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Clifford Algebra $Cl_{3,0}$-valued Wavelet Transformation, Clifford Wavelet Uncertainty Inequality and Clifford Gabor Wavelets

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In this paper, it is shown how continuous Clifford $Cl_{3,0}$-valued admissible wavelets can be constructed using the similitude group *SIM(3)*, a subgroup of the affine group of $\mathbb{R}^3$. We express the admissibility condition in terms of a $Cl_{3,0}$ Clifford Fourier transform and then derive a set of important properties such as dilation, translation and rotation covariance, a reproducing kernel, and show how to invert the Clifford wavelet transform of multivector functions. We invent a generalized Clifford wavelet uncertainty principle. For scalar admissibility constant it sets bounds of accuracy in multivector wavelet signal and image processing. As concrete example we introduce multivector Clifford Gabor wavelets, and describe important properties such as the Clifford Gabor transform isometry, a reconstruction formula, and an uncertainty principle for Clifford Gabor wavelets.

Keywords: Similitude group, Clifford Fourier transform, Clifford wavelet transform, Clifford Gabor wavelets, uncertainty principle.

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1. Introduction

Transformations such as the Fourier transformation are powerful methods for signal representations and feature detection in signals. The signals are transformed from the original domain to the spectral or frequency domain. In the frequency domain many characteristics of a signal are seen more clearly. In contrast to the Fourier kernel, wavelet basis functions are localized in both spatial and frequency domains and thus yield very sparse and well-structured representations of *piecewise smooth signals* (signals that are smooth except for a finite number of discontinuous jumps), important facts from a signal processing point of view.

On the other hand Clifford geometric algebra leads to the consequent generalization[1] of real and harmonic analysis to higher dimensions. Clifford algebra accurately treats geometric entities depending on their dimension as scalars, vectors, bivectors (plane area elements), and volume elements, etc. The distinction of axial and polar vectors in physics, e.g. is resolved in the form of vectors and bivectors. The quaternion description of rotations[2] is fully incorporated in the form of rotors. With respect to the geometric product of vectors division by non-zero vectors is defined. Clifford algebra has applications in signal and image processing.[3]

This motivated Mitrea[4] to generalize discrete real wavelets to discrete Clifford algebra wavelets. Some properties of these extended wavelets were also demonstrated. This first work was then followed by Brackx and Sommen[5,6] who proposed an extension of real wavelets to the Clifford algebra $Cl_{0,n}$ called the continuous Clifford wavelet transform. This approach used a group composed of dilations, translations and the Spin-group. Quaternion ($Cl_{0,2}$) wavelets have been studied by Zhao and Peng, [7] and applied by Bayro-Corrochano.[8]

Zhao[9] also constructed continuous Clifford algebra $Cl_{0,n}$-valued wavelets using the semi-direct product of closed $GL(n, \mathbb{R})$ subgroups with the translation subgroup of $\mathbb{R}^n$. Some properties of these extended wavelets were investigated using the classical Fourier transform. The main differences of our present work and Zhao's work are

- the specification and implementation of the underlying transformation group
- the signature of the Clifford algebra
- the use of a Clifford Fourier transformation instead of a mere complex Fourier transformation
- a detailed investigation of the multivector algebra properties of the admissibility constant, including a nontrivial condition on its scalar and vector parts
- the derivation of wavelet uncertainty inequalities.

The purpose of this paper is to construct Clifford algebra $Cl_{3,0}$-valued wavelets using the similitude group *SIM(3)* and then give a detailed explanation of their properties using the Clifford Fourier transform (CFT) described in 10, 11, 12. This form of the CFT has e.g. also been applied by Felsberg [13] as a way to compute monogenic signals[a]. Other variants of the CFT were introduced by Brackx et al. [1] who extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. A related applied approach for hypercomplex Clifford Fourier transformations in $Cl_{0,n}$ was followed by Bülow et al [14]. Ell and Sangwine [15,16] and Le Bihan et al [17] introduced and applied a quaternion Fourier transformation[b]

[a]Monogenic signals are a two-dimensional analogue of the complex analytic signal. They allow to extract a local phase amplitude and a local phase vector for two-dimensional images. [13]
[b]Note that the quaternion algebra is isomorphic to the Clifford algebra $Cl_{0,2}$.

(QFT) for (color) image and signal processing. Buelow [3] used quaternionic Gabor filters based on the QFT to introduce a local quaternionic phase for two-dimensional images. Hitzer [18] deepened the algebraic and geometric properties of the QFT, and generalized the QFT to higher dimensional Clifford Fourier transformations.

These enormous conceptual and practical benefits of using CFTs compared to mere complex Fourier transformations should suffice to warrant the full use of the Clifford geometric algebra framework in wavelet analysis as well. We further emphasize the rigorous derivation of results because the use of non-commutative Clifford algebra in wavelet theory is a non-trivial step compared to commutative complex wavelet theory.

Based on the uncertainty principle for the CFT we derive a generalized Clifford wavelet uncertainty principle. For scalar admissibility constant the interpretation of this uncertainty principle proceeds as usual.

As a concrete example we generalize complex Gabor wavelets to multivector Clifford Gabor wavelets. Next, we describe some of their important properties and we consequently establish an uncertainty principle for Clifford Gabor wavelets.

The outline of this paper is as follows. In section 2, we briefly review Clifford algebra, the CFT, and the similitude group $SIM(3)$. In section 3, we discuss the basic ideas for constructing the Clifford algebra wavelet transform. We then derive important properties of our newly constructed wavelet transform. In section 4, we show how to derive the generalized Clifford wavelet uncertainty principle. In section 5, we present the example of multivector Gabor wavelets and show to what extent the properties of these Clifford Gabor wavelets resemble that of real wavelets. Finally, the uncertainty principle for the Clifford Gabor wavelet transform is presented.

2. Basics: Clifford algebra, Clifford Fourier transform, similitude group

This section introduces the basic concepts[1,19,20,21] of the Clifford geometric algebra $Cl_{3,0}$ and its Clifford Fourier transform [10,11,12]. We also recall the similitude group $SIM(3)$ and its properties from the viewpoint of wavelets.

2.1. *Real Clifford Algebra* $Cl_{3,0}$

Let us briefly review some basic facts of the Clifford geometric algebra $Cl_{3,0}$ of $\mathbb{R}^3$ (for more details see 1, 19, 20 and 21). Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be an orthonormal basis of the real 3D Euclidean vector space $\mathbb{R}^3$. The associative geometric multiplication of the basis vectors is governed by

$$\boldsymbol{e}_k^2 = 1, \qquad \boldsymbol{e}_k\,\boldsymbol{e}_l = -\boldsymbol{e}_l\,\boldsymbol{e}_k \qquad \text{for} \quad l \neq k, \qquad k,l = 1,2,3. \tag{2.1}$$

The Clifford geometric algebra over $\mathbb{R}^3$ denoted by $Cl_{3,0}$ then has the graded $2^3 = 8$-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \boldsymbol{e}_{123}\}, \tag{2.2}$$

Table 1. $Cl_{3,0}$ basis in terms of real and dual vectors and scalars, reverted basis, ordered by grade.

Grade	Name	Basis	Reverted Basis
0	real scalar	1	1
1	real vectors	$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$	$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
2	bivectors / dual vectors	$\{i_3\boldsymbol{e}_1, i_3\boldsymbol{e}_2, i_3\boldsymbol{e}_3\}$	$\{-i_3\boldsymbol{e}_1, -i_3\boldsymbol{e}_2, -i_3\boldsymbol{e}_3\}$
3	trivector / dual scalar	i_3	$-i_3$

where 1 is the real scalar identity element (grade 0), $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathbb{R}^3$ are vectors (grade 1),

$$\boldsymbol{e}_{12} = \boldsymbol{e}_1\boldsymbol{e}_2 = i_3\boldsymbol{e}_3, \quad \boldsymbol{e}_{31} = \boldsymbol{e}_3\boldsymbol{e}_1 = i_3\boldsymbol{e}_2, \quad \boldsymbol{e}_{23} = \boldsymbol{e}_2\boldsymbol{e}_3 = i_3\boldsymbol{e}_1 \tag{2.3}$$

are a basis of bivectors (grade 2), and

$$\boldsymbol{e}_{123} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = i_3, \qquad i_3^2 = -1 \tag{2.4}$$

defines the unit oriented pseudoscalar[c] (grade 3), i.e. the highest grade blade element in $Cl_{3,0}$. i_3 is central in $Cl_{3,0}$. These properties allow us to rewrite the basis multivectors in terms of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, and i_3 as in table 1.

