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Windowed Fourier Transform of Two-Dimensional Quaternionic Signals*

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Abstract

In this paper, we generalize the classical windowed Fourier transform (WFT) to quaternion-valued signals, called the *quaternionic windowed Fourier transform* (QWFT). Using the spectral representation of the quaternionic Fourier transform (QFT), we derive several important properties such as reconstruction formula, reproducing kernel, isometry, and orthogonality relation. Taking the Gaussian function as window function we obtain quaternionic Gabor filters which play the role of coefficient functions when decomposing the signal in the quaternionic Gabor basis. We apply the QWFT properties and the (right-sided) QFT to establish a Heisenberg type uncertainty principle for the QWFT. Finally, we briefly introduce an application of the QWFT to a linear time-varying system.

Keywords : quaternionic Fourier transform, quaternionic windowed Fourier transform, signal processing, Heisenberg type uncertainty principle

1 Introduction

One of the basic problems encountered in signal representations using the conventional Fourier transform (FT) is the ineffectiveness of the Fourier kernel to represent and compute location information. One method to overcome such a problem is the windowed Fourier transform (WFT). Recently, some authors [6, 9, 23] have extensively studied the WFT and its properties from a mathematical point of view. In [17, 24] the WFT has been successfully applied as a tool of spatial-frequency analysis which is able to characterize the local frequency at any location in a fringe pattern.

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On the other hand, the quaternionic Fourier transform (QFT), which is a nontrivial generalization of the real and complex Fourier transform (FT) using the quaternion algebra [10], has been of interest to researchers for some years (see, for example, [1, 2, 4, 5, 12, 16, 19, 20]). They found that many FT properties still hold and others have to be modified. Based on the (right-sided) QFT, one may extend the WFT to quaternion algebra while enjoying similar properties as in the classical case.

The idea of extending the WFT to the quaternion algebra setting has been recently studied by Bülow and Sommer [1, 2]. They introduced a special case of the QWFT known as quaternionic Gabor filters. They applied these filters to obtain a local two-dimensional quaternionic phase. Their generalization is obtained using the inverse (two-sided) quaternion Fourier kernel. Hahn [11] constructed a Fourier-Wigner distribution of 2-D quaternionic signals which is in fact closely related to the QWFT. In [18], the extension of the WFT to Clifford (geometric) algebra was discussed. This extension used the kernel of the Clifford Fourier transform (CFT) [13]. In general a CFT replaces the complex imaginary unit $i \in \mathbb{C}$ by a geometric root [14, 15] of -1, i.e. any element of a Clifford (geometric) algebra squaring to -1.

The main goal of this paper is to thoroughly study the generalization of the classical WFT to quaternion algebra, which we call the *quaternionic windowed Fourier transform* (QWFT), and investigate important properties of the QWFT such as (specific) shift, reconstruction formula, reproducing kernel, isometry, and orthogonality relation. We emphasize that the QWFT proposed in the present work is significantly different from [18] in the definition of the exponential kernel. In the present approach, we use the kernel of the (right-sided) QFT. We present several examples to show the differences between the QWFT and the WFT. Using the (right-sided) QFT properties and its uncertainty principle [19] we establish a generalized QWFT uncertainty principle. We will also study an application of the QWFT to a linear time-varying system.

The organization of the paper is as follows. The remainder of this section briefly reviews quaternions and the (right-sided) QFT. In section 2, we discuss the basic ideas for the construction of the QWFT and derive several important properties of the QWFT using the (right-sided) QFT. We also give some examples of the QWFT. In section 4, an application of the QWFT to a linear time varying system is presented.

The concept of the quaternion algebra [8, 10] was introduced by Sir Hamilton in 1842 and is denoted by $\mathbb{H}$ in his honor. It is an extension of the complex numbers to a four-dimensional (4-D) algebra. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three imaginary units $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ with real coefficients,

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_i + \boldsymbol{j}q_j + \boldsymbol{k}q_k \mid q_0, q_i, q_j, q_k \in \mathbb{R}\}, \tag{1}$$

which obey Hamilton's multiplication rules

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \quad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{2}$$

For simplicity, we express a quaternion q as sum of a scalar q_0 and a *pure* 3D quaternion $\boldsymbol{q}$,

$$q = q_0 + \boldsymbol{q} = q_0 + \boldsymbol{i}q_i + \boldsymbol{j}q_j + \boldsymbol{k}q_k, \tag{3}$$

where the *scalar* part q_0 is also denoted by $Sc(q)$.

It is convenient to introduce an inner product for two functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{H}$ as follows:

$$\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, d^2\boldsymbol{x}, \tag{4}$$

where the overline indicates the quaternion conjugation of the function. In particular, if $f = g$, we obtain the associated norm

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = \langle f, f\rangle^{1/2}_{L^2(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x}\right)^{1/2}. \tag{5}$$

As a consequence of the inner product (4) we obtain the *quaternion Cauchy-Schwarz* inequality

$$|Sc\langle f, g\rangle| \leq \|f\|_{L^2(\mathbb{R}^2;\mathbb{H})}\|g\|_{L^2(\mathbb{R}^2;\mathbb{H})}, \qquad \forall f, g \in L^2(\mathbb{R}^2;\mathbb{H}). \tag{6}$$

In the following we introduce the (right-sided) QFT. This will be needed in section 2 to establish the QWFT.

Definition 1.1 (Right-sided QFT) *The (right sided) quaternion Fourier transform (QFT) of $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is the function $\mathcal{F}_q\{f\}\colon \mathbb{R}^2 \to \mathbb{H}$ given by*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} \, d^2\boldsymbol{x}, \tag{7}$$

where $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$, and the quaternion exponential product $e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}$ is the quaternion Fourier kernel.

Theorem 1.1 (Inverse QFT) *Suppose that $f \in L^2(\mathbb{R}^2;\mathbb{H})$ and $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$. Then the QFT of f is an invertible transform and its inverse is given by*

$$\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1} \, d^2\boldsymbol{\omega}, \tag{8}$$

where the quaternion exponential product $e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1}$ is called the inverse (right-sided) quaternion Fourier kernel.

Detailed information about the QFT and its properties can be found in [1, 2, 5, 12, 19].

2 Quaternionic Windowed Fourier Transform (QWFT)

This section generalizes the classical WFT to quaternion algebra. Using the definition of the (right-sided) QFT described before, we extend the WFT to the QWFT. We shall later see how some properties of the WFT are extended in the new definition. For this purpose we briefly review the 2-D WFT.

2.1 2-D WFT

The FT is a powerful tool for the analysis of stationary signals but it is not well suited for the analysis of non-stationary signals because it is a global transformation with poor spatial localization [24]. However, in practice, most natural signals are non-stationary. In order to characterize a non-stationary signal properly, the WFT is commonly used.

Definition 2.1 (WFT) *The WFT of a two-dimensional real signal $f \in L^2(\mathbb{R}^2)$ with respect to the window function $g \in L^2(\mathbb{R}^2) \setminus \{0\}$ is given by*

$$\mathcal{G}_g f(\boldsymbol{\omega}, \boldsymbol{b}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(\boldsymbol{x}) \, \overline{g_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})} \, d^2\boldsymbol{x}, \tag{9}$$

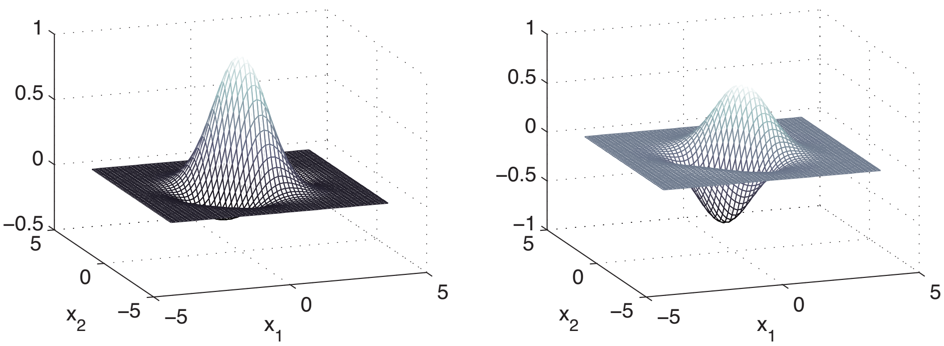

Figure 1: Representation of complex Gabor filter for $\sigma_1 = \sigma_2 = 1, u_0 = v_0 = 1$ in the spatial domain with its real part (*left*) and imaginary part (*right*).

where the window daughter function $g_{\boldsymbol{\omega},\boldsymbol{b}}$ *is called the windowed Fourier kernel defined by*

$$g_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x}) = g(\boldsymbol{x} - \boldsymbol{b})e^{\sqrt{-1}\,\boldsymbol{\omega}\cdot\boldsymbol{x}}. \tag{10}$$

Equation (9) shows that the image of a WFT is a complex 4-D coefficient function.

