \dot{\vec{\partial}} \vec{a} \cdot \vec{\partial} \dot{F}(\vec{x}) \overset{def1}{=} \vec{b} \cdot \dot{\vec{\partial}} \frac{d\dot{F}(\vec{x} + \tau\vec{a})}{d\tau}\Bigg|_{\tau=0} =$$

$$\frac{d^2 F(\vec{x} + \tau\vec{a} + \sigma\vec{b})}{d\sigma d\tau}\Bigg|_{\substack{\tau=0\\ \sigma=0}} = \lim_{\sigma\to 0} \lim_{\tau\to 0}$$

$$\frac{\dfrac{F(\vec{x} + \tau\vec{a} + \sigma\vec{b}) - F(\vec{x} + \sigma\vec{b})}{\tau} - \dfrac{F(\vec{x} + \tau\vec{a}) - F(\vec{x})}{\tau}}{\sigma}$$

which is symmetric under $(\vec{a}, \tau) \leftrightarrow (\vec{b}, \sigma)$. Hence

$$F_{\vec{a}\vec{b}}(\vec{x}) \equiv \vec{b} \cdot \dot{\vec{\partial}} \vec{a} \cdot \vec{\partial} \dot{F}(\vec{x}) = \vec{a} \cdot \dot{\vec{\partial}} \vec{b} \cdot \vec{\partial} \dot{F}(\vec{x}) = F_{\vec{b}\vec{a}}(\vec{x}).$$

The bilinearity follows from the linearity in each argument (P2, P3 and P14).

Proposition 43 (*differential of identity function*)

$$\vec{a} \cdot \vec{\partial}_{\vec{x}} \vec{x} = P(\vec{a}) = \vec{\partial}_{\vec{x}} (\vec{x} \cdot \vec{a})$$

Proof 43

$\vec{a} \cdot \vec{\partial}_{\vec{x}} \vec{x} \overset{P38}{=} P_I(\vec{a}) \cdot \vec{\partial}_{\vec{x}} \vec{x} \overset{P8}{=} P_I(\vec{a})$ (first identity).

Especially for base vectors $\vec{a}_k \in \mathcal{A}_n = \mathcal{G}^1(I)$:

$$\vec{a}_k \cdot \vec{\partial}_{\vec{x}} \vec{x} = P_I(\vec{a}_k) = \vec{a}_k ,$$

$$\vec{\partial}_{\vec{x}} (\vec{x} \cdot \vec{a}) \overset{P18}{=} \sum_k \vec{a}^k \vec{a}_k \cdot \vec{\partial}_{\vec{x}} (\vec{x} \cdot \vec{a}) \overset{P21}{=} \sum_k \vec{a}^k \vec{a}_k \cdot \vec{a}$$

$= P_I(\vec{a})$ (second identity).

Proposition 44 (*operator identity*)

$$\vec{\partial}_{\vec{x}} = P_I(\vec{\partial}_{\vec{x}}) = \vec{\partial}_{\vec{a}} \vec{a} \cdot \vec{\partial}_{\vec{x}} .$$

Proof 44

Propositions 18 and 43.

Proposition 45 (*derivative from differential*)

$$\vec{\partial}_{\vec{x}} F(\vec{x}) = \vec{\partial}_{\vec{a}} \vec{a} \cdot \vec{\partial}_{\vec{x}} F(\vec{x}) = \vec{\partial}_{\vec{a}} \underline{F}(\vec{x}, \vec{a}).$$

Proof 45

Proposition 44 and definition 13.

Definition 46

$$\vec{\partial} F = \vec{\partial}_{\vec{x}} F(\vec{x}) \overset{P45}{=} \vec{\partial}_{\vec{a}} \underline{F}(\vec{x}, \vec{a}) = \underline{\vec{\partial}} \underline{F} ,$$

where $\underline{\vec{\partial}}$ is the derivative with respect to the differential argument $\vec{a}$ of $\underline{F}(\vec{x}, \vec{a})$.

Proposition 47

$$\vec{\partial} F = \vec{\partial} \cdot F + \vec{\partial} \wedge F .$$

Proof 47

Vector property (P. 18) of $\vec{\partial} = \vec{\partial}_{\vec{x}}$ and (18).

Proposition 48 (*gradient*)

For scalar $F = \Phi(\vec{x})$:

$$\vec{\partial} \cdot \Phi = 0 , \; \vec{\partial} \Phi = \vec{\partial} \wedge \Phi = \dot{\Phi} \dot{\vec{\partial}}$$

Proof 48

(21), P. 47 and (23).

Remark 49

In proposition 48 the special definition of Hestenes and Sobczyk[2] in (20) and (21) for the inner product becomes important. It should be possible to make it more intuitive by replacing the inner product with the contraction [4].

Definition 50

Divergence of *F*: $\vec{\partial} \cdot F$,

Curl of *F*: $\vec{\partial} \wedge F$

(Full vector derivative of *F*: $\vec{\partial} F$.)

