PLAY WITH GEOMETRY –
ANIMATED AND INTERACTIVE, FREE, INSTANT ACCESS,
ONLINE GEOMETRIC ALGEBRA JAVA APPLETS
WITH CINDERELLA

Eckhard M.S. Hitzer and Luca Redaelli *

Department of Mechanical Engineering, Fukui University, Fukui 910-8507, Japan
* Department of Mechanical Engineering, Politecnico di Milano, Milano, Italy

ABSTRACT

Conventional illustrations of elementary relations and physical applications of geometric algebra are helpful, but restricted in communicating full generality and time dependence. The main restrictions are one special perspective in each graph and the static character of such illustrations. Several attempts have been made to overcome such restrictions. But up till now very little animated and interactive, free, instant access, online material is available.

This talk presents therefore a set of well over 60 newly developed (freely online accessible[1]) JAVA applets. These applets range from the elementary concepts of vector, bivector, outer product and rotations to triangle relationships, oscillations and polarized waves. A special group of 21 applets illustrates three geometrically different approaches to the representation of conics; and even more ways to describe ellipses. Finally Clifford's circle chain theorem is illustrated for two to eight primary circles. The interactive geometry software Cinderella[2] was used for creating these applets. Some construction principles will be explained and a number of applets will be demonstrated. The interactive features of many of the applets invite the user to freely explore by a few mouse clicks as many different special cases and perspectives as he likes. This is of great help in "visualizing" the geometry encoded in the concepts and formulas of Geometric Algebra.

1. INTRODUCTION

1.1 Geometric Algebra

I do think that at the beginning of the 21st century, we have strong reasons to believe, that all of mathematics can be formulated in a single unified universal[8] way, with concrete geometrical foundations. Why is geometry so important? Because it is that aspect of mathematics, which we can imagine and visualize. The branch of mathematics, which Grassmann said "far surpasses" all others [9] is now known under the name universal *geometric calculus*. The algebraic "grammar" geometric calculus uses is geometric algebra. Some of the applications of geometric algebra in the engineering sciences are: computer vision, graphics and reconstruction, robotics, signal and image processing, structural dynamics, control theory, quantum computing, bioengineering and molecular design, space dynamics, elasticity and solid mechanics, electromagnetism and wave propagation, geometric and Grassmann algebras, quaternions and screw theory, automated theorem proving, symbolic algebra and numerical algorithms.[10]

1.2 Cinderella created JAVA applets

I originally learned about the JAVA based interactive geometry software Cinderella[2] from Leo Dorst[3]. He also used it to do some geometric algebra illustrations. Here I have elaborated his approach, producing a variety of online JAVA applets, part of them animated and part of them interactive. Especially the interactive applets invite to explore the full meaning of geometric relationships in a visual way. With permission, I have freely drawn on the material of David Hestenes' *New Foundations for Classical Mechanics*[4]. You can tour the applets without using this book. But if you have it, it might increase the fun of reading it. Another important source is Dorst, Mann and Bouma's geometric algebra MATLAB tutorial GABLE[5].

Particularly in the section on conics I have compiled an instructive variety of ways to obtain conics (points, pairs of intersection lines, circles, ellipses, hyperbolas and parabolas).

Finally, W.K. Clifford's circle chain theorem in the ordinary Euclidean plane refers to a "chain of theorems" of increasing complexity. Every one of this infinite

sequence of theorems must be true for the whole to be true. You will find the illustrations for n=2 to n=8 primary circles through one point O.

The applets work with Netscape 6.2, Explorers 5 and 6, but not with Netscape 4.7. In each group the first applet may take some time, because your browser has to load the 412k cindyrun.jar file. Later on some of the more involved applets may also take a few minutes to appear on your screen. One needs to take into account that Cinderella is inherently two dimensional. That is three dimensional projections may sometimes get disarranged. The easiest way to resolve this is to use the refresh button of the browser to start a particular applet afresh.

2. VECTORS AND BIVECTORS

2.1 Vectors

This first group of applets illustrates how vectors can be represented by arrows, which are identical up to translations. The vector length and orientation (opposite directions) can be changed by dilations. The addition of vectors is defined as attaching one vector to the tip of another and drawing the result from the beginning of the first to the end of the second. The additive inverse of a vector brings it back to zero.