The general elements of a geometric algebra are called multivectors. Every multivector $f \in Cl_{3,0}$ can be expressed as

$$f = \sum_A f_A \boldsymbol{e}_A, \qquad f_A \in \mathbb{R}, \quad A \in \{0, 1, 2, 3, 12, 31, 23, 123\}. \tag{2.5}$$

The reverse $\tilde{f}$ (i.e. vector factors in reverse order) of a multivector f (an anti-automorphism that corresponds to complex conjugation in table 1) is given by

$$\tilde{f} = \sum_A f_A \widetilde{\boldsymbol{e}_A} = \langle f \rangle_0 + \langle f \rangle_1 - \langle f \rangle_2 - \langle f \rangle_3, \tag{2.6}$$

where $\langle \ldots \rangle_k$ indicates grade k selection. Note that we often write $\langle \ldots \rangle$ for $\langle \ldots \rangle_0$.

The symmetric *scalar* product[d] of multivectors $f, \tilde{g}$ is defined as the scalar (grade 0, indicated by $\langle \ldots \rangle_0 = \langle \ldots \rangle$) part of the geometric product $f\tilde{g}$ of multivectors

$$f * \tilde{g} = \langle f\tilde{g} \rangle_0 = \sum_A f_A g_A. \tag{2.7}$$

The modulus (or magnitude) $|f|$ of a multivector $f \in Cl_{3,0}$ is defined as

$$|f|^2 = f * \tilde{f} = \sum_A f_A^2. \tag{2.8}$$

[c]Other names in use are *dual scalar*, *trivector* or *unit oriented volume element*.
[d]The use of the $*$ symbol for the scalar product is well-established standard in Clifford geometric algebra literature [21].

Replacing in (2.5) the real components f_A by real functions $f_A(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^3$ yields a multivector-valued function $f : \mathbb{R}^3 \to Cl_{3,0}$ of the form

$$f(\boldsymbol{x}) = \sum_A f_A(\boldsymbol{x})\boldsymbol{e}_A. \tag{2.9}$$

It is convenient to introduce an inner product of $\mathbb{R}^3 \to Cl_{3,0}$ functions f, g as follows

$$(f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})} = \int_{\mathbb{R}^3} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\, d^3\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A\widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^3} f_A(\boldsymbol{x})g_B(\boldsymbol{x})\, d^3\boldsymbol{x}. \tag{2.10}$$

Note that[e]

$$d^3\boldsymbol{x} = \frac{d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge d\boldsymbol{x_3}}{i_3} \tag{2.11}$$

is scalar valued ($d\boldsymbol{x}_k = dx_k\boldsymbol{e}_k$, $k = 1, 2, 3$, no summation). In (2.10) the inner product $(\ ,\ )_{L^2(\mathbb{R}^3;Cl_{3,0})}$ satisfies the following conditions[1]

$$\begin{aligned}
(f, g+h)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= (f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})} + (f,h)_{L^2(\mathbb{R}^3;Cl_{3,0})}, \\
(f, \lambda g)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= (f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})}\tilde{\lambda}, \\
(f\lambda, g)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= (f, g\tilde{\lambda})_{L^2(\mathbb{R}^3;Cl_{3,0})}, \\
(f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})} &= \widetilde{(g,f)}_{L^2(\mathbb{R}^3;Cl_{3,0})}.
\end{aligned} \tag{2.12}$$

where $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$, and the constant multivector $\lambda \in Cl_{3,0}$. The scalar part of the inner product gives the L^2-norm

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} &= \langle (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})} \rangle \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{x}) * \tilde{f}(\boldsymbol{x}) d^3\boldsymbol{x} \overset{(2.7)}{=} \int_{\mathbb{R}^3} \sum_A f_A^2(\boldsymbol{x}) d^3\boldsymbol{x}.
\end{aligned} \tag{2.13}$$

In particular for $g = \boldsymbol{a}f,\ f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$, $\boldsymbol{a} \in \mathbb{R}^3$ we get because of $\langle \boldsymbol{a}f\widetilde{\boldsymbol{a}f}\rangle_0 = \langle \boldsymbol{a}f\tilde{f}\boldsymbol{a}\rangle_0 = \langle \boldsymbol{a}^2 f\tilde{f}\rangle_0 = \boldsymbol{a}^2 f * f$

$$\|\boldsymbol{a}f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} = \int_{\mathbb{R}^3} \boldsymbol{a}^2 f(\boldsymbol{x}) * \tilde{f}(\boldsymbol{x}) d^3\boldsymbol{x} = \int_{\mathbb{R}^3} \boldsymbol{a}^2 \sum_A f_A^2(\boldsymbol{x}) d^3\boldsymbol{x}. \tag{2.14}$$

Definition 2.1 (Clifford module). *Let $Cl_{3,0}$ be the real Clifford algebra of 3D Euclidean space $\mathbb{R}^3$. A Clifford algebra module $L^2(\mathbb{R}^3; Cl_{3,0})$ is defined by*

$$L^2(\mathbb{R}^3; Cl_{3,0}) = \{f : \mathbb{R}^3 \longrightarrow Cl_{3,0} \mid \|f\|_{L^2(\mathbb{R}^3;Cl_{3,0})} < \infty\}. \tag{2.15}$$

[e]The division by the geometric algebra unit volume element i_3 in (2.11) to obtain a scalar infinitesimal volume is a matter of choice. Defining instead the pseudoscalar $d^3\boldsymbol{x}_p = d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge d\boldsymbol{x_3}$ would work equally well. It would simply mean, that all integrals using $d^3\boldsymbol{x}_p$ instead of $d^3\boldsymbol{x}$ in this paper would have to be multiplied by $-i_3 = \frac{1}{i_3}$, which is central in $Cl_{3,0}$.

2.2. $Cl_{3,0}$ Clifford Fourier Transform (CFT)

Let us first define[10,11,12] the Clifford Fourier transform (CFT) on $L^2(\mathbb{R}^3; Cl_{3,0})$ as follows

Definition 2.2 (Clifford Fourier transform (CFT)). *The Clifford Fourier transform (CFT) of $f(\boldsymbol{x}) \in L^2(\mathbb{R}^3; Cl_{3,0})$, with $\int_{\mathbb{R}^3} |f(\boldsymbol{x})| d^3\boldsymbol{x} < \infty$ is the function $\mathcal{F}\{f\}\colon \mathbb{R}^3 \to Cl_{3,0}$ given by*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\, e^{-i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}, \tag{2.16}$$

where $\boldsymbol{\omega} \in \mathbb{R}^3$, $\boldsymbol{\omega} \cdot \boldsymbol{x} = \boldsymbol{\omega} * \boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3$.

Because i_3 is central in $Cl_{3,0}$, the Clifford Fourier kernel $e^{-i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}}$ will also commute with every multivector of $Cl_{3,0}$.

Theorem 2.1 (Inverse CFT). *The Clifford Fourier transform $\mathcal{F}\{f\}$ of $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ with $\int_{\mathbb{R}^3} |f(\boldsymbol{x})| d^3\boldsymbol{x} < \infty$ is invertible with inverse*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{\omega}. \tag{2.17}$$

The following theorem shows that a rotation of the argument of f leads to the same rotation of the argument of $\mathcal{F}\{f\}$ (see table 2).

Theorem 2.2 (Rotation property). *If $r_{\boldsymbol{\theta}} \in S0(3)$, then the Clifford Fourier transform of $f(r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{x}))$ is given by*

$$\mathcal{F}\{f(r_{\boldsymbol{\theta}}^{-1}(.))\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})), \quad \boldsymbol{\omega} \in \mathbb{R}^3. \tag{2.18}$$

Proof Equation (2.16) immediately gives

$$\begin{aligned} \mathcal{F}\{f(r_{\boldsymbol{\theta}}^{-1}(.))\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{x})) e^{-i_3 \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{y}) e^{-i_3 \boldsymbol{\omega}\cdot(r_{\boldsymbol{\theta}}\boldsymbol{y})} \det(r_{\boldsymbol{\theta}})\, d^3\boldsymbol{y} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{y}) e^{-i_3 (r_{\boldsymbol{\theta}}^{-1}\boldsymbol{\omega}) \cdot \boldsymbol{y}}\, d^3\boldsymbol{y}. \end{aligned} \tag{2.19}$$

□

Table 2 summarizes basic properties of the CFT. See 11 for more details and proofs.

