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Classical and Semi-Classical Optics with MAPLE: Investigating a Two-dimensional Light Ray Geometry

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Abstract

1. The Geometry of a Two-dimensional Equiangular Spiral

The easiest way to describe two-dimensional geometry in physics is in a real two-dimensional Euclidean vector space $\mathcal{E}_2$ representing[†] a plane and its real geometric algebra $\mathcal{G}_2$. Such an algebra is equipped with an associative geometric product of vectors $\mathbf{ab} = <\mathbf{ab}> + <\mathbf{ab}>_2 = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = ab\exp(\mathbf{i}\Phi) = ab(\cos\Phi + \mathbf{i}\sin\Phi)$. $\mathbf{a}\wedge\mathbf{b}$ denotes a bivector or in two dimensions a pseudoscalar of maximal rank in $\mathcal{G}_2$, $\mathbf{i}$ is the real, oriented unit bivector, a,b the lengths of $\mathbf{a}$ and $\mathbf{b}$, respectively, and Φ the angle enclosed by $\mathbf{a}$ and $\mathbf{b}$. An equiangular spiral can then be described by $\mathbf{x} = \mathbf{x}_0\exp(\mathbf{i}\Phi + t\Phi)$, and $t = \tan\delta$, where $\frac{\pi}{2} + \delta$ describes the constant angle between radius vector and tangent of the equiangular spiral.[‡]

The sinus law for two successive reflections at $\mathbf{x}$ and $\mathbf{x}'$ on the equiangular spiral boundary reads $<(\mathbf{x}'-\mathbf{x})\mathbf{x}'>_2 = <(\mathbf{x}'-\mathbf{x})\mathbf{x}>_2$ or in terms of angles $\sin\Theta' = \exp(t\Delta)\sin(\Delta + \Theta')$ with $\Delta = \Phi' - \Phi$ and $\Theta' = \Theta + 2\delta$ (T). Θ' is the angle between the radius vector $\mathbf{x}$ and the ray path vector $\mathbf{x}' - \mathbf{x}$, and Θ the angle of incidence at $\mathbf{x}$. A closed path with, e.g., $n = 3$ reflections at the equiangular spiral boundary and one reflection at the gap, i.e., the line segment $\overline{\mathbf{x}_0\mathbf{x}_{2\pi}}$, is described by[§]

$$\begin{aligned} \sin(\Theta_1 + 2\delta) &= \exp(t\Delta_{21})\sin(\Theta_1 + 2\delta + \Delta_{21}) \\ \sin(\Theta_1 + \Delta_{21}) &= \exp(t\Delta_{32})\sin(\Theta_1 + \Delta_{21} - \Delta_{32}) \quad \text{(M)} \\ -\sin\Theta_1 &= \exp[t(\Delta_{21} + \Delta_{32})]\sin(\Theta_1 + 2\delta + \Delta_{21} - \Delta_{32}) \end{aligned}$$

The third equation describes the reflection at the gap.

2. Solving the Transcendental Light Path Equation (T) with MAPLE's fsolve

fsolve is the Newton method based numeric equivalent of MAPLE's general-purpose equation solver *solve*.[¶] It produces approximate (floating-point) solutions. *fsolve* takes as arguments the (list of) equation(s) to be solved, the list of unknown variables to be solved for, together with an optional list of ranges in which *fsolve* should look for the solution. δ is defined as a MAPLE constant and $\Phi \to \exp(\tan\delta\Phi)$ as a new MAPLE function. By specifying Θ_1 as well *fsolve* is left with Δ as the only unknown in equation (T). It returns Δ, which because of the selfsimilarity of the equiangular spiral, is completely general for any polar position Φ of $\mathbf{x}$. This generality is only restricted by IF conditions that describe the escape through (or reflection at) the gap. The MAPLE code for each definition, for the solution with *fsolve* and for the gap conditions will be explained briefly.

3. MAPLE Programming of a Simple, Nontrivial Ray Tracing Algorithm

In order to develop the above described numerical solution of (T) into a full ray tracing algorithm a MAPLE procedure[‖] is written. The arguments of this procedure and its structure will be described. Then the basic input data of the ray tracing algorithm, i.e., the starting point and the number of reflections to be calculated are described. The major computational part consists of the part computing the initial reflection and the main FOR loop, which repeatedly calls the procedure. The main FOR loop uses the same IF conditions for the gap as mentioned above. The final part is the creation of a spatial ray path coordinate list and the creation of a plot.

4. Interactive, Iterative Minimization of Errors

Trying to use *fsolve* on a set of equations like (M) usually produces no result. The reason is that the highly transcendental character of the equations demands a very narrow specification of the intervalls of the unknowns where *fsolve* should look for the solution. In order to get an intuitive understand-

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†D. Hestenes, *New Foundations for Classical Mechanics* (D. Reidel, Dordrecht, 1987).

