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Geometric Calculus for Engineers

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This paper treats important questions at the interface of mathematics and the engineering sciences. It starts off with a quick quotation tour through 2300 years of mathematical history. At the beginning of the 21st century, technology has developed beyond every expectation. But do we also learn and practice an adequately modern form of mathematics? The paper argues that this role is very likely to be played by (universal) *geometric calculus*. The fundamental *geometric product* of vectors is introduced. This gives a quick-and-easy description of rotations as well as the ultimate geometric interpretation of the famous quaternions of Sir W.R. Hamilton. Then follows a one page review of the historical roots of geometric calculus. In order to exemplify the role geometric calculus for the engineering sciences three representative examples are looked at in some detail: elasticity, image geometry and pose estimation. Finally the value of geometric calculus for teaching, research and development and its worldwide impact are commented.

***Key Words* :** Geometric calculus, applied geometric calculus , engineering mathematics, design of mathematics, teaching of mathematics for engineering students

1. Introduction

1.1 Mathematicians life

1. A point is that which has no part. 2. A line is breadthless length. … *Euclid*[1]

…never to accept anything as true if I did not have evident knowledge of its truth; …We have an idea of that which has infinite perfection. … The origin of the idea could only be the real existence of the infinite being that we call God. *Rene Descartes*[2]

But on the 16th day of the same month … an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. … Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; *Sir William R. Hamilton*[3]

…extension theory, which extends and intellectualizes the sensual intuitions of geometry into general, logical concepts, and, with regard to abstract generality, is not simply one among other branches of mathematics, such as algebra, combination theory, but rather far surpasses them, in that all fundamental elements are unified under this branch, which thus as it were forms the keystone of the entire structure of mathematics. *Hermann Grassmann*[4]

… for geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child. *William K. Clifford*[5]

The symbolical method, however, seems to go more deeply into the nature of things. It … will probably be increasingly used in the future as it becomes better understood and its own special mathematics gets developed. *Paul A.M. Dirac* [6]

The geometric operations in question can in an efficient way be expressed in the language of Clifford algebra. *Marcel Riesz*[7]

This was Grassmann's great goal, and he would surely be pleased to know that it has finally been achieved, although the path has not been straightforward. *David Hestenes*[8]

1.2 Design of mathematics

Over a span of more than 2300 years, Euclid, Descartes, Hamilton, Grassmann, Clifford, Dirac, Riesz, Hestenes and others all contributed significantly to the development of modern mathematics. Today we enjoy more than ever the fruits of their creative work. Nobody can think of science and technology, research and development, without acknowledging the great reliance on mathematics from beginning to end.

Many forms of mathematics have been developed over thousands of years: geometry, algebra, calculus, matrices, vectors, determinants, etc. All of which find rich applications in the engineering sciences as well. But it takes many years in school and university to train students until they reach the level of mathematics needed for today's advanced requirements.

Yet very important questions seem to largely go unnoticed: Is the present way we learn, exercise, apply and research mathematics really the most efficient and satisfying way there is? In an age, where we can double the speed of computers every 3 years, is there no room for improvement for the teaching and application of one of our most fundamental tools mathematics? How should mathematics be designed, so that students, researchers and engineers alike will benefit most from it?

I do think that at the beginning of the 21st century, we have strong reasons to believe, that all of mathematics can be formulated in a single unified universal[9] way, with concrete geometrical foundations. Why is geometry so important? Because it is that aspect of mathematics, which we can imagine and visualize. The branch of mathematics, which Grassmann said far surpasses all others [4] is now known under the name (universal) *geometric calculus*.

Its formulation is at the same time surprisingly simply, clear and straightforward in teaching and applications. In my experience it is also of great appeal for students.

The rest of this paper is divided into four major sections. In the next section we will see how geometric calculus defines a new way to multiply vectors. This immediately gives us a new method to do rotations and teaches us the nature of Hamilton's famous quaternions.

Section three briefly reviews the history of geometric calculus.

Section four takes up three examples of geometric calculus applied to elasticity, image geometry and pose estimation. Many other applications more closely related to other fields of engineering exist as well.