Proposition 51 (*vector derivative of sums*)

$$\vec{\partial}(F + G) = \vec{\partial} F + \vec{\partial} G .$$

Proof 51

$$\vec{\partial}(F + G) \overset{def46}{=} \underline{\vec{\partial}}(\underline{F + G}) \overset{P4}{=} \underline{\vec{\partial}}(\underline{F} + \underline{G}) \overset{P18}{=}$$

$$\sum_k \vec{a}^k \vec{a}_k \cdot \vec{\partial}_{\vec{a}} (\underline{F} + \underline{G}) \overset{P4}{=}$$

$$\sum_k \vec{a}^k (\vec{a}_k \cdot \vec{\partial}_{\vec{a}} \underline{F} + \vec{a}_k \cdot \vec{\partial}_{\vec{a}} \underline{G}) \underset{(19)}{\overset{distributivity}{=}}$$

$$\sum_k \vec{a}^k \vec{a}_k \cdot \vec{\partial}_{\vec{a}} \underline{F} + \sum_k \vec{a}^k \vec{a}_k \cdot \vec{\partial}_{\vec{a}} \underline{G} \underset{def46}{\overset{P18}{=}} \vec{\partial} F + \vec{\partial} G.$$

Note that, geometric multiplication is *distributive* with respect to addition.

Proposition 52 (*vector derivative of products*)

$$\vec{\partial}(FG) = \dot{\vec{\partial}}\dot{F}G + \dot{\vec{\partial}}F\dot{G}.$$

Proof 52

$$\vec{\partial}(FG) \overset{def\,46}{=} \underline{\vec{\partial}}(\underline{FG}) \overset{P5}{=} \underline{\vec{\partial}}(\underline{F}G + F\underline{G}) \underset{P51}{\overset{def\,46}{=}}$$

$$\underline{\vec{\partial}}\underline{F}G + \underline{\vec{\partial}}F\underline{G} \overset{def\,46}{=} \dot{\vec{\partial}}\dot{F}G + \dot{\vec{\partial}}F\dot{G}.$$

The third equality is a special case of P. 51, if we take the definition of $\underline{\vec{\partial}}$ in Def. 46 into account. The last term is to be interpreted as: $\dot{\vec{\partial}}F\dot{G} = \vec{\partial}_{\vec{y}}\left(F(\vec{x})G(\vec{y})\right)\big|_{\vec{y}=\vec{x}}$.

Proposition 53

$$\vec{\partial}\vec{x}^2 = 2\vec{x}.$$

Proof 53

$$\vec{\partial}\vec{x}^2 = (\vec{\partial}\vec{x})\vec{x} + \dot{\vec{\partial}}\vec{x}\dot{\vec{x}} \overset{P18}{=}$$

$$\left(\sum_k \vec{a}^k\vec{a}_k\cdot\vec{\partial}_{\vec{x}}\vec{x}\right)\vec{x} + \sum_k \vec{a}^k\vec{x}\vec{a}_k\cdot\vec{\partial}_{\vec{x}}\vec{x} \overset{P43}{=}$$

$$\left(\sum_k \vec{a}^k\vec{a}_k\right)\vec{x} + \sum_k \vec{a}^k\vec{x}\vec{a}_k \overset{(7)}{=} 2\sum_k \vec{a}^k\vec{x}\cdot\vec{a}_k \overset{(32)}{=} 2\vec{x}.$$

Proposition 54

$$\vec{\partial}|\vec{x}| = \hat{x}.$$

Proof 54

$$\vec{\partial}|\vec{x}| \overset{P18}{=} \sum_k \vec{a}^k\vec{a}_k\cdot\vec{\partial}_{\vec{x}}|\vec{x}| \overset{P10}{=} \sum_k \vec{a}^k\vec{a}_k\cdot\hat{x} \overset{(32)}{=} \hat{x}.$$

Proposition 55

$$\text{For } F = F(|\vec{x}|): \quad \vec{\partial}F = \hat{x}\frac{dF}{d|\vec{x}|}.$$

Proof 55

$$\vec{a}\cdot\vec{\partial}F \overset{P7}{=} \vec{a}\cdot\vec{\partial}|\vec{x}|\frac{dF}{d|\vec{x}|} \overset{P54}{=} \vec{a}\cdot\hat{x}\frac{dF}{d|\vec{x}|}.$$

$$\Rightarrow \vec{\partial}F \overset{def\,46}{=} \vec{\partial}_{\vec{a}}\vec{a}\cdot\vec{\partial}F = \vec{\partial}_{\vec{a}}\vec{a}\cdot\hat{x}\frac{dF}{d|\vec{x}|} \overset{P43}{=} \hat{x}\frac{dF}{d|\vec{x}|}.$$

Definition 56 (*sides of differentiation*)

Only right side differentiation:

$$F\vec{\partial}G \overset{def\,46}{=} F\underline{\vec{\partial}}\underline{G} = F\dot{\vec{\partial}}\dot{G}.$$

Left and right side differentiation (another form of the product rule P. 52):

$$\dot{F}\dot{\vec{\partial}}\dot{G} = \dot{F}\dot{\vec{\partial}}G + F\dot{\vec{\partial}}\dot{G}.$$

Definition 57 (*adjoint*)

For $f : \vec{x}\in\mathcal{A}_n = \mathcal{G}^1(I) \to f(\vec{x}) \in \mathcal{A}'_n = \mathcal{G}^1(I')$

$$\overline{f}(\vec{a}') \equiv \underline{\vec{\partial}}(\underline{f}\cdot\vec{a}'),$$

is the *adjoint* of f or explicitly:

$$\overline{f}(\vec{x},\vec{a}') \equiv \vec{\partial}_{\vec{a}}\left[\left\{(\vec{a}\cdot\vec{\partial}_{\vec{x}})f(\vec{x})\right\}\cdot\vec{a}'\right] \underset{37}{\overset{def}{=}} \vec{\partial}_{\vec{a}}\left[\underline{f}(\vec{x},\vec{a})\cdot\vec{a}'\right].$$