2.3. Similitude group

We consider *the similitude* group *SIM (3)* denoted by $\mathcal{G}$, a subgroup of the affine group of motion on $\mathbb{R}^3$ associated with wavelets as follows (for more details see 23)

$$\mathcal{G} = \mathbb{R}^+ \times SO(3) \otimes \mathbb{R}^3 = \{(a, r_{\boldsymbol{\theta}}, \boldsymbol{b}) | a \in \mathbb{R}^+, r_{\boldsymbol{\theta}} \in SO(3), \boldsymbol{b} \in \mathbb{R}^3\}, \tag{2.20}$$

Table 2. Properties of the Clifford Fourier transform (CFT)

Property	**Multivector function**	**Clifford Fourier transform**
Linearity	$\alpha f(\boldsymbol{x}) + \beta\, g(\boldsymbol{x})$	$\alpha\mathcal{F}\{f\}(\boldsymbol{\omega}) + \beta\mathcal{F}\{g\}(\boldsymbol{\omega})$ constants $\alpha, \beta \in Cl_{3,0}$
Delay	$f(\boldsymbol{x} - \boldsymbol{a})$	$e^{-i_3 \boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega})$
Shift	$e^{i_3 \boldsymbol{\omega}_0\cdot\boldsymbol{x}} f(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0), \quad \boldsymbol{\omega}_0 \in \mathbb{R}^3$
Scaling	$f(a\boldsymbol{x})$	$\frac{1}{\lvert a\rvert^3}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a}), \quad a \in \mathbb{R} \setminus \{0\}$
Rotation	$f(r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{x}))$	$\mathcal{F}\{f\}(r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))$
Convolution	$[f\star g](\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\,\mathcal{F}\{g\}(\boldsymbol{\omega})$
Planch. Th.	$(f_1, f_2)_{L^2(\mathbb{R}^3;Cl_{3,0})}$	$= \frac{1}{(2\pi)^3}(\mathcal{F}\{f_1\}, \mathcal{F}\{f_2\})_{L^2(\mathbb{R}^3;Cl_{3,0})}$
Parseval Th.	$\lVert f\rVert_{L^2(\mathbb{R}^3;Cl_{3,0})}$	$= \frac{1}{(2\pi)^{\frac{3}{2}}}\lVert\mathcal{F}\{f\}\rVert_{L^2(\mathbb{R}^3;Cl_{3,0})}$
Gaussian	$e^{-(a\boldsymbol{x})^2/2}$	$(2\pi)^{3/2}\lvert a\rvert^{-3}e^{-(\boldsymbol{\omega}/a)^2/2}$

where $SO(3)$ is the special orthogonal group of $\mathbb{R}^3$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ with $\theta_1 \in [0, \pi], \theta_2, \theta_3 \in [0, 2\pi]$. Instead of $(a, r_{\boldsymbol{\theta}}, \boldsymbol{b})$ we often write simply $(a, \boldsymbol{\theta}, \boldsymbol{b})$. More precisely, we represent $SO(3)$ of $\mathbb{R}^3$ by rotors R

$$SO(3) = \{r \,|\, r(\boldsymbol{x}) = \tilde{R}\boldsymbol{x}R, R \in Cl^+_{3,0}/\{\pm 1\}, \tilde{R}R = R\tilde{R} = 1\}. \tag{2.21}$$

Any $r \in SO(3)$ has a unique Euler angle representation with rotors of the form

$$R = R_z(\theta_3)R_y(\theta_1)R_z(\theta_2), \tag{2.22}$$

where R_z, R_y denote rotors about the z- and y-axes, respectively. Note that the group $\mathcal{G}$ includes dilations, rotations and translations. The representation defined by (2.20) is consistent with the group action $(a, \boldsymbol{\theta}, \boldsymbol{b})$ on $\mathbb{R}^3$ as follows

$$\begin{aligned} (a, \boldsymbol{\theta}, \boldsymbol{b}) : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \boldsymbol{x} &\mapsto a\tilde{R}(\boldsymbol{\theta})\boldsymbol{x}R(\boldsymbol{\theta}) + \boldsymbol{b}. \end{aligned} \tag{2.23}$$

The above leads to two important propositions.

Proposition 2.1. *With respect to the representation defined by (2.20), $\mathcal{G}$ is a non-abelian group in which* $(1, 1, 0)$ *and* $(a^{-1}, r^{-1}, -a^{-1}r^{-1}(\boldsymbol{b}) = -R\boldsymbol{b}\tilde{R}/a)$ *are its identity element and inverse element, respectively.*

Proposition 2.2. *The left Haar measure*[f] *on $\mathcal{G}$ (see 24) is given by*

$$d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b}) = d\mu(a, \boldsymbol{\theta})d^3\boldsymbol{b},$$
$$d\mu(a, \boldsymbol{\theta}) = \frac{da d\boldsymbol{\theta}}{a^4}, \quad d\boldsymbol{\theta} = \frac{1}{8\pi^2}\sin\theta_1 d\theta_1 d\theta_2 d\theta_3, \tag{2.24}$$

[f]The right Haar measure on $\mathcal{G}$ is $d\delta(a, \boldsymbol{\theta}, \boldsymbol{b}) = \frac{da d\boldsymbol{\theta}}{a}d^3\boldsymbol{b}$.

where $d\boldsymbol{\theta}$ is the Haar measure on SO(3) (see 25).

We often abbreviate $d\mu = d\mu(a, \boldsymbol{\theta}),\ d\lambda = d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b})$. Similar to (2.10) the inner product of $f(a, \boldsymbol{\theta}, \boldsymbol{b}), g(a, \boldsymbol{\theta}, \boldsymbol{b}) \in L^2(\mathcal{G}; Cl_{3,0})$ is defined by

$$(f, g)_{L^2(\mathcal{G};Cl_{3,0})} = \int_{\mathcal{G}} f(a, \boldsymbol{\theta}, \boldsymbol{b})\widetilde{g(a, \boldsymbol{\theta}, \boldsymbol{b})}\; d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b}), \tag{2.25}$$

and its associated scalar norm

$$\|f\|^2_{L^2(\mathcal{G};Cl_{3,0})} = \left\langle (f, f)_{L^2(\mathcal{G};Cl_{3,0})} \right\rangle = \int_{\mathcal{G}} f(a, \boldsymbol{\theta}, \boldsymbol{b}) * \tilde{f}(a, \boldsymbol{\theta}, \boldsymbol{b}) d\boldsymbol{\mu}. \tag{2.26}$$

3. Clifford algebra $Cl_{3,0}$-valued wavelet transform

3.1. *Basics*

Based on the concepts of Clifford algebra, one can extend the real continuous wavelet transform to a continuous Clifford wavelet transform. This section constructs the Clifford algebra $Cl_{3,0}$-valued wavelets from a group theoretical point of view. We will see how some properties of the classical wavelet transform are extended in the new construction. In particular we look at the admissibility condition, inner product and norm identities, and a reproducing kernel. We define the unitary linear operator

$$\begin{aligned} U_{a,\boldsymbol{\theta},\boldsymbol{b}} : L^2(\mathbb{R}^3; Cl_{3,0}) &\longrightarrow L^2(\mathcal{G}; Cl_{3,0}) \\ \psi(\boldsymbol{x}) \longrightarrow U_{a,\boldsymbol{\theta},\boldsymbol{b}}\, \psi(\boldsymbol{x}) \quad &= \quad \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) \\ &= \frac{1}{a^{3/2}} \psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x} - \boldsymbol{b}}{a})). \end{aligned} \tag{3.1}$$

The family of wavelets $\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}$ are so-called *daughter Clifford wavelets* with $a \in \mathbb{R}^+$ as dilation parameter, $\boldsymbol{b} \in \mathbb{R}^3$ as the translation vector parameter, and $\boldsymbol{\theta}$ as the $SO(3)$ rotation parameters. The normalization constant $a^{-3/2}$ ensures that the norm of $\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}$ is independent of a, i.e.

$$\|\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\|_{L^2(\mathbb{R}^3;Cl_{3,0})} = ||\psi||_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{3.2}$$