Section five outlines the general benefits for teaching, research and development. It also comments on the situation we face in Asia.

2. New vector product makes rotations easy

At the (algebraic) foundations of Geometric Calculus [9] lies a new definition of vector multiplication, the geometric product. It was introduced by Grassmann [4] and Clifford [14] as a combination of inner product and outer product. The outer product was invented by Grassmann before that. The outer product $\vec{a} \wedge \vec{b}$ of two vectors is the (oriented) parallelogram area spanned by two vectors $\vec{a}$ and $\vec{b}$, illustrated in Fig. 2.1.

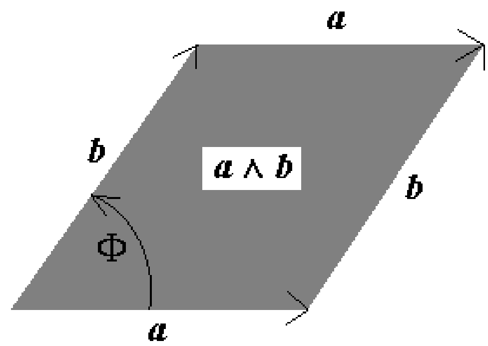

Fig. 2.1 Oriented parallelogram area $\vec{a} \wedge \vec{b}$

The oriented unit area is denoted by **i**. ***But a warning is in order:* i *is NOT to be confused with the imaginary unit of the complex numbers introduced by Gauss!*** In two dimensions the area unit **i** is of similar importance as the unit length 1 is for one dimension.

The "new" geometric product then simply reads

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b} \,. \qquad (2.1)$$

Yes here we add scalar numbers (inner product) and areas (outer product), but nobody has a problem to put balls and discs in one box, without confusing them. The usual multiplication of real numbers is associative, i.e.

$$(2*3)*4 = 6*4 = 2*(3*4) = 2*12 = 24 \,. \qquad (2.2)$$

It simply doesn't matter where you put the brackets, the result is the same. The same is true for the geometric product of vectors.

Let us now again take two vectors, but of unit length: $\hat{a}$, $\hat{b}$. Multiplying their geometric product $\hat{a}\hat{b}$ once more with $\hat{a}$ we get $\hat{b}$ again:

$$\hat{a}(\hat{a}\hat{b}) = (\hat{a}\hat{a})\hat{b} = 1\hat{b} = \hat{b} \qquad (2.3)$$

What we have just done is to rotate the vector $\hat{a}$ into the vector $\hat{b}$ by multiplying it with $\hat{a}\hat{b}$. This is a rotation by the angle Φ as seen in Fig. 2.1. This is indeed a very general description of rotations in the plane of the rotation. It can be applied to any vector in order to rotate it in the plane of $\hat{a}$ and $\hat{b}$ by the angle Φ. The product $\hat{a}\hat{b}$ deserves thus a separate name

$$R_{ab} \equiv \hat{a}\hat{b} = \cos\Phi + \mathbf{i}\sin\Phi \,. \qquad (2.4)$$

Remember that the inner product of two unit vectors is just cosΦ and the area of the parallelogram they span is base*height = 1*sinΦ = sinΦ. **i** and Φ describe the rotation as good as $\hat{a}$ and $\hat{b}$. A Φ=90° rotation with cos90°=0 and sin90°=1 is therefore given by

$$R(90^o) \equiv \mathbf{i} \,. \qquad (2.5)$$

Rotating twice by 90° gives 180°, i.e. turns each vector into the opposite direction. We therefore have:

$$R(90^o)R(90^o) = \mathbf{ii} = \mathbf{i}^2 = -1 \,. \qquad (2.6)$$

Independent of this, the Irish mathematician Sir William R. Hamilton was thinking in 1843 about how to describe rotations in three dimensions in the most simple way. While making a walk he suddenly found the answer

$$\mathbf{i}^2 = -1,\ \mathbf{j}^2 = -1,\ \ \mathbf{k}^2 = -1, \qquad (2.7a)$$

$$\mathbf{ijk} = 1 \,. \qquad (2.7b)$$

Hamilton was so happy that he carved (2.7) immediately into a stone bridge. He called the four entities {1, **i**, **j**, **k**} *quaternions* (=fourfold). [3,10]