Proposition 58

$$\overline{f}(\vec{a}') = \vec{\partial}(f\cdot\vec{a}'),$$

or explicitly:

$$\overline{f}(\vec{x},\vec{a}') = \vec{\partial}_{\vec{x}}\left(f(\vec{x})\cdot\vec{a}'\right).$$

Proof 58

$$\overline{f}(\vec{x},\vec{a}') \overset{def\,57}{=} \vec{\partial}_{\vec{a}}\left[\left\{(\vec{a}\cdot\vec{\partial}_{\vec{x}})f(\vec{x})\right\}\cdot\vec{a}'\right] =$$

$$\vec{\partial}_{\vec{a}}(\vec{a}\cdot\vec{\partial}_{\vec{x}})\left[f(\vec{x})\cdot\vec{a}'\right] \overset{P45}{=} \vec{\partial}_{\vec{x}}\left(f(\vec{x})\cdot\vec{a}'\right).$$

([2] p. 50; [5] p. 23 (1.109), p. 24 (1.118), p. 104 (5.11).)

Proposition 59

$$\overline{f}(\vec{a}'+\vec{b}') = \overline{f}(\vec{a}') + \overline{f}(\vec{b}').$$

$$\overline{f}(\alpha\vec{a}') = \alpha\overline{f}(\vec{a}'), \quad \alpha \text{ scalar.}$$

Proof 59

Linearity of the inner product (9).

Proposition 60

$$P\left(\overline{f}(\vec{a}')\right) = \overline{f}(\vec{a}').$$

Proof 60

$$P\left(\overline{f}(\vec{a}')\right) \overset{def\,57,P58}{=} P\left(\vec{\partial}_{\vec{x}}\underbrace{\left(f(\vec{x})\cdot\vec{a}'\right)}_{scalar}\right) =$$

$$P(\vec{\partial}_{\vec{x}})f(\vec{x})\cdot\vec{a}' \overset{P44}{=} \vec{\partial}_{\vec{x}}(f(\vec{x})\cdot\vec{a}') \overset{P58}{=} \overline{f}(\vec{a}').$$

Proposition 61

$$\overline{f}(\vec{a}') = \vec{\partial}_{\vec{a}}(\underline{f}(\vec{a})\cdot\vec{a}') = \overline{f}(P'(\vec{a}')),$$

with *P’* the projection into the range of f and $\underline{f}$, i.e. into $\mathcal{A}'_n \equiv \mathcal{G}^1(I')$.

Proof 61

$$\overline{f}(\vec{a}') \overset{def\,57}{=} \vec{\partial}_{\vec{a}}(\underline{f}(\vec{a})\cdot\vec{a}') \overset{P58}{=} \vec{\partial}_{\vec{x}}(f(\vec{x})\cdot\vec{a}') =$$

$$\vec{\partial}_{\vec{x}}(P'f(\vec{x})\cdot\vec{a}') = \vec{\partial}_{\vec{x}}(f(\vec{x})\cdot P'(\vec{a}')) \overset{P58}{=} \overline{f}(P'(\vec{a}')).$$

Proposition 62

$$P\overline{f}P'(\vec{a}') = \overline{f}(\vec{a}'),$$

$$P'\underline{f}P(\vec{a})=\underline{f}(\vec{a}).$$

Proof 62

Line 1: Propositions 57,58.

Line 2: P. 39 and because the range of f and $\underline{f}$ is $A_n' = G^1(I')$.

Proposition 63 (*change of variables*)

For $F(\vec{x}) = G(f(\vec{x}))$, i.e. $\vec{x}\to\vec{x}' = f(\vec{x})$:

$$\vec{\partial}_{\vec{x}}F(\vec{x}) = \overline{f}(\vec{\partial}_{\vec{x}'})G(\vec{x}'), \text{ i.e. } \vec{\partial}_{\vec{x}} = \overline{f}(\vec{\partial}_{\vec{x}'}).$$

Proof 63

$$\vec{\partial}_{\vec{x}}F(\vec{x}) \overset{P44}{=} \vec{\partial}_{\vec{a}}\left(\vec{a}\cdot\vec{\partial}_{\vec{x}}F(\vec{x})\right) = \vec{\partial}_{\vec{a}}\left[\vec{a}\cdot\vec{\partial}_{\vec{x}}G\left(f(\vec{x})\right)\right] \overset{P40}{=}$$
$$\vec{\partial}_{\vec{a}}[\underbrace{\underline{f}(\vec{a})\cdot\vec{\partial}_{\vec{x}'}}_{scalar}G(\vec{x}')]\big|_{\vec{x}'=f(\vec{x})} =$$
$$[\vec{\partial}_{\vec{a}}\left(\underline{f}(\vec{a})\cdot\vec{\partial}_{\vec{x}'}\right)G(\vec{x}')]\big|_{\vec{x}'=f(\vec{x})} \overset{def\,57}{=}$$
$$[\overline{f}(\vec{\partial}_{\vec{x}'})G(\vec{x}')]\big|_{\vec{x}'=f(\vec{x})} = \overline{f}(\vec{\partial}_{\vec{x}'})G(\vec{x}').$$

Proposition 64 (*second differential*)

$$\vec{\partial}^2{}_{\vec{x}}F(\vec{x}) = \vec{\partial}_{\vec{b}}\vec{\partial}_{\vec{a}}F_{\vec{a}\vec{b}} = (\vec{\partial}_{\vec{b}}\cdot\vec{\partial}_{\vec{a}} + \vec{\partial}_{\vec{b}}\wedge\vec{\partial}_{\vec{a}})F_{\vec{a}\vec{b}}.$$