Most applications make use of the Gaussian window function g which is non-negative and well localized around the origin in both spatial and frequency domains. The Gaussian window function can be expressed as

$$g(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{-\left[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2\right]/2}, \tag{11}$$

where σ_1 and σ_2 are the standard deviations of the Gaussian function and determine the width of the window. We call (10), for fixed $\boldsymbol{\omega} = \boldsymbol{\omega}_0 = u_0\boldsymbol{e}_1 + v_0\boldsymbol{e}_2$, and $b_1 = b_2 = 0$, a complex Gabor filter as shown in Figure 1 if g is the Gaussian function (11), i.e.

$$g_{c,\boldsymbol{\omega}_0}(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{\sqrt{-1}\,(u_0x_1 + v_0x_2)}g(\boldsymbol{x}, \sigma_1, \sigma_2). \tag{12}$$

In general, when the Gaussian function (11) is chosen as the window function, the WFT in (9) is called *Gabor transform*. We observe that the WFT localizes the signal f in the neighbourhood of $\boldsymbol{x} = \boldsymbol{b}$. For this reason, the WFT is often called *short time Fourier transform*.

2.2 Definition of the QWFT

Bülow [1] extended the complex Gabor filter (12) to quaternion algebra by replacing the complex kernel $e^{\sqrt{-1}(u_0x_1+v_0x_2)}$ with the inverse (two-sided) quaternion Fourier kernel $e^{\boldsymbol{i}u_0x_1}\,e^{\boldsymbol{j}v_0x_2}$. His extension then takes the form

$$g_q(\boldsymbol{x}, \sigma_1, \sigma_2) = e^{\boldsymbol{i}u_0x_1}e^{\boldsymbol{j}v_0x_2}e^{-\left[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2\right]/2}, \tag{13}$$

which he called *quaternionic Gabor filter*[1] as shown in Figure 2 and applied it to get the local quaternionic phase of a two-dimensional real signal. Bayro et al. [4] also used quaternionic Gabor filters for the preprocessing of 2D speech representations.

[1] If we would have interchanged the order of the two exponentials in Definition 1.1, which we are always free to do, then (13) and (15) would agree fully, except for the factor $(2\pi)^{-2}$. Figures 2 and 3 illustrate the two different kinds of quaternionic Gabor filters that arise. The differences can be made obvious by decomposition of the two exponential products $e^{\boldsymbol{i}u_0x_1}e^{\boldsymbol{j}v_0x_2}$ and $e^{\boldsymbol{j}v_0x_2}e^{\boldsymbol{i}u_0x_1}$. The imaginary $\boldsymbol{i}$-part of Figure 2 is the imaginary $\boldsymbol{j}$-part of Figure 3 and vice versa. Note also that the imaginary $\boldsymbol{k}$-parts of Figures 2 and 3 are essentially the same, because they only have different signs.

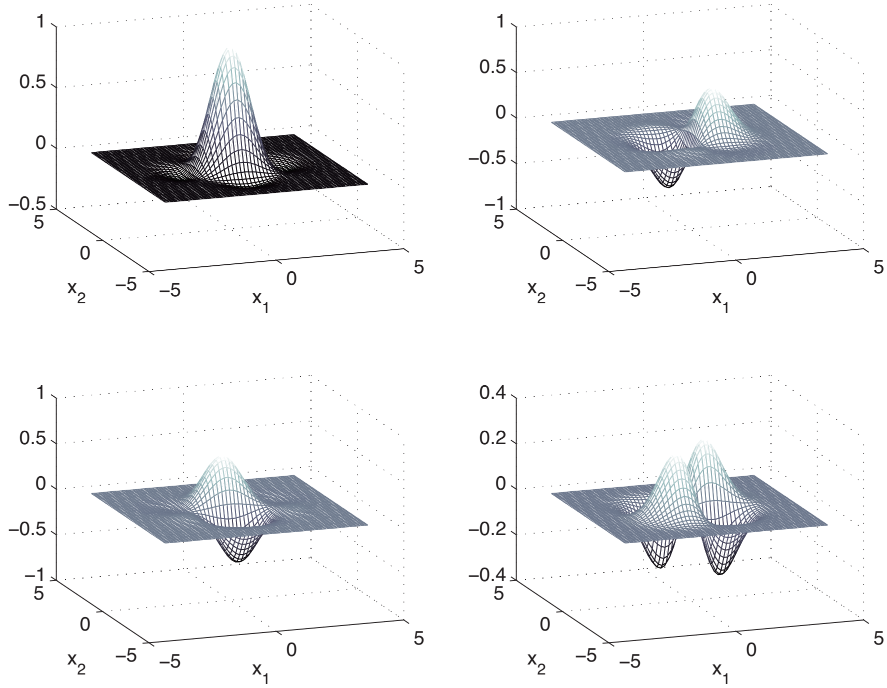

Figure 2: Bülow's quaternionic Gabor filter (13) ($\sigma_1 = \sigma_2 = 1, u_0 = v_0 = 1$) in the spatial domain with real part (*top left*) and imaginary $\boldsymbol{i}$-part (*top right*), $\boldsymbol{j}$-part (*bottom left*), and $\boldsymbol{k}$-part (*bottom right*).

The extension of the WFT to quaternion algebra using the (two-sided) QFT is rather complicated, due to the non-commutativity of quaternion functions. Alternatively, we use the (right-sided) QFT to define the QWFT. We therefore introduce the following general QWFT of a two-dimensional quaternion signal $f \in L^2(\mathbb{R}^2;\mathbb{H})$ in Def. 2.3.

Definition 2.2 *A quaternion window function is a function $\phi \in L^2(\mathbb{R}^2;\mathbb{H})\backslash\{0\}$ such that $|\boldsymbol{x}|^{1/2}\phi(\boldsymbol{x}) \in L^2(\mathbb{R}^2;\mathbb{H})$ too. We call*

$$\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x}) = \frac{1}{(2\pi)^2} e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1} \phi(\boldsymbol{x}-\boldsymbol{b}), \tag{14}$$

a quaternionic window daughter function.

If we fix $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, and $b_1 = b_2 = 0$, and take the Gaussian function as the window function of (14), then we get the quaternionic Gabor filter shown in Figure 3,

$$g_q(\boldsymbol{x}, \sigma_1, \sigma_2) = \frac{1}{(2\pi)^2}\, e^{\boldsymbol{j} v_0 x_2} e^{\boldsymbol{i} u_0 x_1} e^{-\left[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2\right]/2}. \tag{15}$$

Definition 2.3 (QWFT) *Denote the QWFT on $L^2(\mathbb{R}^2;\mathbb{H})$ by G_ϕ. Then the QWFT of $f \in$*

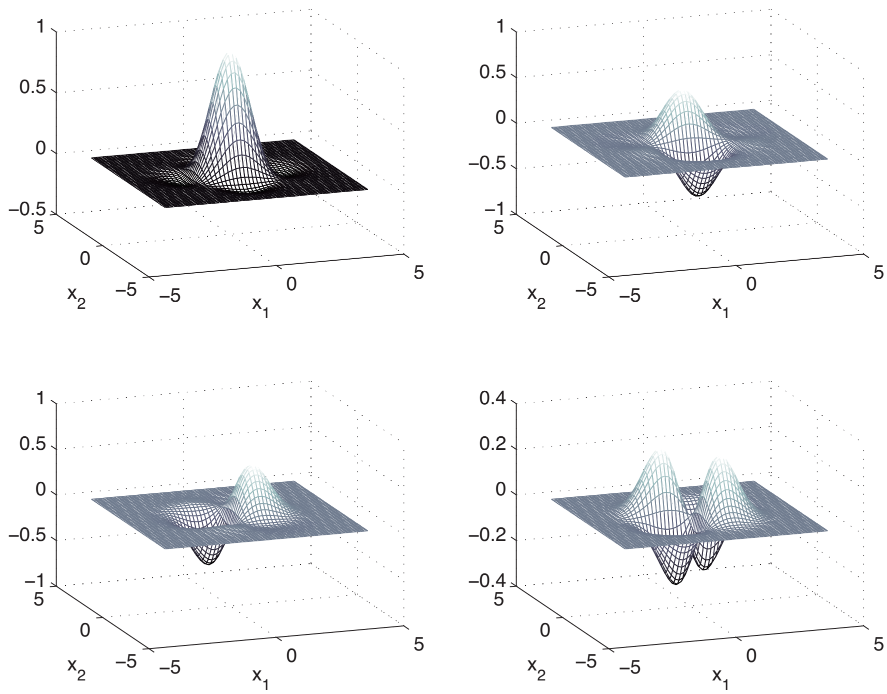

Figure 3: The real part (*top left*) and imaginary $\boldsymbol{i}$-part (*top right*), $\boldsymbol{j}$-part (*bottom left*), and $\boldsymbol{k}$-part (*bottom right*) of a quaternionic Gabor filter ($\sigma_1 = \sigma_2 = 1, u_0 = v_0 = 1$) in the spatial domain.