For describing a rotation with a quaternion *q*, we just need to choose the angle of rotation ϑ and the axis [unit vector (u_1, u_2, u_3) in the direction of the axis]:

$$q \equiv 1\cos\frac{\vartheta}{2} + (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\sin\frac{\vartheta}{2}, \qquad (2.8a)$$

$$\tilde{q} \equiv 1\cos\frac{\vartheta}{2} - (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\sin\frac{\vartheta}{2} \,. \qquad (2.8b)$$

The rotation of any vector $\vec{x}$ is then given as [11]

$$\vec{x}' = \tilde{q}\vec{x}q \,, \qquad (2.9)$$

which obviously is much more direct, simpler and computationally more efficient than the usual 3 by 3 matrix notation. [In (2.3) the rotation operation was one sided, here it is two sided, because the part of $\vec{x}$ not in the rotation plane must not change.] Instead of *nine* matrix elements, we need only *four* parameters ϑ, u_1, u_2, u_3 in (2.9).

Sir Hamilton knew that his new description of rotations was revolutionary, but what he did not know and even many of today's scientists do not yet know is the geometric meaning of { **i**, **j**, **k** }. But given that **i** represents in two dimensions the (oriented) unit area element, it is natural to take { **i**, **j**, **k** } to represent the three mutually perpendicular (oriented) unit area elements of a cube, as in Fig. 2.2.

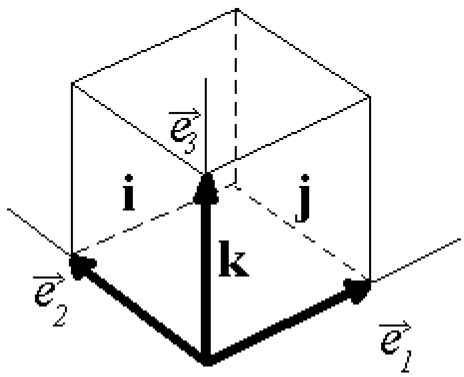

Fig. 2.2 Oriented unit area elements **i**, **j**, **k** of a cube.

This interpretation is indeed consistent and valid in the framework of Geometric Calculus.[12] In three dimensions, adding plane area elements, is quite similar to adding vectors. The result is a new area element. The sum

$$\mathbf{u} \equiv (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \qquad (2.10)$$

in (2.8) is therefore just a new (oriented unit) area element perpendicular to the axis

$$\vec{u} \equiv (u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3) \,. \qquad (2.11)$$

Just like as in (2.4) each quaternion *q* (2.8) can therefore be written as a product of two unit vectors in the plane

$$R_q \equiv q = \cos\frac{\vartheta}{2} + \mathbf{u}\sin\frac{\vartheta}{2} = \hat{a}_\mathbf{u}\hat{b}_\mathbf{u} \,. \qquad (2.12)$$

3. Creation of Geometric Calculus

300 BC	Euclid	*Geometry*
250 AD	Diophantes	*Algebra*
1637	Descartes	*Coordinates*
1798	Gauss	*Complex Algebra*
1843	Hamilton	*Quaternions*
1844	Grassmann	*Extensive Algebra*
1854	Cayley	*Matrix Algebra*
	Boole	
1878	Clifford	*Geometric Algebra*
	Sylvester	*Determinants*
1881	Gibbs	*Vector Calculus*
1890	Ricci	*Tensors*
1908	Cartan	*Differential Forms*
1928	Dirac	*Spin Algebra*
	Pauli	
1957	Riesz	*Clifford Numb., Spinors*
1966	Hestenes	*Space-Time Algebra*
now		*Geometric Calculus*

Fig. 3.1 History of Geometric Calculus [13]

2300 years ago the ancient Greek scholar Euclid described (synthetic) geometry in his famous *13 books of the elements*. 50 years later (syncopated) algebra entered the stage through the work of Diophantes. Euclid's work [1] was first printed in 1482. But it took yet another 150 years until the French Jesuit monk Rene Descartes [2] invented analytic geometry. Every student knows him through his introduction of rectangular Cartesian coordinates. After the French revolution, Gauss and Wessel introduced the algebra of complex numbers.