Proof 64

$$\vec{\partial}_{\vec{x}}F(\vec{x}) \overset{P54}{=} \vec{\partial}_{\vec{a}}\left(\vec{a}\cdot\vec{\partial}_{\vec{x}}F(\vec{x})\right) \overset{def13}{=} \vec{\partial}_{\vec{a}}F_{\vec{a}}$$
$$\Rightarrow \vec{\partial}^2{}_{\vec{x}}F(\vec{x}) = \vec{\partial}_{\vec{x}}\left(\vec{\partial}_{\vec{x}}F(\vec{x})\right) = \vec{\partial}_{\vec{x}}\left(\vec{\partial}_{\vec{a}}F_{\vec{a}}\right) \overset{P54}{=}$$
$$\vec{\partial}_{\vec{b}}[\underbrace{\vec{b}\cdot\dot{\vec{\partial}}_{\vec{x}}}_{scalar\ op.}\left(\vec{\partial}_{\vec{a}}\dot{F}_{\vec{a}}\right)] = \vec{\partial}_{\vec{b}}\vec{\partial}_{\vec{a}}[\left(\vec{b}\cdot\vec{\partial}_{\vec{x}}\right)F_{\vec{a}}] \overset{def\,41}{=}$$
$$\vec{\partial}_{\vec{b}}\vec{\partial}_{\vec{a}}F_{\vec{a}\vec{b}} \overset{P18}{=} (\vec{\partial}_{\vec{b}}\cdot\vec{\partial}_{\vec{a}} + \vec{\partial}_{\vec{b}}\wedge\vec{\partial}_{\vec{a}})F_{\vec{a}\vec{b}}.$$

Proposition 65 (*integrability condition for vector derivative*)

$$\vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}} = 0 \Leftrightarrow F_{\vec{a}\vec{b}} = F_{\vec{b}\vec{a}}.$$

Proof 65

$(\Rightarrow)$

$$\vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}} = 0 \Rightarrow 0 = (\vec{a}\wedge\vec{b})\cdot(\vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}})F \overset{(42)}{=}$$
$$\vec{a}\cdot[\vec{b}\cdot(\vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}})]F =$$
$$\vec{a}\cdot[(\vec{b}\cdot\vec{\partial}_{\vec{x}})\vec{\partial}_{\vec{x}}F - \vec{\partial}_{\vec{x}}(\vec{b}\cdot\vec{\partial}_{\vec{x}})]F =$$
$$[(\vec{b}\cdot\vec{\partial}_{\vec{x}})(\vec{a}\cdot\vec{\partial}_{\vec{x}})F - (\vec{a}\cdot\vec{\partial}_{\vec{x}})(\vec{b}\cdot\vec{\partial}_{\vec{x}})]F \overset{def\,41}{=}$$
$$F_{\vec{a}\vec{b}} - F_{\vec{b}\vec{a}},$$

$(\Leftarrow)$

integrability (P. 42): $F_{\vec{a}\vec{b}} = F_{\vec{b}\vec{a}}$

$$0 = \frac{1}{2}(\vec{\partial}_{\vec{b}}\vec{\partial}_{\vec{a}}F_{\vec{a}\vec{b}} - \vec{\partial}_{\vec{a}}\vec{\partial}_{\vec{b}}F_{\vec{b}\vec{a}}) \overset{P42}{=}$$
$$\frac{1}{2}(\vec{\partial}_{\vec{b}}\vec{\partial}_{\vec{a}} - \vec{\partial}_{\vec{a}}\vec{\partial}_{\vec{b}})F_{\vec{a}\vec{b}} = \vec{\partial}_{\vec{b}}\wedge\vec{\partial}_{\vec{a}}F_{\vec{a}\vec{b}} \overset{P64}{=}$$
$$\vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}}F(\vec{x}) \Rightarrow \vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}} = 0.$$

Proposition 66 (*Laplacian*)

$$\text{Integrability of } F \Leftrightarrow \vec{\partial}_{\vec{x}}{}^2 = \vec{\partial}_{\vec{x}}\cdot\vec{\partial}_{\vec{x}}.$$

Proof 66

Integrability of $F \overset{P65}{\Leftrightarrow} \vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}} = 0$

$$\Leftrightarrow \vec{\partial}_{\vec{x}}{}^2 \underset{(6)}{\overset{(18)}{=}} \vec{\partial}_{\vec{x}}\cdot\vec{\partial}_{\vec{x}} + \vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}} = \vec{\partial}_{\vec{x}}\cdot\vec{\partial}_{\vec{x}}.$$

Proposition 67

$$\vec{\partial}_{\vec{x}}\wedge\vec{x} = 0.$$

Proof 67

$$\vec{\partial}_{\vec{x}}\wedge\vec{x} = \frac{1}{2}\vec{\partial}_{\vec{x}}\wedge(2\vec{x}) \overset{P53}{=} \frac{1}{2}(\vec{\partial}_{\vec{x}}\wedge\vec{\partial}_{\vec{x}})\vec{x}^2 \overset{P65}{=} 0,$$

because $F = \vec{x}^2$ is integrable.