This can be seen from

$$\begin{aligned} \|\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} &= \int_{\mathbb{R}^3} \sum_A \frac{1}{a^3} \psi_A^2(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x} - \boldsymbol{b}}{a}))\, d^3\boldsymbol{x} \\ &= \frac{1}{a^3} \int_{\mathbb{R}^3} \sum_A \psi_A^2(\boldsymbol{z}) a^3 \det(r_{\boldsymbol{\theta}})\, d^3\boldsymbol{z} \\ &= \int_{\mathbb{R}^3} \sum_A \psi_A^2(\boldsymbol{z})\, d^3\boldsymbol{z}. \end{aligned} \tag{3.3}$$

Applying (2.13) to the last line of (3.3), we obtain the desired result. □

In the $Cl_{3,0}$ Clifford Fourier domain, equation (3.1) can be represented in the form

$$\mathcal{F}\{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\}(\boldsymbol{\omega}) = e^{-i_3\boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})). \tag{3.4}$$

Substituting $(\boldsymbol{x}-\boldsymbol{b})/a = \boldsymbol{y}$ for the argument of (3.1) under the CFT integral of (3.4) gives

$$\begin{aligned}
\mathcal{F}\{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} \frac{1}{a^{\frac{3}{2}}} \psi(r_{\boldsymbol{\theta}}^{-1}\boldsymbol{y}) e^{-i_3\boldsymbol{\omega}\cdot(\boldsymbol{b}+a\boldsymbol{y})} a^3\, d^3\boldsymbol{y} \\
&= e^{-i_3\boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \int_{\mathbb{R}^3} \psi(r_{\boldsymbol{\theta}}^{-1}\boldsymbol{y}) e^{-i_3 a\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d^3\boldsymbol{y} \\
&= e^{-i_3\boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{3}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})).
\end{aligned}$$

□

3.2. *Admissibility*

In analogy to Zhao[9] we call $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ admissible wavelet if

$$C_\psi = \int_{\mathbb{R}^+} \int_{S0(3)} a^3 \{\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim}\, \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\, d\mu, \tag{3.5}$$

is an invertible multivector constant and finite at a.e. $\boldsymbol{\omega} \in \mathbb{R}^3$. The admissibility condition is important to guarantee that the Clifford wavelet transform is invertible as we will see later. We notice that for $\boldsymbol{\omega} = 0$ we get $\hat{\psi}(0) = \int_{\mathbb{R}^3} \psi(\boldsymbol{x}) e^{i_3 0\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} = 0$ for the scalar part of C_ψ to be finite. Therefore, like classical wavelets (see 26), an admissible Clifford-valued mother wavelet $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ has to satisfy

$$\int_{\mathbb{R}^3} \psi(\boldsymbol{x})\, d^3\boldsymbol{x} = \int_{\mathbb{R}^3} \psi_A(\boldsymbol{x})\boldsymbol{e}_A\, d^3\boldsymbol{x} = 0, \tag{3.6}$$

where $\psi_A(\boldsymbol{x})$ are real-valued wavelets. It means that the integral of every component ψ_A of the Clifford mother wavelet is zero: $\int_{\mathbb{R}^3} \psi_A(\boldsymbol{x}) d^3\boldsymbol{x} = 0$. The admissibility constant (3.5) can be simplified to

$$C_\psi = \int_{\mathbb{R}^3} \frac{\widetilde{\widehat{\psi}}(\boldsymbol{\xi})\widehat{\psi}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^3}\, d^3\boldsymbol{\xi}. \tag{3.7}$$

According to (3.5) or (3.7) it is not difficult to see with (2.6) that $C_\psi = \widetilde{C_\psi}$. Consequently, we have

$$C_\psi = \langle C_\psi \rangle + \langle C_\psi \rangle_1, \tag{3.8}$$

with positive scalar part ($\langle C_\psi \rangle > 0$)

$$\begin{aligned}
\langle C_\psi \rangle &= \int_{\mathbb{R}^3} \langle \{\widehat{\psi}(\boldsymbol{\xi})\}^{\sim} \widehat{\psi}(\boldsymbol{\xi}) \rangle\, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3 = \int_{\mathbb{R}^3} \frac{|\widehat{\psi}(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^3}\, d\boldsymbol{\xi}^3 = \left\| \, |\boldsymbol{\xi}|^{-3/2}\, \widehat{\psi}(\boldsymbol{\xi}) \right\|_{L^2(\mathbb{R}^3; Cl_{3,0})} \\
&= \int_{\mathbb{R}^3} [\langle \widehat{\psi}(\boldsymbol{\xi}) \rangle^2 + \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_1^2 - \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_2^2 - \langle \widehat{\psi}(\boldsymbol{\xi}) \rangle_3^2]\, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3,
\end{aligned} \tag{3.9}$$

and vector part

$$\langle C_\psi \rangle_1 = \int_{\mathbb{R}^3} \langle \{\widehat{\psi}(\boldsymbol{\xi})\}^\sim \widehat{\psi}(\boldsymbol{\xi})\rangle_1 \, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3 \tag{3.10}$$

$$= \int_{\mathbb{R}^3} [\langle \widehat{\psi}(\boldsymbol{\xi})\rangle \langle \widehat{\psi}(\boldsymbol{\xi})\rangle_1 + \langle \widehat{\psi}(\boldsymbol{\xi})\rangle_1 \cdot \langle \widehat{\psi}(\boldsymbol{\xi})\rangle_2 - \langle \widehat{\psi}(\boldsymbol{\xi})\rangle_2 \cdot \langle \widehat{\psi}(\boldsymbol{\xi})\rangle_3] \, \frac{1}{|\boldsymbol{\xi}|^3} d\boldsymbol{\xi}^3,$$

where the dot $\cdot$ indicates the Hestenes inner product [21] or left contraction [22]. The inverse of C_ψ is given by

$$C_\psi^{-1} = \frac{\langle C_\psi \rangle - \langle C_\psi \rangle_1}{\langle C_\psi \rangle^2 - \langle C_\psi \rangle_1^2}. \tag{3.11}$$

This leads to the following theorem.

Theorem 3.1 (Admissibility). *The admissibility constant defined by (3.5) is invertible as in (3.11) if and only if*

$$|C_\psi| < \infty \ \text{ and } \ \langle C_\psi \rangle_1^2 \neq \langle C_\psi \rangle^2. \tag{3.12}$$

3.3. *Clifford Wavelet Transform*

Definition 3.1 (Clifford wavelet transform). *We define the Clifford wavelet transform with respect to the mother wavelet $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ as follows*

$$\begin{aligned} T_\psi : L^2(\mathbb{R}^3; Cl_{3,0}) &\to L^2(\mathcal{G}; Cl_{3,0}) \\ f \to T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) &= \int_{\mathbb{R}^3} f(\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})} \, d^3\boldsymbol{x} \\ &= (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3; Cl_{3,0})}. \end{aligned} \tag{3.13}$$

Note that in general the order of (3.13) is fixed because Clifford multiplication is non-commutative. Alternatively, we may use a convolution ($\star$) to express (3.13) by

$$T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) = \int_{\mathbb{R}^3} f(\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})} \, d^3\boldsymbol{x} = (f \star \psi_{a,\boldsymbol{\theta}})(\boldsymbol{b}) \tag{3.14}$$

where

$$\psi_{a,\boldsymbol{\theta}}(\boldsymbol{x}) = \frac{1}{a^{\frac{3}{2}}} \psi\{(r_{\boldsymbol{\theta}}^{-1}(\frac{-\boldsymbol{x}}{a}))\}^\sim.$$

The Clifford wavelet transform (3.13) has a Clifford Fourier representation of the form

$$T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \widehat{f}(\boldsymbol{\omega}) \, a^{\frac{3}{2}} \{\widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^\sim \, e^{i_3 \boldsymbol{b} \cdot \boldsymbol{\omega}} \, d^3\boldsymbol{\omega} \tag{3.15}$$

Proof We have

$$
\begin{aligned}
&T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\\
&\overset{(3.13)}{=} \quad (f,\psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3;Cl_{3,0})}\\
&\overset{Planc.\,T.}{=} \frac{1}{(2\pi)^3}(\widehat{f},\widehat{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}})_{L^2(\mathbb{R}^3;Cl_{3,0})}\\
&= \quad \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\hat{f}(\boldsymbol{\omega})\left[\widehat{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}}(\boldsymbol{\omega})\right]^\sim d^3\boldsymbol{\omega}\\
&\overset{(3.4)}{=} \quad \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\hat{f}(\boldsymbol{\omega})\,a^{\frac{3}{2}}\left[\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\right]^\sim e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}}\,d^3\boldsymbol{\omega}.
\end{aligned}
$$

This proves (3.15). □

With the inverse CFT (3.15) becomes

$$T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) = \mathcal{F}^{-1}\left\{a^{\frac{3}{2}}\widehat{f}(\boldsymbol{.})[\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{.}))]^\sim\right\}(\boldsymbol{b}), \tag{3.16}$$

or equivalently

$$\mathcal{F}(T_\psi f(a,\boldsymbol{\theta},\boldsymbol{.}))(\boldsymbol{\omega}) = a^{\frac{3}{2}}\widehat{f}(\boldsymbol{\omega})\{\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^\sim. \tag{3.17}$$

3.4. *Properties of the Clifford wavelet transform*

Theorem 3.2 (Left linearity). *Let $f,g\in L^2(\mathbb{R}^3;Cl_{3,0})$ and $\psi\in L^2(\mathbb{R}^3;Cl_{3,0})$ be a Clifford mother wavelet. The Clifford wavelet transform T_ψ is a linear operator, i.e.,*

$$[T_\psi(\lambda f+\mu g)](a,\boldsymbol{\theta},\boldsymbol{b}) = \lambda T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) + \mu T_\psi g(a,\boldsymbol{\theta},\boldsymbol{b}), \tag{3.18}$$

with multivector constants λ,μ in $Cl_{3,0}$.