Special properties of vector addition are commutativity (Fig. 1a), associativity, vector subtraction. The properly defined geometric multiplication of vectors also has a multiplicative inverse, that is in geometric algebra one can naturally divide by vectors as well. One particular application of vector division (Fig. 1b) is the calculation of projection and rejection of one vector with respect to another by using the inner and outer product parts of the geometric product of vectors.

All these properties of vectors are illustrated in a set of interactive applets.
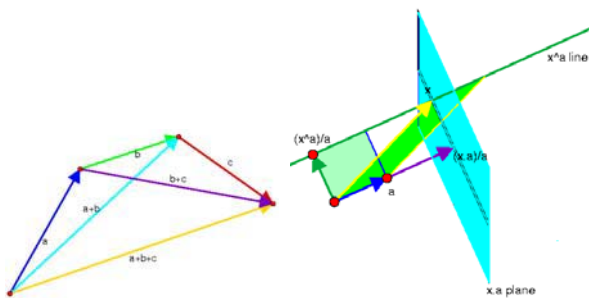

Fig. 1 Commutativity of vector addition. Vector division.

2.2 Bivectors

Area elements, the non scalar parts of the geometric product of vectors are simply called bivectors. Multiplying by a three-volume element results in a dual vector perpendicular to the bivector. The inner product is naturally extended to include products of bivectors and vectors. The inner product of a vector and a bivector is a vector in the bivector plane perpendicular to the projection of the vector into the bivector plane.

The projection and rejection of a vector with respect to a bivector uses the *same* algebraic expressions as that of the vector vector case.

Geometric algebra easily allows to calculate the intersection line vector (meet M in Fig. 2) of two planes without ambiguity in one step.
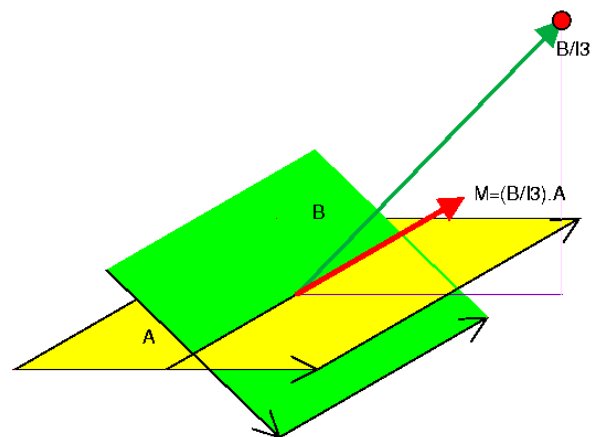

Fig. 2 The intersection (meet M) of two planes A,B.

3. OUTER PRODUCT AND TRIANGLE

The outer product of two vectors (Fig. 3) is the non-scalar part of the geometric product resulting in an oriented area element, which can be defined by one vector sweeping along the other. The JAVA applets make it possible to make this "sweeping" visible as an animated motion. But the outer product is not restricted to two dimensions. Sweeping an area element bivector along a third vector gives an oriented volume element trivector. It does not matter which side face of a parallelepiped is used for sweeping, we always get the same trivector with positive or negative orientation.

The outer product is distributive (Fig. 4). The outer products of the side vectors of a triangle immediately lead to the law of (oriented) sines. Both facts are

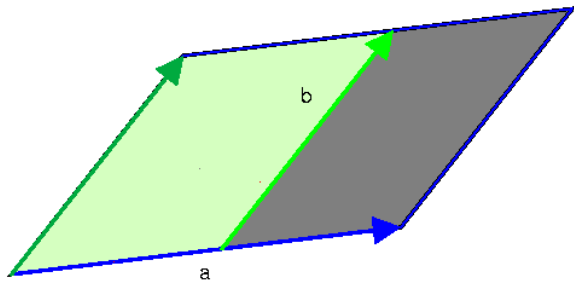illustrated by applets in the triangle category.
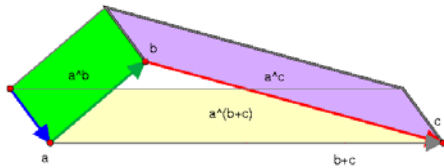

Fig. 3 Outer product of two vectors.

Fig. 4 The outer product is distributive.

4. ROTATIONS

Geometric algebra offers elegant ways to describe rotations without introducing coordinates or matrices. E.g. the full geometric multiplication of a vector with a (unit area) bivector gives a 90 degree rotation (Fig. 5a). Capitalizing on this one can easily demonstrate how the exponential of a bivector yields arbitrary plane rotations (Fig. 5b).