‡E. Hitzer, *The Geometry of Light Paths for Equiangular Spirals*, Advances in Applied Clifford Algebras, 9, No. 2, 261-286 (1999).

§E. Hitzer, *Closed light paths in equiangular spiral disks* to be published.

¶Heal, Hansen, Rickard, *Maple V Learning Guide* Springer, New York (1998).

‖M.B. Monagan et.al. *Maple V Programming Guide* Springer, New York (1998).

ing of the situation it is helpful to put a starting point on the gap and to use the above developed ray tracing algorithm to produce ray paths that nearly close into themselves. This naturally leads to the idea of systematically varying the initial data (x_g on the gap, and φ_0 relative to the gap's vertical) around such points. Then one calculates the differences $(x'_g - x_g, \varphi_n - \varphi_0)$ after n reflections and plots them. One can now interactively reduce $(x'_g - x_g, \varphi_n - \varphi_0)$. With a little experience three to four cycles suffice to get $(x'_g - x_g, \varphi_n - \varphi_0)$ close enough to (0,0), so that it becomes possible to numerically solve the mode equations (M).

5. Solving the Transcendental Mode Equations (M)

The numerical solution of the transcentental mode equations for closed path in equiangular spiral disks with a 100% reflective boundary and gap now becomes straight forward. A MAPLE worksheet of, e.g., 20 commands for the case of $n = 3$ [equations (M)] does the job.

6. Iterative Numerical Continuation of Solutions to Transcendental Mode Equations

In the previous step we obtained a closed path for a single predefined value of $\delta = \delta^{(0)}$. By slightly changing δ to $\delta^{(1)}$ in a little step from $\delta^{(0)}$ and prescribing equally narrow intervalls around the just calculated $\Delta_{21}^{(0)}, \Delta_{32}^{(0)}$ and $\Theta_1^{(0)}$ we can find the solution of (M) for this $\delta^{(1)}$ as well. Repeating this procedure over and over again we get the full dependencies of $\Delta_{21}(\delta), \Delta_{32}(\delta)$ and $\Theta_1(\delta)$ respectively. It will be shown how to put this iterative procedure into semi-algorthmic form and how to suitably visualize and control the results graphically with MAPLE plot structures.

7. Visualizing Results with MAPLE Plot Stuctures

Geometric calculus easily allows to show that for a closed path like the one defined by (M) the polar angle of the first point of reflection $\mathbf{x}_1$ must be $\Phi_1 = (2m+1)\frac{\pi}{2} + \delta - \Delta_{32}$ with $m \in \mathbf{Z}$, such that $0 \leq \Phi_1 < 2\pi$. This allows to calculate all other points of reflection successively and to produce arrays containing the corresponding Φ_1, Φ_2, Φ_3, and the point of reflection x_g on the gap. After producing a list L of spatial two-dimensional coordinates of all four points of reflection for a specific δ one can plot it with MAPLE's ordinary *plot*(L) command and display this plot combined with a polar plot of the equiangular spiral. This allows to conveniently visualize the change of the shape of a mode with changes of the deformation parameter δ. $x_g - \delta$ dependencies can also be visualized by two-dimensional plots.

8. Porting the Results to Standard Image Formats

This section describes how to use the MAPLE *plotsetup* command to port the produced graphics to standard image format image files like gif, ps, jpg, bmp and others.** This is extremely useful for incorporating the results in written documentations, e.g. with Latex or MS Word, etc.

9. Semi-Classical Optics

If time allows I will show how the above described ray tracing algorithm can be expanded in order to yield Poincaré diagrams for any choice of δ. The ray tracing algorithm needs to be enclosed in a double FOR loop that suitably steps through the initial conditions, (e.g., 100 different intitial conditions) and the number of reflections to be traced (e.g., 200 for each ray). The resulting data allow to calculate all angles of reflection γ together with the polar coordinates Φ of the position of the reflection, or x_g in case that the reflection is at the gap. The $\sin\gamma - \Phi$(or x_g) diagram plot is the desired Poincaré diagram. Islands of stability are clearly seen. Picking initial data from these islands allows to visualize the stable path oscillations around those islands in turn through the plotting of spatial paths calculated with the ray tracing algorithm. This allows a hands-on approach for the design, visualization and stability control of modes in equiangular spiral disk shaped laser cavities. This is of great interest for the design of more efficient asymmetric mode micro-disk lasers.††

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**Heal, Hansen, Rickard, *Maple V Learning Guide* Springer, New York (1998).

††J. U. Nöckel, Phys. Blätter, **54**, 927 (1998). C. Gmachl et. al., Science, **280**, 1556 (1998).

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