$L^2(\mathbb{R}^2;\mathbb{H})$ *is defined by*

$$\begin{aligned} f(\boldsymbol{x}) \quad \longrightarrow \quad G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \langle f, \phi_{\boldsymbol{\omega},\boldsymbol{b}} \rangle_{L^2(\mathbb{R}^2;\mathbb{H})} \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \, \overline{\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})} \, d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) \, \overline{e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1} \phi(\boldsymbol{x}-\boldsymbol{b})} d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) \, \overline{\phi(\boldsymbol{x}-\boldsymbol{b})} e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} d^2\boldsymbol{x}. \end{aligned} \tag{16}$$

Please note that the order of the exponentials in (16) is fixed because of the non-commutativity of the product of quaternions. Changing the order yields another quaternion valued function which differs by the signs of the terms. Equation (16) clearly shows that the QWFT can be regarded as the (right-sided) QFT (compare (38)) of the product of a quaternion-valued signal f and a shifted and quaternion conjugate version of the quaternion window function or as an inner product (4) of f and the quaternionic window daughter function. In contrast to the QFT basis $e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_1 x_2}$ which has an infinite spatial extension, the QWFT basis $\phi(\boldsymbol{x}-\boldsymbol{b}) \, e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_1 x_2}$ has a limited spatial extension due to the local quaternion window function $\phi(\boldsymbol{x}-\boldsymbol{b})$.

The energy density is defined as the modulus square of the QWFT (16) given by

$$|G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})|^2 = \frac{1}{(2\pi)^4} \left| \int_{\mathbb{R}^2} f(\boldsymbol{x}) \, \overline{\phi(\boldsymbol{x}-\boldsymbol{b})} e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} \, d^2\boldsymbol{x} \right|^2. \tag{17}$$

Equation (17) is often called a spectrogram which measures the energy of a quaternion-valued function f in the position-frequency neighbourhood of $(\boldsymbol{b}, \, \boldsymbol{\omega})$.

A good choice for the window function ϕ is the Gaussian quaternion function because, according to Heisenberg's uncertainty principle, the Gaussian quaternion signal can simultaneously minimize the spread in both spatial and quaternionic frequency domains, and it is smooth in both domains. The uncertainty principle can be written in the following form [19]

$$\triangle g_{x_1} \triangle g_{x_2} \triangle g_{\omega_1} \triangle g_{\omega_2} \geq \frac{1}{4}, \tag{18}$$

where $\triangle g_{x_k}$, $k = 1, 2$, are the effective spatial widths of the quaternion function g and $\triangle g_{\omega_k}$, $k = 1, 2$, are its effective bandwidths.

2.3 Examples of the QWFT

For illustrative purposes, we shall discuss examples of the QWFT. We begin with a straightforward example.

Example 2.1 *Consider the two-dimensional first order B-spline window function (see [21]) defined by*

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if } -1 \leq x_1 \leq 1 \text{ and } -1 \leq x_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Obtain the QWFT of the function defined as follows:

$$f(\boldsymbol{x}) = \begin{cases} e^{x_1+x_2}, & \text{if } -\infty < x_1 < 0 \text{ and } -\infty < x_2 < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

By applying the definition of the QWFT we have

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{-1+b_1}^{m_1} \int_{-1+b_2}^{m_2} e^{x_1+x_2} e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} dx_1 dx_2, \\ m_1 &= \min(0, 1+b_1), \quad m_2 = \min(0, 1+b_2). \end{aligned} \tag{21}$$

Simplifying (21) yields

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{-1+b_1}^{m_1} \int_{-1+b_2}^{m_2} e^{x_1(1-\boldsymbol{i}\omega_1)} e^{x_2(1-\boldsymbol{j}\omega_2)} \, d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2} \int_{-1+b_1}^{m_1} e^{x_1(1-\boldsymbol{i}\omega_1)} dx_1 \int_{-1+b_2}^{m_2} e^{x_2(1-\boldsymbol{j}\omega_2)} \, dx_2 \\ &= \frac{1}{(2\pi)^2} \, e^{x_1(1-\boldsymbol{i}\omega_1)}(1-\boldsymbol{i}\omega_1)\Big|_{-1+b_1}^{m_1} \, \frac{e^{x_2(1-\boldsymbol{j}\omega_2)}}{(1-\boldsymbol{j}\omega_2)}\Big|_{-1+b_2}^{m_2} \\ &= \frac{(e^{m_1(1-\boldsymbol{i}\omega_1)} - e^{(-1+b_1)(1-\boldsymbol{i}\omega_1)})(e^{m_2(1-\boldsymbol{j}\omega_2)} - e^{(-1+b_2)(1-\boldsymbol{j}\omega_2)})}{(2\pi)^2(1-\boldsymbol{i}\omega_1-\boldsymbol{j}\omega_2+\boldsymbol{k}\omega_1\omega_2)}. \end{aligned} \tag{22}$$

Using the properties of quaternions we obtain

$$G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) = \frac{(e^{m_1(1-\boldsymbol{i}\omega_1)} - e^{(-1+b_1)(1-\boldsymbol{i}\omega_1)})(e^{m_2(1-\boldsymbol{j}\omega_2)} - e^{(-1+b_2)(1-\boldsymbol{j}\omega_2)})(1+\boldsymbol{i}\omega_1+\boldsymbol{j}\omega_2-\boldsymbol{k}\omega_1\omega_2)}{(2\pi)^2(1+\omega_1^2+\omega_2^2+\omega_1^2\omega_2^2)}. \tag{23}$$

Example 2.2 *Given the window function of the two-dimensional Haar function defined by*

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{for } 0 \le x_1 < 1/2 \text{ and } 0 \le x_2 < 1/2, \\ -1, & \text{for } 1/2 \le x_1 < 1 \text{ and } 1/2 \le x_2 < 1, \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

find the QWFT of the Gaussian function $f(\boldsymbol{x}) = e^{-(x_1^2+x_2^2)}$.

From Definition 2.3 we obtain

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x}-\boldsymbol{b})} e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2} \int_{b_1}^{1/2+b_1} e^{-x_1^2} e^{-\boldsymbol{i}\omega_1 x_1}\, dx_1 \int_{b_2}^{1/2+b_2} e^{-x_2^2} e^{-\boldsymbol{j}\omega_2 x_2}\, dx_2 \\ &\quad - \frac{1}{(2\pi)^2} \int_{1/2+b_1}^{1+b_1} e^{-x_1^2} e^{-\boldsymbol{i}\omega_1 x_1}\, dx_1 \int_{1/2+b_2}^{1+b_2} e^{-x_2^2} e^{-\boldsymbol{j}\omega_2 x_2}\, dx_2. \end{aligned} \tag{25}$$

By completing squares, we have

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{b_1}^{1/2+b_1} e^{-(x_1+\boldsymbol{i}\omega_1/2)^2-\omega_1^2/4} dx_1 \int_{b_2}^{1/2+b_2} e^{-(x_2+\boldsymbol{j}\omega_2/2)^2-\omega_2^2/4} dx_2 \\ &\quad - \frac{1}{(2\pi)^2} \int_{1/2+b_1}^{1+b_1} e^{-(x_1+\boldsymbol{i}\omega_1/2)^2-\omega_1^2/4} dx_1 \int_{1/2+b_2}^{1+b_2} e^{-(x_2+\boldsymbol{j}\omega_2/2)^2-\omega_2^2/4} dx_2. \end{aligned} \tag{26}$$

Making the substitutions $y_1 = x_1 + \boldsymbol{i}\,\frac{\omega_1}{2}$ and $y_2 = x_2 + \boldsymbol{j}\,\frac{\omega_2}{2}$ in the above expression we immediately obtain

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\pi)^2} \int_{b_1+\boldsymbol{i}\omega_1/2}^{1/2+b_1+\boldsymbol{i}\omega_1/2} e^{-y_1^2}\, dy_1 \int_{b_2+\boldsymbol{j}\omega_2/2}^{1/2+b_2+\boldsymbol{j}\omega_2/2} e^{-y_2^2}\, dy_2 \\ &\quad - \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\pi)^2} \int_{1/2+b_1+\boldsymbol{i}\omega_1/2}^{1+b_1+\boldsymbol{i}\omega_1/2} e^{-y_1^2}\, dy_1 \int_{1/2+b_2+\boldsymbol{j}\omega_2/2}^{1+b_2+\boldsymbol{j}\omega_2/2} e^{-y_2^2}\, dy_2 \\ &= \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\pi)^2} \left[\left(\int_0^{b_1+\boldsymbol{i}\omega_1/2} (-e^{-y_1^2})\, dy_1 + \int_0^{1/2+b_1+\boldsymbol{i}\omega_1/2} e^{-y_1^2}\, dy_1 \right) \right. \\ &\quad \times \left(\int_0^{b_2+\boldsymbol{j}\omega_2/2} (-e^{-y_2^2})\, dy_2 + \int_0^{1/2+b_2+\boldsymbol{j}\omega_2/2} e^{-y_2^2}\, dy_2 \right) \\ &\quad - \left(\int_0^{1/2+b_1+\boldsymbol{i}\omega_1/2} (-e^{-y_1^2})\, dy_1 + \int_0^{1+b_1+\boldsymbol{i}\omega_1/2} e^{-y_1^2}\, dy_1 \right) \\ &\quad \left. \times \left(\int_0^{1/2+b_2+\boldsymbol{j}\omega_2/2} (-e^{-y_2^2})\, dy_2 + \int_0^{1+b_2+\boldsymbol{j}\omega_2/2} e^{-y_2^2}\, dy_2 \right) \right]. \end{aligned} \tag{27}$$