The following 19^{th} century proved very fruitful for the development of modern mathematics. The Irish mathematician Sir William R. Hamilton discovered the quaternions [3,10] in 1843, providing a most elegant way to describe rotations. One year later published the German mathematician Herrmann Grassmann his now famous work on extensive algebra.[4] Yet at first only few mathematicians like Hamilton, later Clifford [14] and Klein and a growing number of others took notice. 10 years later showed G. Boole how algebra can be used to study locigal operations. In the same year, Cayley continued the coordinate approach of Descartes by introducing matrix algebra. Something which Grassmann had no need of in the first place.

Then came the year 1878, when Clifford [14] created the *geometric product*, in Grassmann's work it appeared as central product. After Clifford's early death (supposedly because he overworked himself repeatedly), the algebra based on the geometric product became to be known as Clifford algebra, yet following his original intent, it should better be named *geometric algebra*. Again in the same year, Sylvester continued to develop matrix algebra in the form of introducing determinants. In 1881 Gibbs' vector calculus followed, which Ricci enhanced in 1890 to tensor calculus.

In the first half of the 20^{th} century, the names of Cartan (differential forms, 1908) and of Dirac and Pauli (Spin Algebra, 1928) deserve to be mentioned. In the second half of the 20^{th} century (1957), Marcel Riesz [7] gave some lectures on Clifford Numbers and Spinors. Early in his career (1966), a young American David Hestenes came across Riesz lecture notes and created the socalled Space-Time Algebra [15], integrating classical and quantum physics. This marked the beginning of renewed interest in geometric algebra, combined with calculus. Sobczyk and Hestenes published in the early 1980ies a modern classic[9]: Clifford Algebra to Geometric Calculus – A Unified Language for Mathematics and Physics. By the beginning of the 21^{st} century it has become a truly universal geometric calculus, incorporating more or less all areas of mathematics, and starting to be extensively applied in science and technology. [16,17,29]

The proponents of geometric calculus have no doubt, that this new language for mathematics will make its way into undergraduate syllabi and even school education. Mathematics will thus become easier to understand, teach, learn and apply. As for the applications, the next section will show how geometric calculus is successfully used in engineering.

4. Geometric Calculus for Engineers

4.1 Overview

In order to get an overview of how geometric calculus supports engineering applications, let me first list some relevant topics from a recent conference[18] on applied geometric algebras in computer science and engineering:

- Computer vision, graphics and reconstruction
- Robotics
- Signal and image processing
- Structural dynamics
- Control theory
- Quantum computing
- Bioengineering and molecular design
- Space dynamics
- Elasticity and solid mechanics
- Electromagnetism and wave propagation
- Geometric and Grassmann algebras

- Quaternions and screw theory
- Automated theorem proving
- Symbolic Algebra
- Numerical Algorithms

One should note that the organizers cautioned: “Topics covered will include (but are not limited to):” and that geometric algebra itself is only the algebraic fraction of the full-blown geometric *calculus* [9]. Limitations of space prohibit any complete listing here.

4.2 Three examples of engineering applications

Trying to choose what to present from the recent engineering applications of geometric calculus is a very tough choice, because there are many good applications.

I have chosen three dealing with elasticity, image geometry and pose estimation.

4.2.1 Example 1: Elastically coupled rigid bodies[19]

Modelling elastically coupled rigid bodies is an important problem in multibody dynamics. A flexural joint has two rigid bodies coupled by a more elastic body. Such a system is shown in Fig. 4.1.

Fig. 4.1 Elastically coupled rigid bodies. Source: [19].