Proposition 68

$$\vec{\partial}_{\vec{x}}\vec{x} = \vec{\partial}_{\vec{x}}\cdot\vec{x} \overset{(33)}{=} n.$$

Proof 68

$$\vec{\partial}_{\vec{x}}\vec{x} \overset{P47}{=} \vec{\partial}_{\vec{x}}\cdot\vec{x} + \vec{\partial}_{\vec{x}}\wedge\vec{x} \overset{P67}{=} \vec{\partial}_{\vec{x}}\cdot\vec{x} \overset{P18}{=}$$
$$\left(\sum_{k=1}^{n}\vec{a}^k\vec{a}_k\cdot\vec{\partial}_{\vec{x}}\right)\cdot\vec{x} = \sum_{k=1}^{n}\vec{a}^k\cdot(\vec{a}_k\cdot\vec{\partial}_{\vec{x}})\vec{x} \overset{P43}{=}$$
$$\sum_{k=1}^{n}\vec{a}^k\cdot P(\vec{a}_k) = \sum_{k=1}^{n}\vec{a}^k\cdot\vec{a}_k = n.$$

Proposition 69

$$\vec{\partial}_{\vec{x}}|\vec{x}|^k = k|\vec{x}|^{k-2}\vec{x}.$$

Proof 69

$$\vec{\partial}_{\vec{x}}|\vec{x}|^k \overset{P55}{=} \hat{x}\frac{d|\vec{x}|^k}{d|\vec{x}|} = \hat{x}k|\vec{x}|^{k-1} = k|\vec{x}|^{k-2}|\vec{x}|\hat{x} \overset{(11)}{=} k|\vec{x}|^{k-2}\vec{x}.$$

Proposition 70

$$\vec{\partial}_{\vec{x}}\left(\frac{\vec{x}}{|\vec{x}|^k}\right) = \frac{n-k}{|\vec{x}|^k}.$$

Proof 70

$$\vec{\partial}_{\vec{x}}\left(\frac{\vec{x}}{|\vec{x}|^k}\right) \overset{P52}{=} \vec{\partial}_{\vec{x}}(\vec{x})\frac{1}{|\vec{x}|^k} + \vec{x}\vec{\partial}_{\vec{x}}|\vec{x}|^{-k} \overset{P68,69}{=}$$

$$n\frac{1}{|\vec{x}|^k} + \vec{x}(-k)|\vec{x}|^{-k-2}\vec{x} = n\frac{1}{|\vec{x}|^k} - k|\vec{x}|^{-k-2}\vec{x}^2 =$$

$$\frac{n}{|\vec{x}|^k} - k|\vec{x}|^{-k} = \frac{n-k}{|\vec{x}|^k}.$$

Proposition 71

$$\vec{\partial}_{\vec{x}} \log|\vec{x}| = \frac{\vec{x}}{|\vec{x}|^2} = \vec{x}^{-1}.$$

Proof 71

$$\vec{\partial}_{\vec{x}} \log|\vec{x}| \overset{P55}{=} \hat{x}\frac{d\log|\vec{x}|}{d|\vec{x}|} = \hat{x}\frac{1}{|\vec{x}|} \overset{(12)}{=} \frac{\vec{x}}{|\vec{x}|^2} \overset{(12)}{=} \vec{x}^{-1}.$$

Proposition 72

For $A = P(A) = \langle A \rangle_r$:

$$\dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \cdot A) = A \cdot \vec{\partial}_{\vec{x}}\vec{x} = rA.$$

Proof 72

If A is a simple r-blade, then

$$\dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \cdot A) \overset{(39)}{=} \dot{\vec{\partial}}_{\vec{x}}(A \cdot \dot{\vec{x}})(-1)^{r-1} =$$

$$\frac{1}{2}[\dot{\vec{\partial}}_{\vec{x}} A\dot{\vec{x}} - (-1)^r \underbrace{(\vec{\partial}_{\vec{x}}\vec{x})}_{=n(P68)} A](-1)^{r-1} =$$

$$\frac{1}{2}[\dot{\vec{\partial}}_{\vec{x}} A\dot{\vec{x}} - (-1)^r A(\vec{\partial}_{\vec{x}}\vec{x})](-1)^{r-1} =$$

$$(\dot{\vec{\partial}}_{\vec{x}} \cdot A)\dot{\vec{x}}(-1)^{r-1} \overset{(39)}{=} (A \cdot \vec{\partial}_{\vec{x}})\vec{x},$$

$$\overset{(30)}{\Rightarrow} A^{-1}\dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \cdot A) = A^{-1}(A \cdot \vec{\partial}_{\vec{x}})\vec{x} \overset{(35)}{=} P_A(\vec{\partial}_{\vec{x}})\vec{x}$$

$$\overset{P18}{=}\left(\sum_{k=1}^{r} \vec{a}^k \vec{a}_k \cdot \partial_{\vec{x}}\right)\vec{x} = \sum_{k=1}^{r} \vec{a}^k(\vec{a}_k \cdot \partial_{\vec{x}})\vec{x} \overset{P43}{=} \sum_{k=1}^{r} \vec{a}^k \vec{a}_k$$

$$\overset{(33)}{=} r, \underset{(30)}{\overset{AA^{-1}=1}{\Rightarrow}} \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \cdot A) = (A \cdot \vec{\partial}_{\vec{x}})\vec{x} = rA.$$

Last step: Multiplication with A from the left. Distributivity (19), (25) gives the same result even for non-simple A.