Theorem 3.3 (Translation covariance). *Let $\psi\in L^2(\mathbb{R}^3;Cl_{3,0})$ be a Clifford mother wavelet. If the argument of $T_\psi f(\boldsymbol{x})$ is translated by a constant $\boldsymbol{x}_0\in\mathbb{R}^3$ then*

$$[T_\psi f(\cdot-\boldsymbol{x}_0)](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}-\boldsymbol{x}_0). \tag{3.19}$$

Proof Equation (3.13) gives

$$
\begin{aligned}
[T_\psi f(\cdot-\boldsymbol{x}_0)](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}} f(\boldsymbol{x}-\boldsymbol{x}_0)\widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\,d^3\boldsymbol{x}\\
&= \int_{\mathbb{R}} f(\boldsymbol{x}-\boldsymbol{x}_0)\frac{1}{a^{3/2}}\left[\psi(a^{-1}r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{x}-\boldsymbol{b})\right]^\sim d^3\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})\frac{1}{a^{3/2}}\left[\psi\left(a^{-1}r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{y}-(\boldsymbol{b}-\boldsymbol{x}_0))\right)\right]^\sim d^3\boldsymbol{y}\\
&= T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}-\boldsymbol{x}_0).
\end{aligned}
$$

Theorem 3.4 (Dilation covariance). *Let $\psi\in L^2(\mathbb{R}^3;Cl_{3,0})$ be a Clifford mother wavelet. If c is a real positive constant, then*

$$[T_\psi f(c\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) = \frac{1}{c^{3/2}} T_\psi f(ac,\boldsymbol{\theta},\boldsymbol{b}c). \tag{3.20}$$

Proof Equation (3.13) gives again

$$\begin{aligned}
[T_\psi f(c\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}^3} f(c\boldsymbol{x}) \frac{1}{a^{3/2}} \left[\psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-b}{a}))\right]^\sim d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y}) \frac{1}{a^{3/2}} \left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{y}/c-\boldsymbol{b}}{a})\right)\right]^\sim \frac{1}{c^3} d^3\boldsymbol{y} \\
&= \frac{1}{c^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(\boldsymbol{y}) \frac{1}{(ac)^{3/2}} \left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{y}-\boldsymbol{b}c}{ac})\right)\right]^\sim d^3\boldsymbol{y} \\
&= \frac{1}{c^{3/2}} T_\psi f(ac,\boldsymbol{\theta},\boldsymbol{b}c).
\end{aligned}$$

Theorem 3.5 (Rotation covariance). *Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford mother wavelet. If $r_{\boldsymbol{\theta}}$ and $r_{\boldsymbol{\theta}_0}$ are both rotations, then*

$$[T_\psi f(r_{\boldsymbol{\theta}_0}\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta}', r_{\boldsymbol{\theta}_0}\boldsymbol{b}), \tag{3.21}$$

with rotors $R_{\boldsymbol{\theta}'} = R_{\boldsymbol{\theta}_0} R_{\boldsymbol{\theta}}$.

Proof Applying equation (3.13) and using the fact that the product of two rotations is always a rotation,[20] we obtain

$$\begin{aligned}
[T_\psi f(r_{\boldsymbol{\theta}_0}\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}^3} f(r_{\boldsymbol{\theta}_0}\boldsymbol{x}) \widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})}\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(r_{\boldsymbol{\theta}_0}\boldsymbol{x}) \left[\psi(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-b}{a}))\right]^\sim d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y}) \left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{r_{\boldsymbol{\theta}_0}^{-1}\boldsymbol{y}-\boldsymbol{b}}{a})\right)\right]^\sim \det{}^{-1}(r_{\boldsymbol{\theta}})\, d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y}) \left[\psi\left(r_{\boldsymbol{\theta}}^{-1} r_{\boldsymbol{\theta}_0}^{-1}(\frac{\boldsymbol{y}-r_{\boldsymbol{\theta}_0}\boldsymbol{b}}{a})\right)\right]^\sim d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y}) \left[\psi\left((r_{\boldsymbol{\theta}_0} r_{\boldsymbol{\theta}})^{-1}(\frac{\boldsymbol{y}-r_{\boldsymbol{\theta}_0}\boldsymbol{b}}{a})\right)\right]^\sim d^3\boldsymbol{y} \\
&= T_\psi f(a,\boldsymbol{\theta}', r_{\boldsymbol{\theta}_0}\boldsymbol{b}),
\end{aligned}$$

where we omit brackets like $r_{\boldsymbol{\theta}_0}\boldsymbol{x} = r_{\boldsymbol{\theta}_0}(\boldsymbol{x})$. This proves (3.21). □

These four properties above correspond to classical wavelet transform properties. Now we will see the differences between the Clifford and the classical wavelet transforms.

Theorem 3.6 (Inner product relation). *Let $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ be an admissible Clifford mother wavelet and $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$ arbitrary. Then we have*

$$\begin{aligned}
(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{3,0})} &= (f C_\psi, g)_{L^2(\mathbb{R}^3;Cl_{3,0})} \\
&= \langle C_\psi \rangle (f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})} + (f \langle C_\psi \rangle_1, g)_{L^2(\mathbb{R}^3;Cl_{3,0})}.
\end{aligned} \tag{3.22}$$

Before proving theorem 3.6 we remark that for $\langle C_\psi\rangle_1 = 0$ the operator $\langle C_\psi\rangle^{-1/2}T_\psi$ is an *isometry* from $L^2(\mathbb{R}^3;Cl_{3,0})$ to $L^2(\mathcal{G};Cl_{3,0})$.

Proof By inserting (3.15) into the left side of (3.22), we obtain

$$\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{3,0})} \\
&= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta})\,\{T_\psi g(a,\boldsymbol{b},\boldsymbol{\theta})\}^{\sim}\, d^3\boldsymbol{b}d\mu \\
&= \int_{\mathbb{R}^+}\int_{S0(3)} \frac{a^3}{(2\pi)^6}(\int_{\mathbb{R}^3}[\int_{\mathbb{R}^3} \hat{f}(\boldsymbol{\omega})\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}} d^3\boldsymbol{\omega} \\
&\quad \int_{\mathbb{R}^3}\left\{(\hat{g}(\boldsymbol{\omega}')\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}'))\}^{\sim} e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}'}\right\}^{\sim} d^3\boldsymbol{\omega}']d^3\boldsymbol{b})d\mu.
\end{aligned} \tag{3.23}$$

For abbreviation, we use the notation

$$F_a(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega})\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim},\; G_a(\boldsymbol{\omega}') = \hat{g}(\boldsymbol{\omega}')\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}'))\}^{\sim}.$$