Another simple way to describe rotations is through two successive reflections at planes which comprise half of the angle of the resulting rotation (Fig. 6a). These planes of reflection can be denoted by unit vectors $\vec{n}, \vec{m}$ perpendicular to them. The geometric product $\vec{n}\vec{m}$ of these two unit bivectors is the desired rotation operator (rotor), which must be applied in both orders $\vec{n}\vec{m}$ and $\vec{m}\vec{n}$ from the left and right. Rotors can be written in terms of the angle of the two vectors $\vec{n}, \vec{m}$ and of the area element of the plane defined by $\vec{n}, \vec{m}$. This brings us full swing back to the exponential description of rotations already mentioned, this time in the full general form proper for the rotation of objects in higher dimensions. The same coordinate free rotor rotation can be applied to vectors and bivectors (Fig. 6b).

The exponential rotor form of rotations gives us a coordinate and matrix free understanding of the well known Euler angles (Fig. 7a). And it leads on to an incredibly simple algorithm for interpolating between arbitrary orientations (Fig. 7b), e.g. of plane area elements. This is of great use for both computer graphics and robot manipulations.
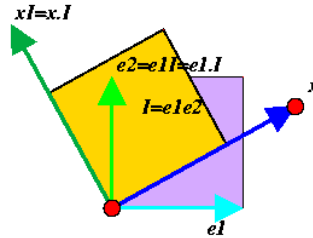

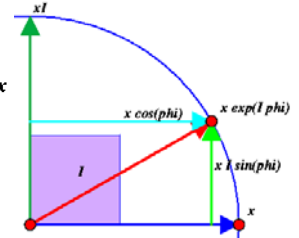Fig. 5a) Right angle rotation with unit area element.

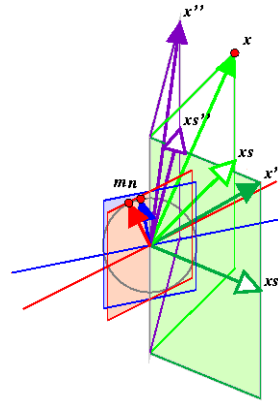Fig. 5b) Rotation with exponential of unit area element.
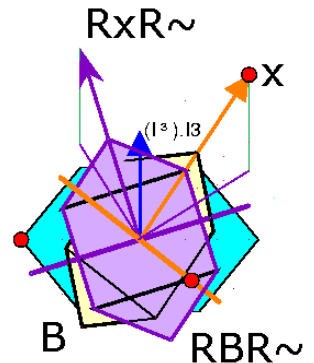

Fig. 6a) Rotation by two plane reflections.
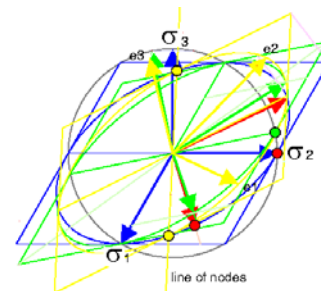

Fig. 6b) General rotation by rotor R operation.

Fig. 7a) Euler angles of rotation.
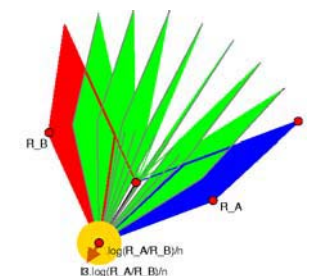

Fig. 7b) Rotation interpolation.

5. OSCILLATIONS AND WAVES

5.1 Oscillations

All basic forms of symmetric and antisymmetric oscillations in longitudinal (Fig. 8), transverse (Fig. 9) and circular (Fig. 10) directions are easily formulated by geometric algebra and visualized by animated applets. As one may expect by now, geometric algebra can again do it without coordinates. It really brings out the full

geometric nature of oscillations without artificially resorting to a Cartesian coordinate system.
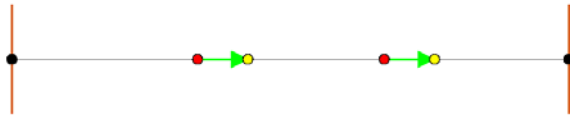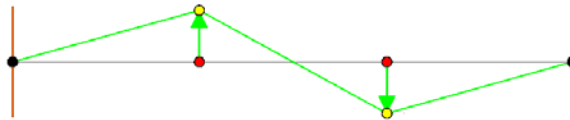

Fig. 8 Symmetric longitudinal oscillation.