It is convenient to avoid specifying an origin, i.e. use a new homogeneous formulation.[8,19] Rotations R and translations T are fully integrated as twistors in screw theory. That is, any relative displacement D of two bodies can be written as

$$D \equiv TR\,,\ \ \vec{x}' = D\vec{x}\widetilde{D}\,. \tag{4.1}$$

R is the rotation of section 2 and

$$T \equiv \exp(\tfrac{1}{2}\vec{n}\vec{e}) = 1 + \tfrac{1}{2}\vec{n}\vec{e} \tag{4.2}$$

$\vec{n}$ is the translation vector and $\vec{e}$ represents an infinitely far away point in (conformal) geometric algebra [8,19]. Motion, momentum and kinetic energy are than given as

$$\frac{\partial}{\partial t}\vec{x} = V\vec{x}\,, \tag{4.3}$$

$$P = \underline{M}V\,, \tag{4.4}$$

$$E = \frac{1}{2}V \cdot P\,. \tag{4.5}$$

V is defined by

$$\frac{\partial}{\partial t}D = \frac{1}{2}VD\,. \tag{4.6}$$

Finally the potential energy of the elasticity problem can be written as a sum of basically three kinds of terms, depending on $\vartheta\mathbf{u}$ and $\vec{n}$. The first term depends only on $\vartheta\mathbf{u}$, the second on $\vartheta\mathbf{u}$ and $\vec{n}$, and the third only on $\vec{n}$. The three terms are therefore the potential energies of pure rotation, coupled rotation and translation, and pure translation.

The method described here is invariant, unambiguous, has a clear geometric interpretation and is very efficient in symbolic computation. Two researchers have applied for a patent on the use of the method described here in software for modeling and simulation.

4.2.2 Example 2: Image geometry[20]

Image processing commonly considers “Euclidean differential invariants” of the image space (picture plane times intensity). But this makes not much sense, because one can’t rotate the image surface to see its “other side”, but invariants are supposed not to change under such transformations. It also makes no sense to “mix” the physical dimensions of the picture plane with the intensity dimension by transforming one into the other.

But these inconsistencies can be helped by first introducing a log intensity domain and second making new definitions for the basic formulas of measuring angles and distances in the image space.

The log intensity means to divide by a fiducial intensity and take the logarithm

$$z(\vec{r}) = \log\left({}^{I(\vec{r})}\!\big/_{I_0}\right). \tag{4.7}$$

A definition very well adapted to the human eye functions.

The definition of measurements is not changed, considering only the picture plane. But if we look at a plane in the image space perpendicular to the picture plane, a *rotation* becomes a *shear* as shown in Fig. 4.2. In geometric algebra one continues to use a description of rotation as given by (2.9) and (2.10), but the square of $\mathbf{u}$ will be zero instead of -1.

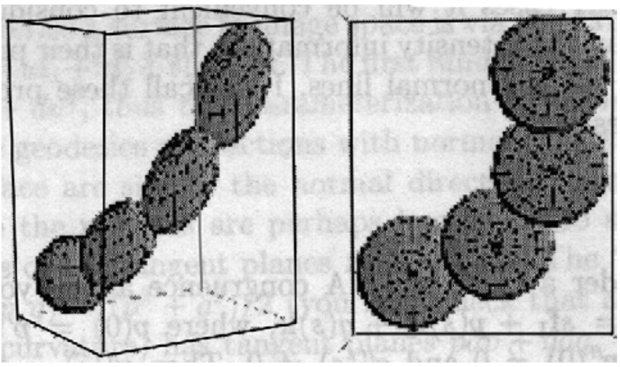

Fig. 4.2 An object is moved through image space by a parabolic rotation (left). Perpendicular view on the picture plane (right). Source: [20].

Analyzing the image surface curvature in the image space gives very natural descriptions of ruts and ridges. A ridge point, e.g. is "an extremum of (principle) curvature along the direction of the other (principle) curvature." I should be clear that when a curve has one curvature, a surface (e.g. a saddle) must have two (principal) curvatures. Fig. 4.3 shows a variety of common image transformations easily implemented with our new definition of image space.