Proposition 73

For $A = P(A) = \langle A \rangle_r$:

$$\dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \wedge A) = A \wedge \vec{\partial}_{\vec{x}}\vec{x} = (n-r)A.$$

Proof 73

If A is a simple r-blade, then

$$\dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \wedge A) \overset{(40)}{=} \dot{\vec{\partial}}_{\vec{x}}(A \wedge \dot{\vec{x}})(-1)^r =$$

$$= \frac{1}{2}[\dot{\vec{\partial}}_{\vec{x}} A\dot{\vec{x}} + (-1)^r \underbrace{(\vec{\partial}_{\vec{x}}\vec{x})}_{=n(P68)} A](-1)^r =$$

$$\frac{1}{2}[\dot{\vec{\partial}}_{\vec{x}} A\dot{\vec{x}} + (-1)^r A(\vec{\partial}_{\vec{x}}\vec{x})](-1)^r =$$

$$(\dot{\vec{\partial}}_{\vec{x}} \wedge A)\dot{\vec{x}}(-1)^r \overset{(40)}{=} (A \wedge \vec{\partial}_{\vec{x}})\vec{x},$$

$$\overset{(30)}{\Rightarrow} A^{-1}\dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \wedge A) = A^{-1}(A \wedge \vec{\partial}_{\vec{x}})\vec{x} =$$

$$A^{-1}I^{-1}I(A \wedge \vec{\partial}_{\vec{x}})\vec{x} \underset{(40)}{\overset{I\ \ pseudoscalar}{=}}$$

$$A^{-1}I^{-1}[\underbrace{I}_{n} \cdot \underbrace{(\vec{\partial}_{\vec{x}} \wedge A)}_{r+1}(-1)^r]\vec{x} \overset{(39)}{=}$$

grades:

$$A^{-1}I^{-1}[(\vec{\partial}_{\vec{x}} \wedge A) \cdot I]\vec{x}(-1)^{r+(r+1)(n-r-1)} \overset{(38)}{=}$$

$$= A^{-1}I^{-1}[\vec{\partial}_{\vec{x}} \cdot (AI)]\vec{x}(-1)^{r+(r+1)(n-r)-r-1} \overset{(39)}{=}$$

$$= A^{-1}I^{-1}[\underbrace{(AI)}_{n-r} \cdot \vec{\partial}_{\vec{x}}]\vec{x}(-1)^{(r+1)(n-r)-1+n-r-1} \overset{(40)}{=}$$

$$= A^{-1}I^{-1}[\underbrace{(IA)}_{n-r} \cdot \vec{\partial}_{\vec{x}}]\vec{x}\underbrace{(-1)^{r(n-r)+r(n-r)}}_{=+1} \overset{(35),(36)}{=}$$

$$P_{IA}(\vec{\partial}_{\vec{x}})\vec{x} \overset{proof\ 72}{=} (n-r)$$

$$\underset{(30)}{\overset{AA^{-1}=1}{\Rightarrow}} \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \wedge A) = (A \wedge \vec{\partial}_{\vec{x}})\vec{x} = (n-r)A.$$

Last step: Multiplication with A from the left. The distributive rule for the inner product gives the same result even for non-simple A.

Proposition 74

For $A = P(A) = \langle A \rangle_r$:

$$\dot{\vec{\partial}}_{\vec{x}} A\dot{\vec{x}} = \sum_{k=1}^{n} \vec{a}^k A\vec{a}_k = (-1)^r(n-2r)A.$$

Proof 74

$$\dot{\vec{\partial}}_{\vec{x}} A\dot{\vec{x}} \overset{P18}{=} \sum_{k=1}^{n} \vec{a}^k(\vec{a}_k \cdot \dot{\vec{\partial}}_{\vec{x}})A\dot{\vec{x}} = \sum_{k=1}^{n} \vec{a}^k A(\vec{a}_k \cdot \dot{\vec{\partial}}_{\vec{x}})\dot{\vec{x}}$$

$$\overset{P43}{=} \sum_{k=1}^{n} \vec{a}^k A\vec{a}_k,$$

$$\dot{\vec{\partial}}_{\vec{x}} A\dot{\vec{x}} = \dot{\vec{\partial}}_{\vec{x}}[A \cdot \dot{\vec{x}} + A \wedge \dot{\vec{x}}] \overset{(39),(40)}{=}$$

$$\dot{\vec{\partial}}_{\vec{x}}[\dot{\vec{x}} \cdot A(-1)^{r-1} + \dot{\vec{x}} \wedge A(-1)^r] =$$

$$(-1)^r[-\dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \cdot A) + \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{x}} \wedge A)] \underset{P73}{\overset{P72}{=}}$$

$$(-1)^r[-rA + (n-r)A] = (-1)^r(n-2r)A.$$

Proposition 75

For $\vec{a} = \vec{a}(\vec{x})$, $\vec{b} = \vec{b}(\vec{x})$:

$$\vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{b})$$
$$= \vec{a} \cdot \vec{\partial}_{\vec{x}}\vec{b} + \vec{b} \cdot \vec{\partial}_{\vec{x}}\vec{a} - \vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b}) - \vec{b} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a}).$$

Proof 75

$$\vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b}) \underset{(41)}{\overset{P18}{=}} \vec{a} \cdot \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{b}} - \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{b}} \cdot \vec{a}),$$

$$\vec{b} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a}) \underset{(41)}{\overset{P18}{=}} \vec{b} \cdot \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{a}} - \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{a}} \cdot \vec{b}),$$

$$\overset{addition}{\Rightarrow} \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{b}) = \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{a}} \cdot \vec{b}) + \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{b}} \cdot \vec{a}) =$$
$$\vec{a} \cdot \vec{\partial}_{\vec{x}}\vec{b} + \vec{b} \cdot \vec{\partial}_{\vec{x}}\vec{a} - \vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b}) - \vec{b} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a}).$$