Equation (27) can be written in the form

$$
\begin{aligned}
G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) &= \frac{e^{-(\omega_1^2+\omega_2^2)/4}}{(2\sqrt{\pi})^3}\left\{\left[-\mathrm{erf}\left(b_1+\frac{\boldsymbol{i}}{2}\omega_1\right)+\mathrm{erf}\left(\frac{1}{2}+b_1+\frac{\boldsymbol{i}}{2}\omega_1\right)\right]\right.\\
&\quad\times\left[-\mathrm{erf}\left(b_2+\frac{\boldsymbol{j}}{2}\omega_2\right)+\mathrm{erf}\left(\frac{1}{2}+b_2+\frac{\boldsymbol{j}}{2}\omega_2\right)\right]\\
&\quad-\left[-\mathrm{erf}\left(\frac{1}{2}+b_1+\frac{\boldsymbol{i}}{2}\omega_1\right)+\mathrm{erf}\left(1+b_1+\frac{\boldsymbol{i}}{2}\omega_1\right)\right]\\
&\quad\left.\times\left[-\mathrm{erf}\left(\frac{1}{2}+b_2+\frac{\boldsymbol{j}}{2}\omega_2\right)+\mathrm{erf}\left(1+b_2+\frac{\boldsymbol{j}}{2}\omega_2\right)\right]\right\}, \qquad (28)
\end{aligned}
$$

where $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$.

2.4 Properties of the QWFT

In this subsection, we describe the properties of the QWFT. We must exercise care in extending the properties of the WFT to the QWFT because of the general non-commutativity of quaternion multiplication. We will find most of the properties of the WFT are still valid for the QWFT, however with some modifications.

Theorem 2.1 (Left linearity) *Let $\phi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion window function. The QWFT of $f,g \in L^2(\mathbb{R}^2;\mathbb{H})$ is a left linear operator, which means*

$$[G_\phi(\lambda f+\mu g)](\boldsymbol{\omega},\boldsymbol{b}) = \lambda G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) + \mu G_\phi g(\boldsymbol{\omega},\boldsymbol{b}), \qquad (29)$$

for arbitrary quaternion constants $\lambda,\mu \in \mathbb{H}$.

Proof. This follows directly from the linearity of the quaternion product and the integration involved in Definition 2.3.

Remark 2.1 *Restricting the constants in Theorem 2.1 to $\lambda,\mu \in \mathbb{R}$ we get both left and right linearity of the QWFT.*

Theorem 2.2 (Parity) *Let $\phi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion window function. Then we have*

$$G_{P\phi}\{Pf\}(\boldsymbol{\omega},\boldsymbol{b}) = G_\phi f(-\boldsymbol{\omega},-\boldsymbol{b}), \qquad (30)$$

where $P\phi(\boldsymbol{x})=\phi(-\boldsymbol{x}), \forall \phi \in L^2(\mathbb{R}^2;\mathbb{H})$.

Proof. A direct calculation gives for every $f \in L^2(\mathbb{R}^2;\mathbb{H})$

$$
\begin{aligned}
G_{P\phi}\{Pf\}(\boldsymbol{\omega},\boldsymbol{b}) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(-\boldsymbol{x})\overline{\phi(-(\boldsymbol{x}-\boldsymbol{b}))}\,e^{-\boldsymbol{i}\omega_1 x_1}e^{-\boldsymbol{j}\omega_2 x_2}\,d^2\boldsymbol{x}\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(-\boldsymbol{x})\overline{\phi(-\boldsymbol{x}-(-\boldsymbol{b}))}\,e^{-\boldsymbol{i}(-\omega_1)(-x_1)}e^{-\boldsymbol{j}(-\omega_2)(-x_2)}\,d^2\boldsymbol{x}, \qquad (31)
\end{aligned}
$$

which proves the theorem according to Definition 2.3. □

Theorem 2.3 (Specific shift) *Let ϕ be a quaternion window function. Assume that*

$$f = f_0 + \boldsymbol{i} f_1 \quad \text{and} \quad \phi = \phi_0 + \boldsymbol{i}\phi_1. \tag{32}$$

Then we obtain

$$G_\phi T_{\boldsymbol{x}_0} f(\boldsymbol{\omega}, \boldsymbol{b}) = e^{-\boldsymbol{i}\omega_1 x_0} \left(G_\phi f(\boldsymbol{\omega}, \boldsymbol{b} - \boldsymbol{x}_0)\right) e^{-\boldsymbol{j}\omega_2 y_0}, \tag{33}$$

where $T_{\boldsymbol{x}_0}$ denotes the translation operator by $\boldsymbol{x}_0 = x_0\boldsymbol{e}_1 + y_0\boldsymbol{e}_2$, i.e. $T_{\boldsymbol{x}_0} f = f(\boldsymbol{x} - \boldsymbol{x}_0)$.

Proof. By (16) we have

$$G_\phi(T_{\boldsymbol{x}_0} f)(\boldsymbol{\omega}, \boldsymbol{b}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\boldsymbol{x} - \boldsymbol{x}_0) \overline{\phi(\boldsymbol{x} - \boldsymbol{b})}\, e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x}. \tag{34}$$

We substitute $\boldsymbol{t}$ for $\boldsymbol{x} - \boldsymbol{x}_0$ in the above expression and get, with $d^2\boldsymbol{x} = d^2\boldsymbol{t}$,

$$\begin{aligned} G_\phi(T_{\boldsymbol{x}_0} f)(\boldsymbol{\omega}, \boldsymbol{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\boldsymbol{t}) \overline{\phi(\boldsymbol{t} - (\boldsymbol{b} - \boldsymbol{x}_0))}\, e^{-\boldsymbol{i}\omega_1 (t_1 + x_0)} e^{-\boldsymbol{j}\omega_2 (t_2 + y_0)}\, d^2\boldsymbol{t} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\boldsymbol{t}) \overline{\phi(\boldsymbol{t} - (\boldsymbol{b} - \boldsymbol{x}_0))}\, e^{-\boldsymbol{i}\omega_1 x_0} e^{-\boldsymbol{i}\omega_1 t_1} e^{-\boldsymbol{j}\omega_2 t_2} e^{-\boldsymbol{j}\omega_2 y_0}\, d^2\boldsymbol{t} \\ &\overset{(32)}{=} \frac{1}{(2\pi)^2} e^{-\boldsymbol{i}\omega_1 x_0} \int_{\mathbb{R}^2} f(\boldsymbol{t}) \overline{\phi(\boldsymbol{t} - (\boldsymbol{b} - \boldsymbol{x}_0))}\, e^{-\boldsymbol{i}\omega_1 t_1} e^{-\boldsymbol{j}\omega_2 t_2} d^2\boldsymbol{t}\;\; e^{-\boldsymbol{j}\omega_2 y_0}. \end{aligned} \tag{35}$$

The theorem has been proved. □

Equation (33) describes that if the original function $f(\boldsymbol{x})$ is shifted by $\boldsymbol{x}_0$, its window function will be shifted by $\boldsymbol{x}_0$, the frequency will remain unchanged, and the phase will be changed by the left and right phase factors $e^{-\boldsymbol{i}\omega_1 x_0}$ and $e^{-\boldsymbol{j}\omega_2 y_0}$.

Remark 2.2 *Like for the (right-sided) QFT, the usual form of the modulation property of the QWFT does not hold [12, 19]. It is obstructed by the non-commutativity of the quaternion exponential product factors*

$$e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} \neq e^{-\boldsymbol{j}\omega_2 x_2} e^{-\boldsymbol{i}\omega_1 x_1}. \tag{36}$$

The following theorem tells us that the QWFT is invertible, that is, the original quaternion signal f can be recovered simply by taking the inverse QWFT.