Equation (3.23) can then be rewritten as

$$\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{3,0})} \\
&= \frac{1}{(2\pi)^6}\int_{\mathbb{R}^+} a^3 \int_{S0(3)} (\int_{\mathbb{R}^3}[\int_{\mathbb{R}^3} F_a(\boldsymbol{\omega})e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}}\, d^3\boldsymbol{\omega} \\
&\qquad \int_{\mathbb{R}^3}\{G_a(\boldsymbol{\omega}')e^{i_3\boldsymbol{b}\cdot\boldsymbol{\omega}'}\}^{\sim}\, d^3\boldsymbol{\omega}']d^3\boldsymbol{b})d\mu \\
&\overset{(2.16)}{=} \frac{1}{(2\pi)^6}\int_{\mathbb{R}^+} a^3 \int_{S0(3)} \left(\int_{\mathbb{R}^3} \hat{F}_a(-\boldsymbol{b})\,\{\hat{G}_a(-\boldsymbol{b})\}^{\sim} d^3\boldsymbol{b}\right) d\mu \\
&\overset{P.T.}{=} \int_{\mathbb{R}^+}\int_{S0(3)} \frac{a^3}{(2\pi)^3}\left(\int_{\mathbb{R}^3} F_a(\boldsymbol{\xi})\widetilde{G_a(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi}\right)\, d\mu \\
&= \int_{\mathbb{R}^3}\frac{1}{(2\pi)^3}\left(\int_{\mathbb{R}^+} a^3 \int_{S0(3)} \hat{f}(\boldsymbol{\xi})\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))\}^{\sim}\, \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))\widetilde{\hat{g}(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi}\right)\, d\mu \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \hat{f}(\boldsymbol{\xi})\left(\int_{\mathbb{R}^+}\int_{S0(3)} a^3\{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))\}^{\sim}\, \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\xi}))d\mu\right)\widetilde{\hat{g}(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi} \\
&\overset{(3.5)}{=} \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \hat{f}(\boldsymbol{\xi})C_\psi\widetilde{\hat{g}(\boldsymbol{\xi})}\, d^3\boldsymbol{\xi} \\
&\overset{P.T.}{=} \int_{\mathbb{R}^3} f(\boldsymbol{x})C_\psi\widetilde{g(\boldsymbol{x})}\, d^3\boldsymbol{x} = (fC_\psi, g)_{L^2(\mathbb{R}^3;Cl_{3,0})},
\end{aligned}$$

where P.T. denotes the Plancherel theorem of table 2. □

As a consequence of theorem 3.6, we immediately obtain

Corollary 3.1 (Norm relation). *Let* $\psi \in L^2(\mathbb{R}^3;Cl_{3,0})$ *be a Clifford mother wavelet that satisfies the admissibility condition (3.5). Then for any* $f \in$

$L^2(\mathbb{R}^3; Cl_{3,0})$ *we have*

$$\begin{aligned}\|T_\psi f\|^2_{L^2(\mathcal{G};Cl_{3,0})} &= \langle (fC_\psi, f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle \;=\; C_\psi * (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})} \\ &= \langle C_\psi\rangle \|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} + \langle (f\langle C_\psi\rangle_1, f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle \\ &= \langle C_\psi\rangle \|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} + \langle C_\psi\rangle_1 * \langle (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle_1 \quad (3.24)\end{aligned}$$

According to (2.13) we can rewrite the left hand side of (3.24) in the form

$$\|T_\psi f\|^2_{L^2(\mathcal{G};Cl_{3,0})} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^+}\int_{SO(3)} \sum_A \langle T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\rangle_A^2 \, d\mu d^3\boldsymbol{b}. \qquad (3.25)$$

3.5. *Inverse Clifford wavelet transform, reproducing kernel*

In the following we will first derive the important inverse Clifford $Cl_{3,0}$ wavelet transform for multivector functions.

Theorem 3.7 (Inverse Clifford $Cl_{3,0}$ wavelet transform). *Let* $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$ *be a Clifford mother wavelet that satisfies the admissibility condition (3.5). Then any* $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ *can be decomposed as*

$$\begin{aligned}f(\boldsymbol{x}) &= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{b},\boldsymbol{\theta})\,\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}, \\ &= \int_{\mathcal{G}} (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} C_\psi^{-1}\, d\mu d^3\boldsymbol{b}, \qquad (3.26)\end{aligned}$$

the integral converging in the weak sense.

Proof Indeed, we have for every $g \in L^2(\mathbb{R}^3; Cl_{3,0})$

$$\begin{aligned}(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{3,0})} &= \int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\{T_\psi g(a,\boldsymbol{\theta},\boldsymbol{b})\}^\sim \, d\mu d^3\boldsymbol{b} \\ &= \int_{\mathcal{G}}\int_{\mathbb{R}^3} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\, d^3\boldsymbol{x} d\mu d^3\boldsymbol{b} \\ &= \int_{\mathbb{R}^3}\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})\, d\mu\, d^3\boldsymbol{b}\, \widetilde{g(\boldsymbol{x})}\, d^3\boldsymbol{x} \\ &= \left(\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, d\mu d^3\boldsymbol{b}\,,\, g\right)_{L^2(\mathbb{R}^3;Cl_{3,0})}. \quad (3.27)\end{aligned}$$

Applying (3.22) of theorem 3.6 gives for every $g \in L^2(\mathbb{R}^3; Cl_{3,0})$

$$(fC_\psi, g)_{L^2(\mathbb{R}^3;Cl_{3,0})} = \left(\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, d\mu d^3\boldsymbol{b}\,,\, g\right)_{L^2(\mathbb{R}^3;Cl_{3,0})}. \qquad (3.28)$$

Taking the scalar part of (3.28) we obtain

$$\langle (fC_\psi, g)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle = \langle \left(\int_{\mathcal{G}} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, d\mu d^3\boldsymbol{b}\,,\, g\right)_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle. \qquad (3.29)$$

Because the inner product identity (3.29) holds for every $g \in L^2(\mathbb{R}^3; Cl_{3,0})$ (and in particular for all basis elements of the Clifford module of def. 2.1) we conclude that

$$f(\boldsymbol{x})C_\psi = \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{b}, \boldsymbol{\theta}) \psi_{a,\boldsymbol{b},\boldsymbol{\theta}}(\boldsymbol{x})\, d\mu d^3\boldsymbol{b}\,, \tag{3.30}$$

or equivalently, because of the assumed invertibility of C_ψ

$$\begin{aligned} f(\boldsymbol{x}) &= \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{b}, \boldsymbol{\theta}) \psi_{a,\boldsymbol{b},\boldsymbol{\theta}}(\boldsymbol{x})\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}. \\ &\overset{(3.13)}{=} \int_{\mathcal{G}} (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\, C_\psi^{-1}\, d\mu d^3\boldsymbol{b}. \end{aligned} \tag{3.31}$$

which completes the proof. □

Weak convergence of (3.26) means that for all $g \in L^2(\mathbb{R}^3; Cl_{3,0})$ holds

$$(\int_{\mathcal{G}} T_\psi f(a, \boldsymbol{b}, \boldsymbol{\theta})\, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} d\mu d^3\boldsymbol{b}\, C_\psi^{-1}\,,\ g)_{L^2(\mathbb{R}^3;Cl_{3,0})} \to (f, g)_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{3.32}$$

Using the properties of the inner product (2.12), it is not difficult to show that (3.26) can alternatively be rewritten in the form ($C_\psi^{-1} = \widetilde{C_\psi^{-1}}$ because of (3.11))

$$f(\boldsymbol{x}) = C_\psi^{-1} \int_{\mathcal{G}} \{\psi_{a,\boldsymbol{b},\boldsymbol{\theta}}\}^\sim\, (\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\,, \tilde{f}\,)_{L^2(\mathbb{R}^3;Cl_{3,0})}\, d\mu d^3\boldsymbol{b}. \tag{3.33}$$

Theorem 3.8 (Reproducing kernel). *We define for an admissible Clifford mother wavelet* $\psi \in L^2(\mathbb{R}^3; Cl_{3,0})$

$$\mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}') = (\psi_{a,\boldsymbol{\theta},\boldsymbol{b}} C_\psi^{-1}, \psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'})_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{3.34}$$

Then $\mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}')$ *is a reproducing kernel in* $L^2(\mathcal{G}, d\lambda)$*, i.e,*

$$T_\psi f(a', \boldsymbol{\theta}', \boldsymbol{b}') = \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) \mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}')\, d\lambda. \tag{3.35}$$

Proof By inserting (3.26) into the definition of the Clifford wavelet transform (3.13) we obtain

$$\begin{aligned} T_\psi f(a', \boldsymbol{\theta}', \boldsymbol{b}') &= \int_{\mathbb{R}^3} \left\{ \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b})\, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x})\, d\lambda\, C_\psi^{-1} \right\} \widetilde{\psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'}}(\boldsymbol{x})\, d^3\boldsymbol{x} \\ &= \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) \left\{ \int_{\mathbb{R}^3} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) C_\psi^{-1} \{\psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'}(\boldsymbol{x})\}^\sim\, d^3\boldsymbol{x} \right\} d\lambda \\ &= \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{b}, \boldsymbol{\theta}) \mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}')\, d\lambda, \end{aligned} \tag{3.36}$$

which completes the proof. □

4. Uncertainty principles for Clifford algebra $Cl_{3,0}$ wavelets

It is known that uncertainty principles play an important role in the development and understanding of quantum physics. In quantum physics this means that particle momentum and position cannot be simultaneously measured with arbitrary precision. In classical harmonic analysis the uncertainty principle of a function and its Fourier transform establishes a minimum of the products of the variances. The same holds for the multivector CFT.[11 12] The uncertainty principle for the continuous wavelet transforms establishes a lower bound of the product of the variances of the continuous wavelet transform of a function and its Fourier transform (see e.g. 27).