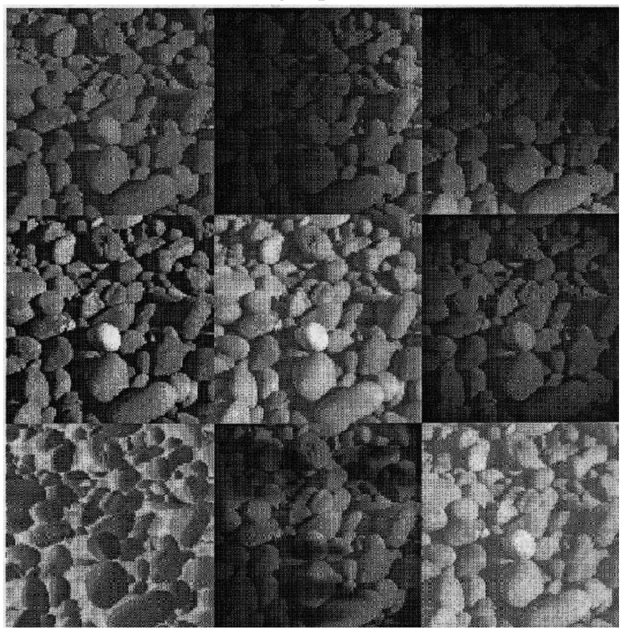

Fig. 4.3 Common image transformations. Source: [20]. Top: original, two gradients, γ transformation; middle: intensity scaling, edge burning; bottom: inversion, dodging and burning, flashing.

Another promising new approach is the *structure multivector* which includes information about local amplitude, local phase, and local geometry of both intrinsically 1D and 2D signals, isotropous even in 2D. It gives the proper generalization to the *analytic signal* (amplitude+phase) of 1D.[21]

4.2.3 Example 3: Monocular pose estimation[22]

(Conformal) geometric algebra [8,19] can be successfully used to formalize algebraic embedding of monocular pose estimation of kinematic chains. This is helpful for e.g. tracking robot arms or human body movements. As shown in Fig. 4.4 one relates positions of a 3D object to a reference camera coordinate system. The resulting (constraint) equations are compact and clear, and easy to linearize and iterate.

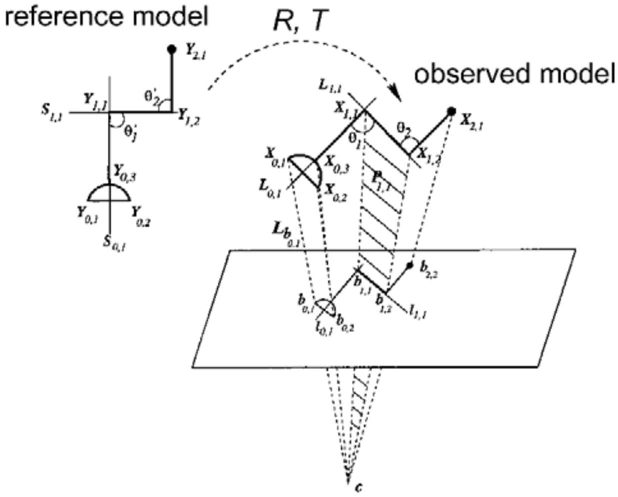

Fig. 4.4 Solid lines: camera model, object model, extracted lines on image plane. Dashed lines: best pose fit. Source: [22].

In a first step the purely kinematic problem of finding the rotation R and the translation T of the observed model in Fig. 4.4 is solved by way of exploiting obvious point-on-line, point-on-plane, and line-on-plane constraints. Points, lines and planes are defined by the outer product in (conformal) geometric algeba as

$$X \equiv \vec{e} \wedge \vec{x}\,, \tag{4.8a}$$

$$L \equiv \vec{e} \wedge \vec{x} \wedge \vec{y}\,, \tag{4.8b}$$

$$P \equiv \vec{e} \wedge \vec{x} \wedge \vec{y} \wedge \vec{z}\,. \tag{4.8c}$$

The point-on-plane constraint is e.g. simply given by

$$XL - LX = 0\,. \tag{4.9}$$

To formulation of a *kinematic* constraint is now straightforward by using equation (4.1) for the displacement D, composed of rotation and translation

$$(DX\widetilde{D})L - L(DX\widetilde{D}) = 0\,. \tag{4.10}$$

One now has only to find the best displacement D which satisfies the constraint (4.10).