Theorem 2.4 (Reconstruction formula) *Let ϕ be a quaternion window function. Then every 2-D quaternion signal $f \in L^2(\mathbb{R}^2;\mathbb{H})$ can be fully reconstructed by*

$$f(\boldsymbol{x}) = \frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x})\; d^2\boldsymbol{b}\, d^2\boldsymbol{\omega}. \tag{37}$$

Proof. It follows from the QWFT (16) that

$$G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) = \frac{1}{(2\pi)^2} \mathcal{F}_q\{f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x} - \boldsymbol{b})}\}(\boldsymbol{\omega}). \tag{38}$$

Taking the inverse (right-sided) QFT of both sides of (38) we obtain

$$f(\boldsymbol{x}) \overline{\phi(\boldsymbol{x} - \boldsymbol{b})} = (2\pi)^2 \mathcal{F}_q^{-1}\{G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\}(\boldsymbol{x}) = \frac{(2\pi)^2}{(2\pi)^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\, e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1}\, d^2\boldsymbol{\omega}. \tag{39}$$

Multiplying both sides of (39) from the right by $\phi(\boldsymbol{x}-\boldsymbol{b})$ and integrating with respect to $d^2\boldsymbol{b}$ we get

$$f(\boldsymbol{x})\int_{\mathbb{R}^2}|\phi(\boldsymbol{x}-\boldsymbol{b})|^2 d^2\boldsymbol{b} = (2\pi)^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\,\frac{1}{(2\pi)^2}\,e^{\boldsymbol{j}\omega_2 x_2}e^{\boldsymbol{i}\omega_1 x_1}\phi(\boldsymbol{x}-\boldsymbol{b})\,d^2\boldsymbol{\omega}\,d^2\boldsymbol{b}. \tag{40}$$

Inserting (14) into the right-hand side of (40) we finally obtain

$$f(\boldsymbol{x})\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = (2\pi)^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\,\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\,d^2\boldsymbol{\omega}\,d^2\boldsymbol{b}, \tag{41}$$

which gives (37). □

Set $C_\phi = \|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}$ and assume that $0 < C_\phi < \infty$. Then, the reconstruction formula (37) can also be written as

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{(2\pi)^2}{C_\phi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\,\phi_{\boldsymbol{\omega},\boldsymbol{b}}\,d^2\boldsymbol{b}\,d^2\boldsymbol{\omega} \\ &= \frac{(2\pi)^2}{C_\phi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle f,\phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\phi_{\boldsymbol{\omega},\boldsymbol{b}}\,d^2\boldsymbol{b}\,d^2\boldsymbol{\omega}. \end{aligned} \tag{42}$$

More properties of the QWFT are given in the following theorems.

Theorem 2.5 (Orthogonality relation) *Let ϕ be a quaternion window function and $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ arbitrary. Then we have*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle f,\phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\overline{\langle g,\phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle}_{L^2(\mathbb{R}^2;\mathbb{H})}d^2\boldsymbol{\omega}\,d^2\boldsymbol{b} = \frac{C_\phi}{(2\pi)^2}\langle f,g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{43}$$

Proof. By inserting (16) into the left side of (43), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle f,\phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\overline{\langle g,\phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle}_{L^2(\mathbb{R}^2;\mathbb{H})}d^2\boldsymbol{\omega}\,d^2\boldsymbol{b} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle f,\phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\left(\int_{\mathbb{R}^2}\frac{1}{(2\pi)^2}\,e^{\boldsymbol{j}\omega_2 x_2}e^{\boldsymbol{i}\omega_1 x_1}\phi(\boldsymbol{x}-\boldsymbol{b})\overline{g(\boldsymbol{x})}d^2\boldsymbol{x}\right)d^2\boldsymbol{\omega}\,d^2\boldsymbol{b} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{(2\pi)^4}f(\boldsymbol{x}')\,\overline{\phi(\boldsymbol{x}'-\boldsymbol{b})}e^{-\boldsymbol{i}\omega_1 x_1'}\right. \\ &\qquad\left.\times\, e^{\boldsymbol{j}\omega_2(x_2-x_2')}e^{\boldsymbol{i}\omega_1 x_1}d^2\boldsymbol{\omega}d^2\boldsymbol{x}'\right)\phi(\boldsymbol{x}-\boldsymbol{b})\overline{g(\boldsymbol{x})}d^2\boldsymbol{x}\,d^2\boldsymbol{b} \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}f(\boldsymbol{x}')\overline{\phi(\boldsymbol{x}'-\boldsymbol{b})}\delta^2(\boldsymbol{x}-\boldsymbol{x}')\phi(\boldsymbol{x}-\boldsymbol{b})\overline{g(\boldsymbol{x})}d^2\boldsymbol{x}'\right)d^2\boldsymbol{b}\,d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{b})}\phi(\boldsymbol{x}-\boldsymbol{b})\,\overline{g(\boldsymbol{x})}\,d^2\boldsymbol{b}\,d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}f(\boldsymbol{x})\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}\,\overline{g(\boldsymbol{x})}\,d^2\boldsymbol{x} \\ &= \frac{C_\phi}{(2\pi)^2}\int_{\mathbb{R}^2}f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,d^2\boldsymbol{x}, \end{aligned} \tag{44}$$

where in line five of (44) $\delta^2(\boldsymbol{x}-\boldsymbol{x}') = \delta(x_1-x_1')\delta(x_2-x_2')$. This completes the proof of (43). □

As an easy consequence of the previous theorem, we immediately obtain the following corollary.

Corollary 2.6 *If $f, \phi \in L^2(\mathbb{R}^2;\mathbb{H})$ are two quaternion-valued signals, then*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\, d^2\boldsymbol{b}\, d^2\boldsymbol{\omega} = \frac{1}{(2\pi)^2}\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{45}$$

In particular, if the quaternion window function is normalized so that $\|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})} = 1$, then (45) becomes

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\, d^2\boldsymbol{b}\, d^2\boldsymbol{\omega} = \frac{1}{(2\pi)^2}\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{46}$$

Proof. This identity is based on Theorem 2.5, with $\|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})} = 1$ and $g = f$. □

Equation (46) shows that the QWFT is an *isometry* from $L^2(\mathbb{R}^2;\mathbb{H})$ into $L^2(\mathbb{R}^2;\mathbb{H})$. In other words, up to a factor of $\frac{1}{(2\pi)^2}$ the *total energy* of a quaternion-valued signal computed in the spatial domain is equal to the total energy computed in the quaternionic windowed Fourier domain, compare (17) for the corresponding energy density.

Theorem 2.7 (Reproducing kernel) *Let be $\phi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion window function. If*

$$\mathbb{K}_\phi(\boldsymbol{\omega},\boldsymbol{b};\boldsymbol{\omega}',\boldsymbol{b}') = \frac{(2\pi)^2}{C_\phi}\langle \phi_{\boldsymbol{\omega},\boldsymbol{b}}, \phi_{\boldsymbol{\omega}',\boldsymbol{b}'}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}, \tag{47}$$

then $\mathbb{K}_\phi(\boldsymbol{\omega},\boldsymbol{b};\boldsymbol{\omega}',\boldsymbol{b}')$ is a reproducing kernel, i.e.

$$G_\phi f(\boldsymbol{\omega}',\boldsymbol{b}') = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\mathbb{K}_\phi(\boldsymbol{\omega},\boldsymbol{b};\boldsymbol{\omega}',\boldsymbol{b}')\, d^2\boldsymbol{\omega}\, d^2\boldsymbol{b} \tag{48}$$

Proof. By inserting (42) into the definition of the QWFT (16) we obtain

$$\begin{aligned}
G_\phi f(\boldsymbol{\omega}',\boldsymbol{b}') &= \int_{\mathbb{R}^2} f(\boldsymbol{x})\,\overline{\phi_{\boldsymbol{\omega}',\boldsymbol{b}'}(\boldsymbol{x})}\, d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}^2}\left(\frac{(2\pi)^2}{C_\phi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\,\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x}) d^2\boldsymbol{b}\, d^2\boldsymbol{\omega}\right)\overline{\phi_{\boldsymbol{\omega}',\boldsymbol{b}'}(\boldsymbol{x})}\, d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\frac{(2\pi)^2}{C_\phi}\left(\int_{\mathbb{R}^2}\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\overline{\phi_{\boldsymbol{\omega}',\boldsymbol{b}'}(\boldsymbol{x})}\, d^2\boldsymbol{x}\right) d^2\boldsymbol{b}\, d^2\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\mathbb{K}_\phi(\boldsymbol{\omega},\boldsymbol{b};\boldsymbol{\omega}',\boldsymbol{b}')\, d^2\boldsymbol{b}\, d^2\boldsymbol{\omega},
\end{aligned} \tag{49}$$

which finishes the proof. □

The above properties of the QWFT are summarized in Table 1.

3 Heisenberg's Uncertainty Principle for the QWFT

The classical uncertainty principle of harmonic analysis states that a non-trivial function and its Fourier transform can not both be simultaneously sharply localized [3, 22]. In quantum mechanics an uncertainty principle asserts one can not at the same time be certain of the position and of the velocity of an electron (or any particle). That is, increasing the knowledge of the position decreases the knowledge of the velocity or momentum of an electron. This section extends the uncertainty principle which is valid for the (right-sided) QFT [19] to the setting of the QWFT. A directional QFT uncertainty principle has been studied in [16].