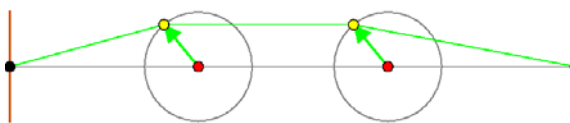Fig. 9 Antisymmetric transverse oscillation.

Fig. 10 Symmetric circular oscillation.

5.2 Circular polarized waves

Circular polarized waves standing, rotating or traveling in arbitrary directions of any wavelengths receive an elegant fully geometric description in geometric algebra. The applets in this category show right and left polarized circular waves (Fig. 11) which can be turned interactively, or observed traveling by animation. The user can continuously choose the wavelength interactively.
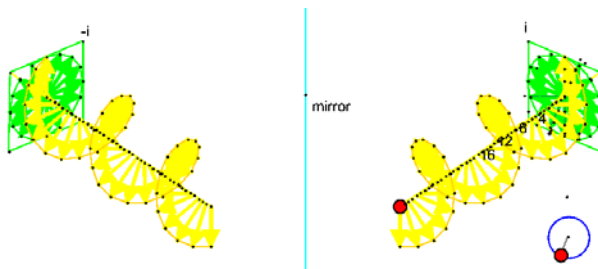

Fig. 11 Right and left circular polarized waves.

6. CONIC INTERSECTIONS

Conic intersections are a vast and immensely important topic in geometry, mathematics and its applications. The geometric and analytic properties of these particular curves are most fascinating, but the incredible variety of equivalent descriptions can be very confusing and rather elusive for the student. To address this subject a comprehensive study has been carried out to first show the "conic" nature of circles, ellipses, parabolas, hyperbolas, points and intersecting pairs of lines, and to fully illustrate with interactive "hands on" the variety of equivalent descriptions. Cinderella even allows to capture the infinite properties of parabolas and hyperbolas by visualizing the spatial infinity in its spherical view.

6.1 Intersecting the cone

This is now meant to be taken literally as the procedure for obtaining all conics. In both finite Euclidean view and in the (infinite) spherical view the position and orientation of a plane intersecting a cone can be freely manipulated interactively (Fig. 12a). This creates before the eyes of the user all known varieties of conics. The spherical view (Fig. 12b) shows how finite ellipses become parabolas closed at infinity and then continuously open up to become hyperbolas. By just dragging the mouse pointer across the screen the user can freely explore, what many pages of text books and many conventional illustrations just cannot show due to their limitations to selective static perspectives.
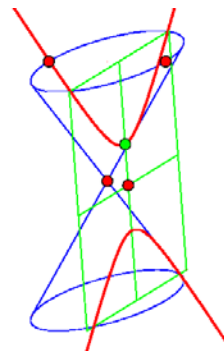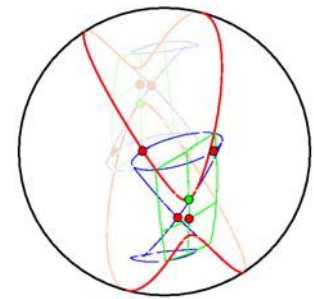

Fig. 12a) Conic intersection. Fig. 12b) Sperical projection.

6.2 Semi-latus rectum formula

A famous formula for all conics is

$$r = \frac{l}{(1 + \vec{\varepsilon} \cdot \hat{r})} , \qquad (1)$$

with l the semi-latus rectum, r the length of the radius vector pointing in the unit vector direction $\hat{r}$, and $\vec{\varepsilon}$ the excentricity vector.

How formula (1) achieves to include circles, ellipses, parabolas (Fig. 13) and hyperbolas is made visible by a variety of applets. In the interactive version, the user can continuously create all conics by dragging the mouse

along the directrix line. Special animated versions show how all values of the formula sweep out the complete conics. At the same time the infinite behavior is captured again with the help of the spherical view (Fig. 14).
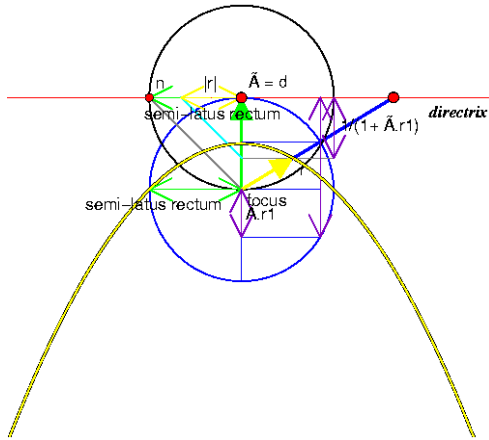