We extend this idea to the Clifford algebra $Cl_{3,0}$ wavelet transform, i.e. we show how the Clifford algebra $Cl_{3,0}$ wavelet transform and the Clifford Fourier transform of a multivector function are related.

4.1. *Uncertainty principles for general admissibility constant*

Let us first formulate a general statement in the following theorem. That this is indeed the generalized form of an uncertainty principle will be seen in the special case of scalar C_ψ in corollary 4.1, which follows in section 4.2.

Theorem 4.1 (Generalized Clifford wavelet uncertainty principle). *Let ψ be a Clifford algebra wavelet that satisfies the admissibility condition (3.7). Then for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, the following inequality holds*

$$\begin{aligned} &\|\boldsymbol{b} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})} \; C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \\ &\quad \geq \frac{3(2\pi)^3}{4} \; \left[C_\psi * (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^2 . \end{aligned} \tag{4.1}$$

Before we attempt the proof of theorem 4.1 we derive the following two useful lemmas.

Lemma 4.1 (Integrated variance of CFT of Cliff. wavelet transf.).

$$\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\, \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\, d\mu = C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{4.2}$$

Proof We observe that

$$\begin{aligned} &\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\, \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\, d\mu \qquad (4.3)\\ &\overset{(2.14)}{=} \int_{\mathbb{R}^3}\int_{\mathbb{R}^+}\int_{SO(3)} \boldsymbol{\omega}^2[\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}(\boldsymbol{\omega})] * \widetilde{\mathcal{F}}\{T_\psi f(a,\boldsymbol{\theta},.)\}(\boldsymbol{\omega})\; d\mu\, d^3\boldsymbol{\omega} \\ &\overset{(3.17)}{=} \int_{\mathbb{R}^3} \underbrace{\int_{\mathbb{R}^+}\int_{SO(3)} a^3[\widetilde{\widehat{\psi}}(ar^{-1}_{\boldsymbol{\theta}}(\boldsymbol{\omega}))\widehat{\psi}(ar^{-1}_{\boldsymbol{\theta}}(\boldsymbol{\omega}))]}\, *[\widetilde{\hat{f}}(\boldsymbol{\omega})\hat{f}(\boldsymbol{\omega})]\; \boldsymbol{\omega}^2 d\mu d^3\boldsymbol{\omega} \\ &= \; C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} . \end{aligned}$$

□

In some cases only the scalar part of the admissibility constant matters on the right hand side of (4.2), as shown in

Lemma 4.2 (With scalar admissibility constant). *If either one of the factors is scalar, or the two vector parts are perpendicular:* $\langle C_\psi\rangle_1 \perp \langle(\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})}\rangle_1$ *we get instead*

$$\int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\,\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu = \langle C_\psi\rangle_0\,\|\boldsymbol{\omega}\hat{f}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}. \tag{4.4}$$

Now we begin with the proof of theorem 4.1.

Proof We apply to $T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})$, where $\boldsymbol{b}\in\mathbb{R}^3$ is the main variable and $a,\boldsymbol{\theta}$ are function parameters, the established uncertainty principle for multivector functions in order to get with (2.13) (compare Theor. 6 of 11 for more details, $\times$ simply represents multiplication of real scalars)

$$\begin{aligned}\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \times \|\boldsymbol{\omega}\,\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\\ \geq \frac{3(2\pi)^3}{4}\|T_\psi f(a,\boldsymbol{\theta},.)\|^4_{L^2(\mathbb{R}^3;Cl_{3,0})}\end{aligned} \tag{4.5}$$

Taking the square root on both sides of (4.5) we obtain

$$\begin{aligned}\Big[\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\Big]^{\frac{1}{2}} \times \Big[\|\boldsymbol{\omega}\,\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\Big]^{\frac{1}{2}}\\ \geq \frac{\sqrt{3}(2\pi)^{3/2}}{2}\|T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\end{aligned} \tag{4.6}$$

Integrating both sides of (4.6) with respect to $d\mu$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{SO(3)}\Bigg(\Big[\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\Big]^{\frac{1}{2}}\\ \times\Big[\|\boldsymbol{\omega}\,\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\Big]^{\frac{1}{2}}\Bigg)\,d\mu\\ \geq \frac{\sqrt{3}(2\pi)^{3/2}}{2}\int_{\mathbb{R}^+}\int_{SO(3)}\|T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu.\end{aligned} \tag{4.7}$$

Applying the multivector Cauchy-Schwartz inequality to the left hand side of (4.7) gives

$$\begin{aligned}\left(\int_{\mathbb{R}^+}\int_{SO(3)}\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu\right)^{\frac{1}{2}}\\ \times\left(\int_{\mathbb{R}^+}\int_{SO(3)}\|\boldsymbol{\omega}\,\mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu\right)^{\frac{1}{2}}\\ \geq \frac{\sqrt{3}(2\pi)^{3/2}}{2}\int_{\mathbb{R}^+}\int_{SO(3)}\|T_\psi f(a,\boldsymbol{\theta},.)\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\,d\mu.\end{aligned} \tag{4.8}$$

Taking the square on both sides of (4.8) and inserting the definitions of the norms of lines 1 and 3 of (4.8) we get with (2.14)

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{SO(3)}\int_{\mathbb{R}^3} \boldsymbol{b}^2 T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \;*\, [T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})]^\sim \, d\mu d^3\boldsymbol{b} \\
&\times \int_{\mathbb{R}^+}\int_{SO(3)} \|\boldsymbol{\omega}\, \mathcal{F}\{T_\psi f(a,\boldsymbol{\theta},.\,)\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\, d\mu \\
&\geq \frac{3(2\pi)^3}{4}\left(\int_{\mathbb{R}^+}\int_{SO(3)}\int_{\mathbb{R}^3} T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) \;*\, [T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})]^\sim \, d\mu d^3\boldsymbol{b}\right)^2 . \qquad (4.9)
\end{aligned}$$

We now recognize the $L^2(\mathcal{G};Cl_{3,0})$-norms in lines 1 and 3 of (4.9) and with lemma 4.1 we replace the second line of (4.9) to become

$$\begin{aligned}
\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})} \; & C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \\
&\geq \frac{3(2\pi)^3}{4}\|T_\psi f\|^4_{L^2(\mathcal{G};Cl_{3,0})}. \qquad (4.10)
\end{aligned}$$

Substituting for the right hand side (3.24) we finally get

$$\begin{aligned}
\|\boldsymbol{b}T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})} \; & C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^3;Cl_{3,0})} \\
&\geq \frac{3(2\pi)^3}{4}\; \left[C_\psi * (f,f)_{L^2(\mathbb{R}^3;Cl_{3,0})}\right]^2, \qquad (4.11)
\end{aligned}$$

which concludes the proof of theorem 4.1. □

4.2. *Uncertainty principle for scalar admissibility constant*

For scalar C_ψ we get due to (4.4) and a similar identity for the right hand side of (4.11) the following corollary

Corollary 4.1 (Uncertainty principle for Clifford wavelet). *Let ψ be a Clifford algebra wavelet that satisfies the admissibility constant (3.7). Then for every $f \in L^2(\mathbb{R}^3;Cl_{3,0})$, the following inequality holds*

$$\|\boldsymbol{b}\,T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})}\; \|\boldsymbol{\omega}\hat{f}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \geq 3C_\psi\frac{(2\pi)^3}{4}\,\|f\|^4_{L^2(\mathbb{R}^3;Cl_{3,0})}. \qquad (4.12)$$

This shows indeed, that theorem 4.1 represents a multivector generalization of the uncertainty principle of corollary 4.1 for Clifford wavelets with scalar admissibility constant.