In [19] a component-wise uncertainty principle for the QFT establishes a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. This uncertainty can be written in the following form.

Table 1: Properties of the QWFT of $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$, where $\lambda, \mu \in \mathbb{H}$ are constants and $\boldsymbol{x}_0 = x_0\boldsymbol{e}_1 + y_0\boldsymbol{e}_2 \in \mathbb{R}^2$.

Property	Quaternion Function	QWFT
Left linearity	$\lambda f(\boldsymbol{x}) + \mu g(\boldsymbol{x})$	$\lambda G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) + \mu G_\phi g(\boldsymbol{\omega}, \boldsymbol{b})$
Parity	$G_{P\phi}\{Pf\}(\boldsymbol{\omega}, \boldsymbol{b})$	$G_\phi f(-\boldsymbol{\omega}, -\boldsymbol{b})$
Specific shift	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$e^{-\boldsymbol{i}\omega_1 x_0}\left(G_\phi f(\boldsymbol{\omega}, \boldsymbol{b} - \boldsymbol{x}_0)\right) e^{-\boldsymbol{j}\omega_2 y_0}$, if $f = f_0 + \boldsymbol{i} f_1$ and $\phi = \phi_0 + \boldsymbol{i}\phi_1$
Formula		
Orthogonality	$\frac{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}}{(2\pi)^2}\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} =$	$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle f, \phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\overline{\langle g, \phi_{\boldsymbol{\omega},\boldsymbol{b}}\rangle}_{L^2(\mathbb{R}^2;\mathbb{H})} d^2\boldsymbol{\omega}\, d^2\boldsymbol{b}$
Reconstruction	$f(\boldsymbol{x}) =$	$\frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\, d^2\boldsymbol{b}\, d^2\boldsymbol{\omega}$
Isometry	$\frac{1}{(2\pi)^2}\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} =$	$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\|G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\|^2\, d^2\boldsymbol{b}\, d^2\boldsymbol{\omega}$, if $\|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})} = 1$
Reproducing Kernel	$G_\phi f(\boldsymbol{\omega}', \boldsymbol{b}') =$	$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}')\, d^2\boldsymbol{\omega}\, d^2\boldsymbol{b}$, $\mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}') = \frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}}\langle \phi_{\boldsymbol{\omega},\boldsymbol{b}}, \phi_{\boldsymbol{\omega}',\boldsymbol{b}'}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}$

Theorem 3.1 (QFT uncertainty principle) *Let $f \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion-valued function. If $\mathcal{F}_q\{f\}(\boldsymbol{\omega}) \in L^2(\mathbb{R}^2;\mathbb{H})$ too, then we have the inequality (no summation over k, $k = 1, 2$)*

$$\int_{\mathbb{R}^2} x_k^2 |f(\boldsymbol{x})|^2\, d^2\boldsymbol{x} \int_{\mathbb{R}^2} \omega_k^2 |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 d^2\boldsymbol{\omega} \geq \frac{(2\pi)^2}{4}\left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d^2\boldsymbol{x}\right)^2. \tag{50}$$

Equality holds if and only if f is the Gaussian quaternion function, i.e.

$$f(\boldsymbol{x}) = C_0\, e^{-(a_1 x_1^2 + a_2 x_2^2)}, \tag{51}$$

where C_0 is a quaternion constant and a_1, a_2 are positive real constants.

Applying the Parseval theorem for the QFT [12] to the right-hand side of (50) we get the following corollary.

Corollary 3.2 *Under the above assumptions, we have*

$$\int_{\mathbb{R}^2} x_k^2 |\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}](\boldsymbol{x})|^2\, d^2\boldsymbol{x} \int_{\mathbb{R}^2} \omega_k^2 |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 d^2\boldsymbol{\omega} \geq \left(\frac{1}{4\pi}\int_{\mathbb{R}^2} |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2\, d^2\boldsymbol{\omega}\right)^2. \tag{52}$$

Let us now establish a generalization of the Heisenberg type uncertainty principle for the QWFT. From a mathematical point of view this principle describes how the spatial extension of a two-dimensional quaternion function relates to the bandwidth of its QWFT.

Theorem 3.3 (QWFT uncertainty principle) *Let $\phi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion window function and let $G_\phi f \in L^2(\mathbb{R}^2;\mathbb{H})$ be the QWFT of f such that $\omega_k G_\phi f \in L^2(\mathbb{R}^2;\mathbb{H})$, $k = 1, 2$. Then for every $f \in L^2(\mathbb{R}^2;\mathbb{H})$ we have the following inequality:*

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega_k^2 |G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})|^2\, d^2\boldsymbol{\omega}\, d^2\boldsymbol{b}\right)^{1/2}\left(\int_{\mathbb{R}^2} x_k^2 |f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\right)^{1/2} \geq \frac{1}{4\pi}\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}\|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{53}$$

In order to prove this theorem, we need to introduce the following lemma.

Lemma 3.4 *Under the assumptions of Theorem 3.3, we have*

$$\frac{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}}{(2\pi)^4}\int_{\mathbb{R}^2} x_k^2|f(\boldsymbol{x})|^2\, d^2\boldsymbol{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} x_k^2|\mathcal{F}_q^{-1}\{G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\}(\boldsymbol{x})|^2\, d^2\boldsymbol{x}d^2\boldsymbol{b},\quad k=1,2. \tag{54}$$

Proof. Applying elementary properties of quaternions, we get

$$\begin{aligned}
\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}\int_{\mathbb{R}^2} x_k^2|f(\boldsymbol{x})|^2\, d^2\boldsymbol{x} &= \int_{\mathbb{R}^2} x_k^2|f(\boldsymbol{x})|^2 d^2\boldsymbol{x}\int_{\mathbb{R}^2}|\phi(\boldsymbol{x}-\boldsymbol{b})|^2\, d^2\boldsymbol{b}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} x_k^2|f(\boldsymbol{x})|^2|\phi(\boldsymbol{x}-\boldsymbol{b})|^2\, d^2\boldsymbol{x}\, d^2\boldsymbol{b}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} x_k^2|f(\boldsymbol{x})|^2|\overline{\phi(\boldsymbol{x}-\boldsymbol{b})}|^2\, d^2\boldsymbol{x}\, d^2\boldsymbol{b}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} x_k^2|f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{b})}|^2\, d^2\boldsymbol{x}\, d^2\boldsymbol{b}\\
&\overset{(39)}{=} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (2\pi)^4 x_k^2|\mathcal{F}_q^{-1}\{G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\}(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\, d^2\boldsymbol{b}.
\end{aligned} \tag{55}$$

□

Let us now begin with the proof of Theorem 3.3.

Proof. Replacing the QFT of f by the QWFT of f on the left-hand side of (52) in Corollary 3.2 we obtain

$$\int_{\mathbb{R}^2}\omega_k^2|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\, d^2\boldsymbol{\omega}\int_{\mathbb{R}^2} x_k^2|\mathcal{F}_q^{-1}\{G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\}(\boldsymbol{x})|^2\, d^2\boldsymbol{x} \geq \left(\frac{1}{4\pi}\int_{\mathbb{R}^2}|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\, d^2\boldsymbol{\omega}\right)^2. \tag{56}$$

This replacement is, according to (38), equivalent to inserting $f'(\boldsymbol{x}) = \frac{1}{(2\pi)^2} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{b})}$ in (52). Taking the square root on both sides of (56) and integrating both sides with respect to $d^2\boldsymbol{b}$ yields

$$\begin{aligned}
\int_{\mathbb{R}^2}\left\{\left(\int_{\mathbb{R}^2}\omega_k^2|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\, d^2\boldsymbol{\omega}\right)^{1/2}\left(\int_{\mathbb{R}^2} x_k^2|\mathcal{F}_q^{-1}\{G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\}(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\right)^{1/2}\right\} d^2\boldsymbol{b}\\
\geq \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\, d^2\boldsymbol{\omega}\, d^2\boldsymbol{b}.
\end{aligned} \tag{57}$$

Now applying the quaternion Cauchy-Schwarz inequality (6) (compare (4.14) of [19]) to the left-hand side of (57) we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\omega_k^2|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2 d^2\boldsymbol{\omega}\, d^2\boldsymbol{b}\right)^{1/2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} x_k^2|\mathcal{F}_q^{-1}\{G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\}(\boldsymbol{x})|^2 d^2\boldsymbol{x}\, d^2\boldsymbol{b}\right)^{1/2}\\
\geq \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2 d^2\boldsymbol{\omega}\, d^2\boldsymbol{b}.
\end{aligned} \tag{58}$$

Inserting Lemma 3.4 into the second term on the left-hand side of (58) and substituting (45) of Corollary 2.6 into the right-hand side of (58), we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\omega_k^2|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2 d^2\boldsymbol{\omega}\, d^2\boldsymbol{b}\right)^{1/2}\left(\frac{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}}{(2\pi)^4}\int_{\mathbb{R}^2} x_k^2|f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\right)^{1/2}\\
\geq \frac{1}{16\pi^3}\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}\|\phi\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}.
\end{aligned} \tag{59}$$

Dividing both sides of (59) by $\frac{\|\phi\|_{L^2(\mathbb{R}^2;\mathbb{H})}}{(2\pi)^2}$, we obtain the desired result. □

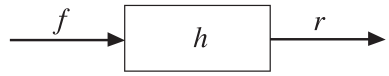

Figure 4: Block diagram of a two-dimensional linear time-varying system.