Definition 76 (*Lie bracket*)

For $\vec{a} = \vec{a}(\vec{x})$, $\vec{b} = \vec{b}(\vec{x})$:

$$[\vec{a},\vec{b}] \equiv \vec{a} \cdot \vec{\partial}_{\vec{x}}\vec{b} - \vec{b} \cdot \vec{\partial}_{\vec{x}}\vec{a}.$$

Proposition 77

$$[\vec{a},\vec{b}] = \vec{\partial}_{\vec{x}} \cdot (\vec{a} \wedge \vec{b}) - \vec{b}\vec{\partial}_{\vec{x}} \cdot \vec{a} + \vec{a}\vec{\partial}_{\vec{x}} \cdot \vec{b}.$$

Proof 77

$$\vec{\partial}_{\vec{x}} \cdot (\vec{a} \wedge \vec{b}) \underset{(41)}{\overset{P18}{=}} (\vec{\partial}_{\vec{x}} \cdot \vec{a})\vec{b} - (\vec{\partial}_{\vec{x}} \cdot \vec{b})\vec{a} =$$
$$\vec{b}(\vec{\partial}_{\vec{x}} \cdot \vec{a}) + (\vec{a} \cdot \vec{\partial}_{\vec{x}})\vec{b} - \vec{a}(\vec{\partial}_{\vec{x}} \cdot \vec{b}) - (\vec{b} \cdot \vec{\partial}_{\vec{x}})\vec{a}$$
$$\Rightarrow [\vec{a},\vec{b}] \overset{def\,76}{=} (\vec{a} \cdot \vec{\partial}_{\vec{x}})\vec{b} - (\vec{b} \cdot \vec{\partial}_{\vec{x}})\vec{a} =$$
$$\vec{\partial}_{\vec{x}} \cdot (\vec{a} \wedge \vec{b}) - \vec{b}\vec{\partial}_{\vec{x}} \cdot \vec{a} + \vec{a}\vec{\partial}_{\vec{x}} \cdot \vec{b}.$$

Proposition 78

For $\vec{a} = \vec{a}(\vec{x})$, $\vec{b} = \vec{b}(\vec{x})$, $\vec{c} = \vec{c}(\vec{x})$:

$$(\vec{c} \wedge \vec{b}) \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a}) = \vec{b} \cdot \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{a}} \cdot \vec{c} - \vec{c} \cdot \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{a}} \cdot \vec{b}$$
$$= \vec{b} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{c}) - \vec{c} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{b}) + [\vec{c},\vec{b}] \cdot \vec{a}.$$

Proof 78

$$(\vec{c} \wedge \vec{b}) \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a}) \overset{(42)}{=} \vec{c} \cdot \Big(\vec{b} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a})\Big) \overset{proof\,75}{=}$$
$$\vec{c} \cdot \Big(\vec{b} \cdot \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{a}} - \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{a}} \cdot \vec{b})\Big) = \vec{b} \cdot \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{a}} \cdot \vec{c} - \vec{c} \cdot \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{a}} \cdot \vec{b}$$
$$= \vec{b} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{c}) - \vec{b} \cdot \dot{\vec{\partial}}_{\vec{x}}(\vec{a} \cdot \dot{\vec{c}})$$
$$- \vec{c} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{b}) + \vec{c} \cdot \dot{\vec{\partial}}_{\vec{x}}(\vec{a} \cdot \dot{\vec{b}}) =$$
$$\vec{b} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{c}) - \vec{c} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{b})$$
$$+ \vec{c} \cdot \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{b}} \cdot \vec{a}) - \vec{b} \cdot \dot{\vec{\partial}}_{\vec{x}}(\dot{\vec{c}} \cdot \vec{a}) \overset{def\,76}{=}$$
$$= \vec{b} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{c}) - \vec{c} \cdot \vec{\partial}_{\vec{x}}(\vec{a} \cdot \vec{b}) + [\vec{c},\vec{b}] \cdot \vec{a}.$$

Proposition 79

For $\vec{a} = \vec{a}(\vec{x})$, $\vec{b} = \vec{b}(\vec{x})$:

$$\vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b}) = \dot{\vec{b}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \vec{a}) + \dot{\vec{\partial}}_{\vec{x}} \cdot (\vec{a} \wedge \dot{\vec{b}}) =$$
$$(\vec{a} \wedge \vec{\partial}_{\vec{x}}) \cdot \vec{b} + \vec{a} \cdot \vec{\partial}_{\vec{x}}\vec{b} - \vec{a}\vec{\partial}_{\vec{x}} \cdot \vec{b}.$$

Proof 79

Jacobi identity (44) $\Rightarrow$

$$\vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b}) + \dot{\vec{\partial}}_{\vec{x}} \cdot (\dot{\vec{b}} \wedge \vec{a}) + \dot{\vec{b}} \cdot (\vec{a} \wedge \dot{\vec{\partial}}_{\vec{x}}) = 0 \Rightarrow$$
$$\vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b}) = \dot{\vec{b}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \vec{a}) + \dot{\vec{\partial}}_{\vec{x}} \cdot (\vec{a} \wedge \dot{\vec{b}}) \underset{(39)}{\overset{(41)}{=}}$$
$$(\vec{a} \wedge \vec{\partial}_{\vec{x}}) \cdot \vec{b} + \vec{a} \cdot \vec{\partial}_{\vec{x}}\vec{b} - \vec{a}\vec{\partial}_{\vec{x}} \cdot \vec{b}.$$