In the field of information theory and image processing corollary 4.1 establishes bounds for the effective width times frequency extension of processed signals or images.

5. Extension of complex Gabor wavelets to multivector Clifford Gabor wavelets

In signal processing complex Gabor (or Morlet[g]) wavelets are used extensively for signal analysis.[29 30 31] Complex Gabor wavelets are well localized in both *space* and *frequency* domains which is very important in understanding signals. Two-dimensional complex Gabor wavelets are composed of a complex exponential function and a Gaussian function. They generally can be written as

$$h(\boldsymbol{x}) = \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}\right)} \left[e^{\boldsymbol{i}(u_0x_1+v_0x_2)} - e^{-\frac{1}{2}(\sigma_1^2u_0^2+\sigma_2^2u_0^2)}\right], \tag{5.1}$$

where σ_1 and σ_1 are the standard deviations of the Gaussian function.

Complex Gabor wavelets can be extended to multivectors. This extension is obtained by replacing the complex kernel $e^{\boldsymbol{i}(u_0x_1+v_0x_2)}$ in the 2D complex Gabor wavelets (5.1) by the Clifford Fourier kernel $e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}$. It then takes the form

$$\begin{aligned}\psi^c(\boldsymbol{x}) &= g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)\,\left(e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}} - e^{-\frac{1}{2}(\sigma_1^2u_0^2+\sigma_2^2u_0^2+\sigma_3^2w_0^2)}\right)\\ &= g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)\,e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}} - \eta(\boldsymbol{x}),\end{aligned} \tag{5.2}$$

where $\boldsymbol{\omega}_0 = u_0\boldsymbol{e}_1 + v_0\boldsymbol{e}_2 + w_0\boldsymbol{e}_3$ denotes a frequency vector. The 3D Gaussian function $g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)$ in (5.2) is defined by

$$g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_1\sigma_2\sigma_3}\, e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}+\frac{x_3^2}{\sigma_3^2}\right)},$$

and

$$\eta(\boldsymbol{x}) = g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)e^{-\frac{1}{2}(\sigma_1^2u_0^2+\sigma_2^2u_0^2+\sigma_3^2w_0^2)}$$

is a correction term in order for equation (3.6) to be satisfied (see 23). Applying the shift and the scaling properties of table 2, we can rewrite the Clifford Gabor wavelets (5.2) in terms of the $Cl_{3,0}$ Clifford Fourier transform as follows

$$\begin{aligned}\mathcal{F}\{\psi^c\}(\boldsymbol{\omega}) = e^{-\frac{1}{2}\left(\sigma_1^2(\omega_1-u_0)^2+\sigma_2^2(\omega_2-v_0)^2+\sigma_3^2(\omega_3-w_0)^2\right)} -\\ e^{-\frac{1}{2}\left(\sigma_1^2(\omega_1^2+u_0^2)+\sigma_2^2(\omega_2^2+v_0^2)+\sigma_3^2(\omega_3^2+w_0^2)\right)}.\end{aligned} \tag{5.3}$$

It is easy to see that $\mathcal{F}\{\psi^c\}(0) = 0$. The representation of the Clifford Gabor wavelets (5.2) shows that they are formally analogous to the 3D complex Gabor

[g] *Gabor paved the time-frequency plane in uniform cells and associated each cell with a wave shape of invariant envelope with a carrier of variable frequency. Morlet kept the constraint resulting from the uncertainty principle applied to time and frequency, but he perceived that it was the wave shape that must be invariant to give uniform resolution in the entire plane. For this he adapted the sampling rate to the frequency, thereby creating, in effect, a changing time scale producing a stretching of the wave shape.* (Goupillaud[28])

wavelets. We can apply the Euler formula to the trivector exponential which gives the Clifford Gabor wavelets (5.2) in the form

$$\psi^c(\boldsymbol{x}) = g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)\cos(\boldsymbol{\omega}_0\cdot\boldsymbol{x}) + i_3 g(\boldsymbol{x};\sigma_1,\sigma_2,\sigma_3)\sin(\boldsymbol{\omega}_0\cdot\boldsymbol{x}) - \eta(\boldsymbol{x}). \quad (5.4)$$

This shows that the resulting wavelets consist of a real scalar part and a trivector part. We note that (5.3) is a real-valued scalar function. As a consequence the admissibility constant (3.5) will also be real. It means that we have

$$0 < C_{\psi^c} = \int_{\mathbb{R}^+}\int_{SO(3)} a^3 \left[\widehat{\psi^c}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\right]^2 d\mu \overset{(3.7)}{=} \int_{\mathbb{R}^3} \frac{(\widehat{\psi^c}(\boldsymbol{\xi}))^2}{|\boldsymbol{\xi}|^3}\, d^3\boldsymbol{\xi} < \infty, \quad (5.5)$$

is a real positive scalar constant and finite at a.e. $\boldsymbol{\omega}\in\mathbb{R}^3$.

We summarize some important properties of Clifford Gabor wavelet transform in the following theorems corresponding to theorem 3.6, corollary 3.1 and theorem 3.7.

Theorem 5.1 (Inner product relation). *Let $\psi^c \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford Gabor wavelet and $f, g \in L^2(\mathbb{R}^3; Cl_{3,0})$ arbitrary. Then we have*

$$(T_{\psi^c} f, T_{\psi^c} g)_{L^2(\mathcal{G};Cl_{3,0})} = C_{\psi^c}(f,g)_{L^2(\mathbb{R}^3;Cl_{3,0})}. \quad (5.6)$$

In other words the operator $C_{\psi^c}^{-\frac{1}{2}} T_{\psi^c}$ is an *isometry* from $L^2(\mathbb{R}^3; Cl_{3,0})$ to $L^2(\mathcal{G}; Cl_{3,0})$. An immediate consequence of (5.6) is

Theorem 5.2 (Norm relation). *Let $\psi^c \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford Gabor wavelet that satisfies the admissibility condition in the sense of (5.5). Then for any $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ we get*

$$\|T_{\psi^c} f\|^2_{L^2(\mathcal{G};Cl_{3,0})} = C_{\psi^c}\|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \quad (5.7)$$

Theorem 5.3 (Reconstruction formula). *Let $\psi^c \in L^2(\mathbb{R}^3; Cl_{3,0})$ be a Clifford Gabor wavelets that satisfies the admissibility condition (5.5). Then any $f \in L^2(\mathbb{R}^3; Cl_{3,0})$ can be decomposed as*

$$f(\boldsymbol{x}) = C_{\psi^c}^{-1}\int_{\mathcal{G}} (f, \psi^c_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^3;Cl_{3,0})}\, \psi^c_{a,\boldsymbol{\theta},\boldsymbol{b}} d\mu d^3\boldsymbol{b}, \quad (5.8)$$

the integral converging in the weak sense.

This theorem shows that any multivector function f can be reconstructed from the Clifford Gabor transform.

As a consequence of the general uncertainty principle for Clifford wavelets with scalar admissibility constant of corollary 4.1 we have

Theorem 5.4 (Uncertainty principle for Clifford Gabor wavelet). *Let ψ^c be a Clifford Gabor wavelet that satisfies the admissibility constant (5.5). Assume $\|f\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} = F < \infty$ for every $f \in L^2(\mathbb{R}^3; Cl_{3,0})$, then the following inequality holds*

$$\|\boldsymbol{b}\, T_{\psi^c} f(a,\boldsymbol{\theta},\boldsymbol{b})\|^2_{L^2(\mathcal{G};Cl_{3,0})}\ \|\boldsymbol{\omega}\hat{f}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})} \geq 3C_{\psi^c}\frac{(2\pi)^3}{4}F^2. \quad (5.9)$$

6. Conclusions

We showed how Clifford algebra $Cl_{3,0}$-valued wavelets extend the classical wavelets on scalar functions to multivector functions. Multivector wavelet admissibility depends on both the scalar and vector parts of the admissibility constant. Important properties such as translation, dilation and rotation covariances, a reproducing kernel, and a reproduction formula for multivector functions were demonstrated.

We established the general form of a new uncertainty principle for Clifford wavelets, which becomes analogous to the usual scalar formulation (corollary 4.1) when the admissibility constant itself is scalar. In the field of information theory and image processing this Clifford wavelet uncertainty principle establishes bounds for the effective width times frequency extension of processed signals or images.

We then applied our formalism by extending complex Gabor wavelets to Gabor multivector wavelets, and looked at some of their important properties. We also established a new uncertainty principle for the Clifford Gabor wavelets.

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