Remark 3.1 *According to the properties of the QFT and its uncertainty principle, Theorem 3.3 does not hold for summation over k. If we introduce summation, we would have to replace the factor $\frac{1}{4\pi}$ on the right hand side of (53) by $\frac{1}{2\pi}$.*

4 Application of the QWFT

The WFT plays a fundamental role in the analysis of signals and linear time-varying (TV) systems [6, 7, 21]. The effectiveness of the WFT is a result of its providing a unique representation for the signals in terms of the windowed Fourier kernel. It is natural to ask whether the QWFT can also be applied to such problems. This section briefly discusses the application of the QWFT to study two-dimensional linear TV systems (see Figure 4). We may regard the QWFT as a linear TV band-pass filter element of a filter-bank spectrum analyzer and, therefore, the TV spectrum obtained by the QWFT can also be interpreted as the output of such a linear TV band-pass filter element. For this purpose let us introduce the following definition.

Definition 4.1 *Consider a two-dimensional linear TV system with $h(\cdot,\cdot,\cdot)$ denoting the quaternion impulse response of the filter. The output $r(\cdot,\cdot)$ of the linear TV system is defined by*

$$r(\boldsymbol{\omega},\boldsymbol{b}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})h(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{b}-\boldsymbol{x})\,d^2\boldsymbol{x} = \int_{\mathbb{R}^2} f(\boldsymbol{b}-\boldsymbol{x})h(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{x})\,d^2\boldsymbol{x}, \tag{60}$$

where $f(\cdot)$ is a two-dimensional quaternion valued input signal.

We then obtain the transfer function $R(\cdot,\cdot)$ of the quaternion impulse response $h(\cdot,\cdot,\cdot)$ of the TV filter as

$$R(\boldsymbol{\omega},\boldsymbol{b}) = \int_{\mathbb{R}^2} h(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{\alpha})\,e^{-\boldsymbol{i}\omega_1\alpha_1}e^{-\boldsymbol{j}\omega_2\alpha_2}d^2\boldsymbol{\alpha}, \quad \boldsymbol{\alpha} = \alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 \in \mathbb{R}^2. \tag{61}$$

The following simple theorem (compare to Ghosh [7]) relates the QWFT to the output of a linear TV band-pass filter.

Theorem 4.1 *Consider a linear TV band-pass filter. Let the TV quaternion impulse response $h_1(\cdot,\cdot,\cdot)$ of the filter be defined by*

$$h_1(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{\alpha}) = \frac{1}{(2\pi)^2}\,\overline{\phi(-\boldsymbol{\alpha})}\,e^{-\boldsymbol{i}\omega_1(b_1-\alpha_1)}e^{-\boldsymbol{j}\omega_2(b_2-\alpha_2)}, \tag{62}$$

where $\phi(\cdot)$ is the quaternion window function. The output $r_1(\cdot,\cdot)$ of the TV system is equal to the QWFT of the quaternion input signal $f(\boldsymbol{x})$.

Proof. Using Definition 4.1, we get the output as follows:

$$\begin{aligned}
r_1(\boldsymbol{\omega},\boldsymbol{b}) &= \int_{\mathbb{R}^2} f(\boldsymbol{x})\,h_1(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{b}-\boldsymbol{x})\,d^2\boldsymbol{x} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(\boldsymbol{x})\,\overline{\phi(\boldsymbol{x}-\boldsymbol{b})}\,e^{-\boldsymbol{i}\omega_1(b_1-(b_1-x_1))}e^{-\boldsymbol{j}\omega_2(b_2-(\omega_2-x_2))}\,d^2\boldsymbol{x} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(\boldsymbol{x})\,\overline{\phi(\boldsymbol{x}-\boldsymbol{b})}\,e^{-\boldsymbol{i}x_1\omega_1}e^{-\boldsymbol{j}x_2\omega_2}\,d^2\boldsymbol{x} \\
&= G_\phi f(\boldsymbol{\omega},\boldsymbol{b}),
\end{aligned} \tag{63}$$

which proves the theorem. □

This shows that the choice of the quaternion impulse response of the filter will determine a characteristic output of the linear TV systems. For example, if we translate the TV quaternion impulse response $h_1(\cdot,\cdot,\cdot)$ by $\boldsymbol{b}_0 = b_{01}\boldsymbol{e}_1 + b_{02}\boldsymbol{e}_2$, i.e.

$$h_1(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{\alpha}) \to h_1(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{\alpha}-\boldsymbol{b}_0) = \frac{1}{(2\pi)^2}\,\overline{\phi(-(\boldsymbol{\alpha}-\boldsymbol{b}_0))}\, e^{-\boldsymbol{i}\omega_1(b_1-(\alpha_1-b_{01}))} e^{-\boldsymbol{j}\omega_2(b_2-(\alpha_2-b_{02}))}, \quad (64)$$

then the output is according to Theorem 2.3

$$r_{1,\boldsymbol{b}_0}(\boldsymbol{\omega},\boldsymbol{b}) = e^{-\boldsymbol{i}\omega_1 b_{01}}\, G_\phi f(\boldsymbol{\omega},\boldsymbol{b}-\boldsymbol{b}_0)\, e^{-\boldsymbol{j}\omega_2 b_{02}}. \quad (65)$$

In this case, we assumed that the input $f\boldsymbol{i} = \boldsymbol{i}f$ and the window function $\phi\boldsymbol{i} = \boldsymbol{i}\phi$.

Theorem 4.2 *Consider a linear TV band-pass filter with the TV quaternion impulse response $h_2(\cdot,\cdot,\cdot)$ defined by*

$$h_2(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{\alpha}) = e^{-\boldsymbol{i}\omega_1(b_1-\alpha_1)} e^{-\boldsymbol{j}\omega_2(b_2-\alpha_2)}, \quad (66)$$

If the input to this system is the quaternion signal $f(\boldsymbol{x})$, its output $r_2(\boldsymbol{\omega}) = r_2(\boldsymbol{\omega},\cdot)$ is, independent of the $\boldsymbol{b}$-argument, equal to the QFT of f:

$$r_2(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega}). \quad (67)$$

Proof. Using Definition 4.1, we obtain

$$\begin{aligned} r_2(\boldsymbol{\omega},\boldsymbol{b}) &= \int_{\mathbb{R}^2} f(\boldsymbol{x})\, h_2(\boldsymbol{\omega},\boldsymbol{b},\boldsymbol{b}-\boldsymbol{x})\, d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{x})\, e^{-\boldsymbol{i}\omega_1(b_1-(b_1-x_1))}\, e^{-\boldsymbol{j}\omega_2(b_2-(b_2-x_2))}\, d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{x})\, e^{-\boldsymbol{i}x_1\omega_1} e^{-\boldsymbol{j}x_2\omega_2}\, d^2\boldsymbol{x} \\ &= \mathcal{F}_q\{f\}(\boldsymbol{\omega}). \end{aligned} \quad (68)$$

Or $r_2(\boldsymbol{\omega}) = r_2(\boldsymbol{\omega},\boldsymbol{b}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega})$, because the right-hand side of (68) is independent of $\boldsymbol{b}$. □

Example 4.1 *Given the TV quaternion impulse response defined by (66). Find the output $r_2(\cdot)$ (see Figure 5) of the following input*

$$f(\boldsymbol{x}) = \begin{cases} e^{-(x_1+x_2)}, & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (69)$$

From Theorem 4.2, we obtain the QFT of f

$$\begin{aligned} r_2(\boldsymbol{\omega}) &= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\infty e^{-x_1(1+\boldsymbol{i}\omega_1)} e^{-x_2(1+\boldsymbol{j}\omega_2)}\, d^2\boldsymbol{x} \\ &= \frac{1}{(2\pi)^2}\,\frac{-1}{1+\boldsymbol{i}\omega_1}\, e^{-\boldsymbol{i}\omega_1 x_1} e^{-x_1}\Big|_0^\infty \,\frac{-1}{(1+\boldsymbol{j}\omega_2)}\, e^{-\boldsymbol{j}\omega_2 x_2} e^{-x_2}\Big|_0^\infty \\ &= \frac{1}{(2\pi)^2}\,\frac{1}{1+\boldsymbol{i}\omega_1+\boldsymbol{j}\omega_2+\boldsymbol{k}\omega_1\omega_2} \\ &= \frac{1}{(2\pi)^2}\,\frac{1-\boldsymbol{i}\omega_1-\boldsymbol{j}\omega_2-\boldsymbol{k}\omega_1\omega_2}{1+\omega_1^2+\omega_2^2+\omega_1^2\omega_2^2}\,. \end{aligned} \quad (70)$$