Pose estimation of kinematic chains is also evident. One simply refines the scheme to include internal displacements. In Fig. 4.4 this can e.g. be internal rotations changing θ_1 and θ_2.

A real application can be seen in Fig. 4.5. The pose of a doll and the angles of the arms are estimated, by labeling one point on each kinematic chain segment. Already few iterations of the linarized problem give a good estimation of the pose and the kinematic chain parameters.

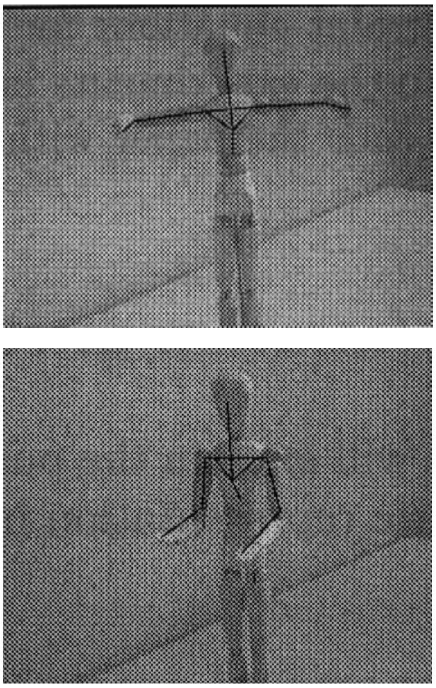

Fig. 4.5 The pose of the doll and the angles of the arms are estimated. Source: [22].

This concludes our short tour through the world of applications of geometric calculus. The literature, the internet, geometric calculus software, pending patents [19,23], etc. contain a lot more.

5. Teaching, Research & Development, Marketing

5.1 Teaching of engineering sciences

Already the teaching of engineering sciences will benefit greatly from making use of the general geometric language of geometric calculus. (Linear) algebra and calculus can be taught in a new unified, easy to understand way. Next all of physics is by now formulated in terms of geometric calculus. [9,12,15,24] The same applies to basic crystal structures, molecular interactions, signal theory, etc. Wherever an engineer employs mechanics, electromagnetism, thermodynamics, solid state matter theory, quantum theory, etc. it can be done in one and the same language of geometric calculus catering for diverse needs. The students will not have to learn new mathematics, whenever they encounter a different part of engineering science.

5.2 Research and development

Research and development do already benefit a great deal from employing geometric calculus. Even the quaternions [3,10] of Hamilton by themselves are already of great advantage for aerospace engineering and virtual reality [11]. Modeling and simulation can now make use of powerful, new methods. Conference participation numbers show that computer vision and graphics people are particularly interested.[18] It also leads to the development of new and very fast computer algorithms both for symbolic and numeric calculations.[16,17] Higher dimensional image geometry may for the first time ever get a solid theoretical footing, enabling systematic study and exploration, not just guessing around.

5.3 Geometric Calculus in Asia?

Both American and European researchers are already applying for patents [19,23] based on new methods developed with the help of geometric calculus. In Asia only few scientists and engineers seem to know about it or even apply it. A noteworthy exception is now a young Chinese mathematician [25] highly successful in automated theorem proving. He was a student of Hestenes in Arizona. It seems that the present generation of foreign Asian students in the Americas and Europe may be the ones to pave the way for its future in Asia as well.

In the *poster session*, I will demonstrate how to visualize geometric calculus with a MATLAB package developed in Amsterdam and Waterloo.[26] I will also demonstrate novel vector field design software, based on geometric calculus, a joint project by researchers in Germany and the US.[27]

6. Acknowledgements

I thank God my creator for the joy of doing research in the wonders of his works: “He (Christ) is before all things, and in him all things hold together.”[28]

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