Proposition 80

For $\vec{a} = \vec{a}(\vec{x})$, $\vec{b} = \vec{b}(\vec{x})$:

$$\dot{\vec{\partial}}_{\vec{x}} \cdot (\vec{a} \wedge \dot{\vec{b}}) = \dot{\vec{a}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{b}}) - \dot{\vec{b}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{a}}) =$$
$$(\vec{b} \wedge \vec{\partial}_{\vec{x}}) \cdot \vec{a} + \vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b})$$
$$- (\vec{a} \wedge \vec{\partial}_{\vec{x}}) \cdot \vec{b} - \vec{b} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a}).$$

Proof 80

Jacobi identity (44) $\Rightarrow$

$$\dot{\vec{\partial}}_{\vec{x}} \cdot (\vec{a} \wedge \dot{\vec{b}}) + \dot{\vec{a}} \cdot (\dot{\vec{b}} \wedge \dot{\vec{\partial}}_{\vec{x}}) + \dot{\vec{b}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{a}}) = 0 \Rightarrow$$
$$\dot{\vec{\partial}}_{\vec{x}} \cdot (\vec{a} \wedge \dot{\vec{b}}) = \dot{\vec{a}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{b}}) - \dot{\vec{b}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{a}}) =$$
$$\dot{\vec{a}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \vec{b}) + \vec{a} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{b}})$$
$$- \dot{\vec{b}} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \vec{a}) - \vec{b} \cdot (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{a}}) \overset{(39)}{=}$$
$$(\vec{b} \wedge \vec{\partial}_{\vec{x}}) \cdot \vec{a} + \vec{a} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{b})$$
$$- (\vec{a} \wedge \vec{\partial}_{\vec{x}}) \cdot \vec{b} - \vec{b} \cdot (\vec{\partial}_{\vec{x}} \wedge \vec{a}).$$

Proposition 81

$$A \times (\vec{\partial}_{\vec{x}} \wedge \vec{b}) =$$
$$A \cdot \vec{\partial}_{\vec{x}}\vec{b} - \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{b}} \cdot A = A \wedge \vec{\partial}_{\vec{x}}\vec{b} - \dot{\vec{\partial}}_{\vec{x}}\dot{\vec{b}} \wedge A.$$

with the commutator product $A \times B$ of multivectors *A*,*B* (45).

Proof 81

$$A \times (\vec{\partial}_{\vec{x}} \wedge \vec{b}) \overset{(45)}{=} -(\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{\vec{b}}) \times A \overset{(46)}{=}$$
$$-\dot{\vec{\partial}}_{\vec{x}}\dot{\vec{b}} \cdot A + A \cdot \vec{\partial}_{\vec{x}}\vec{b} \overset{(46)}{=} -\dot{\vec{\partial}}_{\vec{x}}\dot{\vec{b}} \wedge A + A \wedge \vec{\partial}_{\vec{x}}\vec{b}.$$

Proposition 82

For $A = \langle A \rangle_r = A(\vec{x})$, $B = \langle B \rangle_s = B(\vec{x})$:

$$\dot{A} \wedge \dot{\vec{\partial}}_{\vec{x}} \wedge \dot{B} = (-1)^r \vec{\partial}_{\vec{x}} \wedge (A \wedge B) =$$
$$A \wedge \vec{\partial}_{\vec{x}} \wedge B + (-1)^{r+s(r+1)} B \wedge \vec{\partial}_{\vec{x}} \wedge A.$$

Proof 82

$$\dot{A} \wedge \dot{\vec{\partial}}_{\vec{x}} \wedge \dot{B} \overset{(40)}{=} (-1)^r \vec{\partial}_{\vec{x}} \wedge (A \wedge B),$$

$$\dot{A} \wedge \dot{\vec{\partial}}_{\vec{x}} \wedge \dot{B} = A \wedge \vec{\partial}_{\vec{x}} \wedge B + (\dot{A} \wedge \dot{\vec{\partial}}_{\vec{x}}) \wedge B \overset{(40)}{=}$$
$$A \wedge \vec{\partial}_{\vec{x}} \wedge B + (-1)^r (\dot{\vec{\partial}}_{\vec{x}} \wedge \dot{A}) \wedge B \overset{(40)}{=}$$
$$A \wedge \vec{\partial}_{\vec{x}} \wedge B + (-1)^{r+s(r+1)} B \wedge (\vec{\partial}_{\vec{x}} \wedge A).$$

4. Conclusion

This article first summarized important geometric algebra relationships, which are necessary for the thorough and explicit development of the vector differential calculus part of universal geometric calculus.

It then showed how to differentiate multivector functions by a vector, including the results of standard vector analysis. The vector differential relationships are proven in a very explicit step by step way, enabling the reader, who is unfamiliar with the algebraic techniques to get complete comprehension. It may thus serve as important reference material for studying and applying vector differential calculus.

Future work in a similar manner should be done to elucidate the calculus with *multivector* derivatives.

Acknowledgements

"Soli deo gloria."
(Latin: To God alone be the glory.) [8]

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[7] Hebrews, chapter 11, verses 1 to 3, Holy Bible, New International Version.

[8] Johann Sebastian Bach.