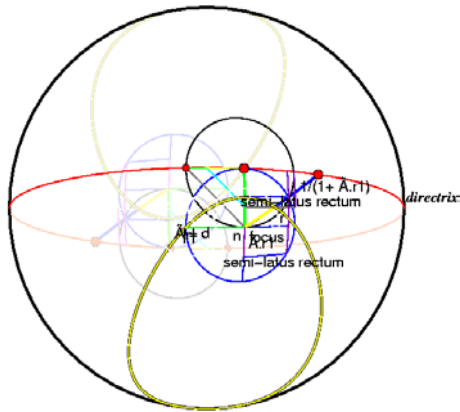Fig. 13 Semi-latus rectus generation of parabola.

Fig. 14 Parabola in spherical view closing at infinity.

6.3 Ellipse in various disguises

One conic, the ellipse has a particular variety of equivalent ways to describe it. There are besides the literal conic intersection and the semi-latus rectus formula three other interesting descriptions (Fig. 15).
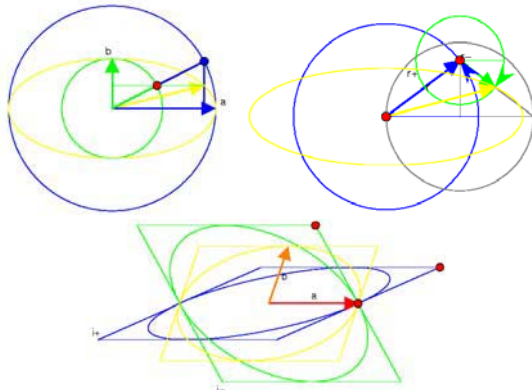

Fig. 15 Description of ellipse by polar angle (top left) two coplanar circles (top right) and two non-coplanar planes.

Yet another way to describe it as a member of the class of second order curves will be treated in the next section. The three descriptions mentioned above are illustrated in both interactive and animated ways by remarkably simple and aesthetic Cinderella applets. Needless to say that the elegant descriptions of planes as bivectors and that of rotations by rotors make all this in the framework of geometric algebra quite easy.

6.4 Second order curves

All curves of the form

$$\vec{r} = \frac{\vec{a}_0 + \vec{a}_1 \lambda + \vec{a}_2 \lambda^2}{\alpha + \lambda^2}, \qquad (2)$$

with any three vectors $\vec{a}_0$, $\vec{a}_1$, $\vec{a}_2$ and scalars λ and

α describe conics. Taking $\vec{r}$ in (2) as functions of λ

yields a specific conic. For $\alpha > 0$ ellipses, for $\alpha = 0$ parabolas and for $\alpha < 0$ hyperbolas. It sounds incredible, but the truth of (2) can be immediately visualized using interactive and animated JAVA applets created with Cinderella (Fig. 16). In the interactive version all three vectors and the values of the scalar parameters can be freely varied to provide full intuition on how the formula works. The spherical view captures again what happens at infinity.
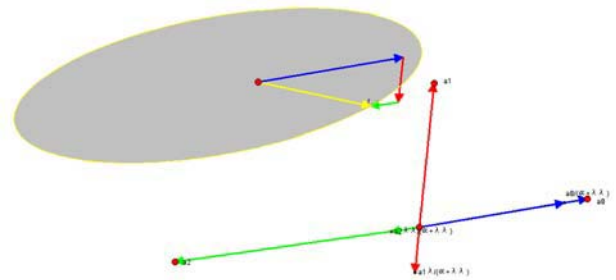

Fig. 16 Ellipse as second order curve.

7. Clifford's chain circle theorem

Clifford's circle chain theorem in the ordinary Euclidean plane refers to a "chain of theorems" of increasing complexity. Every one of this infinite sequence of theorems must be true for the whole to be true. It begins with two circles passing through a common point O (n=2). The next theorem in the chain is for the case of three circles through a common point O (n=3) and so forth for n=4,5,6, ... If one takes the point O to infinity the n circles become n straight lines (circles with infinite

radii.) The interactive JAVA applets show illustrations for the cases n=2, ... ,8 circles through O (Figs. 17, 18). These illustrations are each accompanied by a detailed description. Each illustration shows the magnificent intersection properties that lead in odd cases to a final circle and in even cases to a final point.