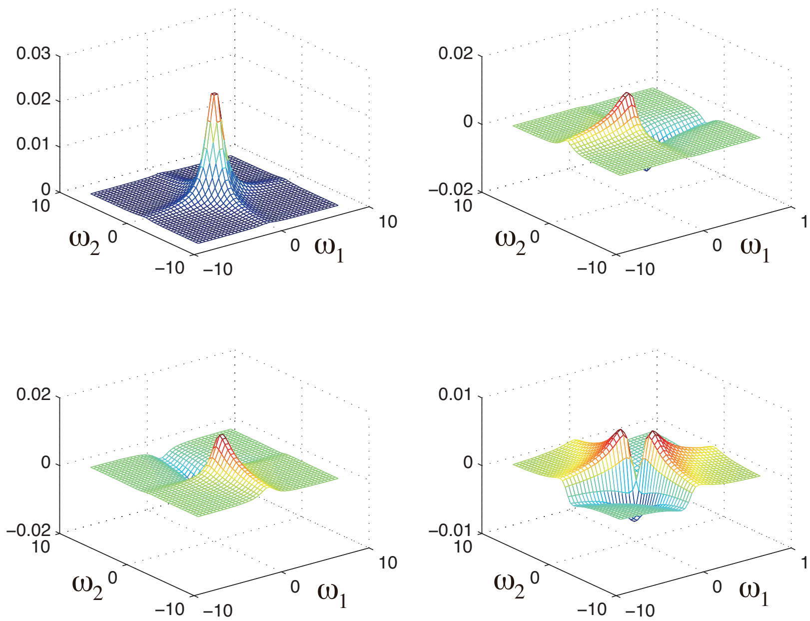

Figure 5: The real part (*top left*) and imaginary $\boldsymbol{i}$-part (*top right*), $\boldsymbol{j}$-part (*bottom left*), and $\boldsymbol{k}$-part (*bottom right*) of the output $r_2(\cdot)$ in Example 4.1, i.e. the QFT (70) of (69).

Example 4.2 *Consider the TV quaternion impulse response defined by (62) with respect to the first order two-dimensional B-spline window function* (19) *in Example 2.1. Find the output* $r_1(\cdot,\cdot)$ *(see Figure 6) of the input* (69) *defined in Example 4.1.*

With $m_1 = \max(0, -1+b_1), m_2 = \max(0, -1+b_2)$, Theorem 4.1 gives

$$\begin{aligned} r_1(\boldsymbol{\omega}, \boldsymbol{b}) &= G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \\ &= \frac{1}{(2\pi)^2} \int_{m_1}^{1+b_1} e^{-x_1} e^{-\boldsymbol{i} x_1 \omega_1} dx_1 \int_{m_2}^{1+b_2} e^{-x_2} e^{-\boldsymbol{j} x_2 \omega_2} dx_2 \\ &= \frac{-1}{(2\pi)^2 (1+\boldsymbol{i}\omega_1)} e^{-x_1(1+\boldsymbol{i}\omega_1)} \Big|_{m_1}^{1+b_1} \frac{-1}{(1+\boldsymbol{j}\omega_2)} e^{-x_2(1+\boldsymbol{j}\omega_1)} \Big|_{m_2}^{1+b_2} \\ &= \frac{(e^{-m_1(1+\boldsymbol{i}\omega_1)} - e^{-(1+b_1)(1+\boldsymbol{i}\omega_1)})(e^{-m_2(1+\boldsymbol{j}\omega_2)} - e^{-(1+b_2)(1+\boldsymbol{j}\omega_2)})}{(2\pi)^2 (1 + \boldsymbol{i}\omega_1 + \boldsymbol{j}\omega_2 + \boldsymbol{k}\omega_1\omega_2)}. \end{aligned} \tag{71}$$

For the sake of simplicity, we take the parameters $b_1 = b_2 = 0 \Rightarrow m_1 = m_2 = 0$, to obtain

$$\begin{aligned} r_1(\boldsymbol{\omega}, \boldsymbol{b} = 0) &= \frac{(1 - e^{-(1+\boldsymbol{i}\omega_1)})(1 - e^{-(1+\boldsymbol{j}\omega_2)})}{(2\pi)^2 (1 + \boldsymbol{i}\omega_1 + \boldsymbol{j}\omega_2 + \boldsymbol{k}\omega_1\omega_2)} \\ &= \frac{1 - e^{-1}\cos\omega_1 - e^{-1}\cos\omega_2 + e^{-2}\cos\omega_1\cos\omega_2 + \boldsymbol{i}(e^{-1}\sin\omega_1 - e^{-2}\sin\omega_1\cos\omega_2)}{(2\pi)^2 (1 + \boldsymbol{i}\omega_1 + \boldsymbol{j}\omega_2 + \boldsymbol{k}\omega_1\omega_2)} \\ &\quad + \frac{\boldsymbol{j}(e^{-1}\sin\omega_2 - e^{-2}\cos\omega_1\sin\omega_2) + \boldsymbol{k}\, e^{-2}\sin\omega_1\sin\omega_2}{(2\pi)^2 (1 + \boldsymbol{i}\omega_1 + \boldsymbol{j}\omega_2 + \boldsymbol{k}\omega_1\omega_2)}. \end{aligned} \tag{72}$$

We may regard the QFT (70) as the QWFT with an infinite window function. Since the integration domain of the QWFT (71) is smaller than that of the QFT (70), the QWFT output (71) is more localized in the base space than the QFT output of (70). In addition, according to the Paley-Wiener theorem the QWFT output of (71) is very smooth. This means that it provides accurate information on the output $r(\cdot,\cdot)$ due to the local window function ϕ.

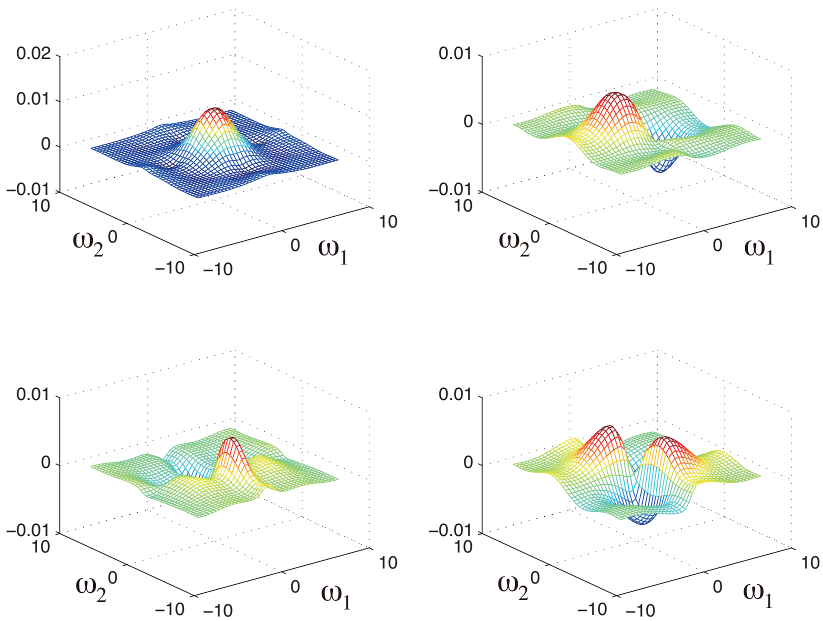

Figure 6: The real part (*top left*) and imaginary $\boldsymbol{i}$-part (*top right*), $\boldsymbol{j}$-part (*bottom left*), and $\boldsymbol{k}$-part (*bottom right*) of the output $r_1(\boldsymbol{\omega}, \boldsymbol{b} = 0)$ in Example 4.2, i.e. the QWFT (71) of (62) with $b_1 = b_2 = 0$.

5 Conclusion

Using the basic concepts of quaternion algebra and the (right-sided) QFT we introduced the QWFT. Important properties of the QWFT such as left linearity, parity, reconstruction formula, reproducing kernel, isometry, and orthogonality relation were demonstrated. Because of the non-commutativity of multiplication in the quaternion algebra $\mathbb{H}$, not all properties of the classical WFT can be established for the QWFT, such as general shift and modulation properties. This generalization also enables us to construct quaternionic Gabor filters (compare to Bülow [1, 2]), which can extend the applications of the 2D complex Gabor filters to texture segmentation and disparity estimation [1].

We have established a new uncertainty principle for the QWFT. This principle is founded on the QWFT properties and the uncertainty principle for the (right-sided) QFT. We also applied the QWFT to a linear time-varying (TV) system. We showed that the output of a linear TV system can result in a QFT or a QWFT of the quaternion input signal, depending on the choice of the quaternion impulse response of the filter.

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