It is most interesting to note that a purely algebraic proof of the case n=5, based on the *conformal* geometric algebra model of the Euclidean plane has been found[7]. With the help of the geometric product of vectors one can construct an algebra of Clifford brackets, with which the proof becomes possible. This illustrates how even Clifford's own inventions submit over time to the descriptive power of his geometric algebra.
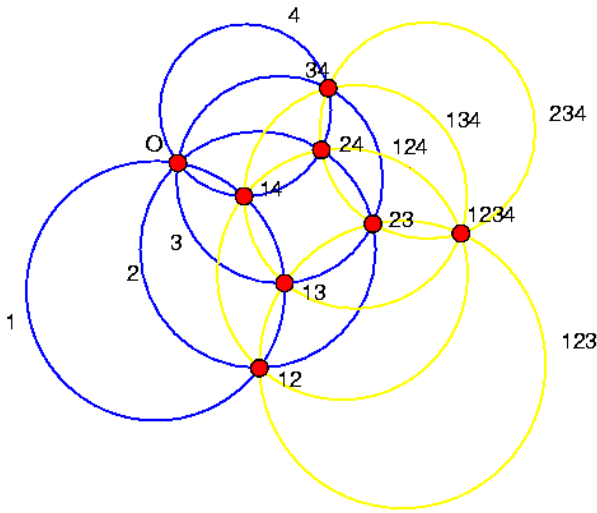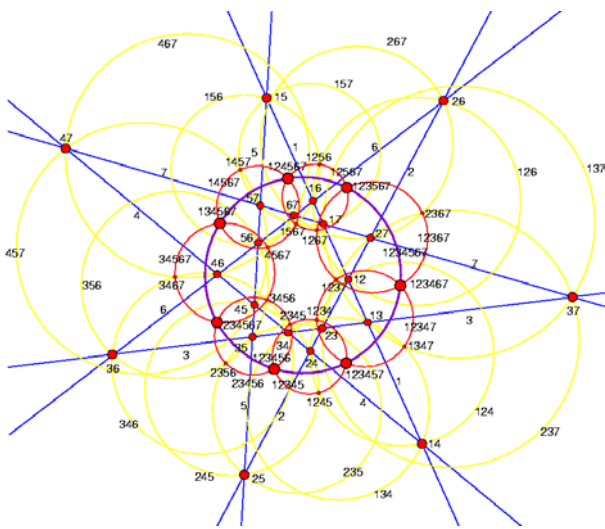

Fig. 17 Clifford's circle chain theorem (n=4).

Fig. 18 Clifford's circle chain theorem (n=7).

REFERENCES

1. http://sinai.mech.fukui-u.ac.jp/gcj/software/GAcindy/GAcindy.htm
2. http://www.cinderella.de
3. http://carol.wins.uva.nl/~leo/cinderella/index.html
4. D. Hestenes, *New Foundations for Classical Mechanics*, Kluwer Academic Publishers, Dordrecht, 1999.
5. http://carol.wins.uva.nl/~leo/GABLE/index.html
6. R. Penrose *The Mathematics of William Kingdon Clifford: A Personal Reflection*, in M. Chisholm *Such Silver Currents - The Story of William and Lucy Clifford 1845-1929*, Lutterworth, Cambridge 2002.
7. Hongbo Li, *Automated Theorem Proving in the Homogeneous Model with Clifford Bracket Algebra*, in L. Dorst, C. Doran, J. Lasenby, Applications of Geometric Algebra in Computer Science and Engineering, Birkhaeuser, Boston, 2002.
8. D. Hestenes, G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Kluwer Academic Publishers, Dordrecht, 1984.
9. H. Grassmann, (F. Engel, editor) *Die Ausdehnungslehre von 1844 und die Geometrische Analyse*, vol. 1, part 1, Teubner, Leipzig, 1894.
10. L. Dorst, C. Doran, J. Lasenby, Applications of Geometric Algebra in Computer Science and Engineering, Birkhaeuser, Boston, 2002.
11. Colossians 1:17, The Bible, New International Version, International Bible Society, Colorado, 1984.
12. Johann Sebastian Bach.