B. Mawardi, E. Hitzer, A. Hayashi, R. Ashino, An Uncertainty Principle for Quaternion Fourier Transform, Computer & Mathematics with Applications, 56, pp. 2398-2410 (2008).

An Uncertainty Principle for Quaternion Fourier Transform

Mawardi Bahri [a,*] Eckhard S. M. Hitzer [a] Akihisa Hayashi [a]
Ryuichi Ashino [b,**]

[a] *Department of Applied Physics, University of Fukui, Fukui 910-8507, Japan*

[b] *Division of Mathematical Sciences, Osaka Kyoiku University, Osaka 582-8582, Japan*

Abstract

We review the *quaternionic Fourier transform* (QFT). Using the properties of the QFT we establish an *uncertainty principle* for the right-sided QFT. This uncertainty principle prescribes a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. It is shown that only a *Gaussian* quaternion signal minimizes the uncertainty.

Key words: Quaternion algebra, Quaternionic Fourier transform, Uncertainty principle, Gaussian quaternion signal, Hypercomplex functions
1991 MSC: 30G35, 42B10, 94A12, 11R52

1 Introduction

Recently it has become popular to generalize the Fourier transform (FT) from real and complex numbers [1] to quaternion algebra. In these constructions many FT properties still hold, others are modified.

The quaternionic Fourier transform (QFT) plays a vital role in the representation of *signals*. It transforms a real (or quaternionic) two-dimensional signal into a

* Current address: School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia.
**Corresponding author.
Email addresses: mawardibahri@gmail.com (Mawardi Bahri), hitzer@mech.fukui-u.ac.jp (Eckhard S. M. Hitzer), hayashi@soliton.fukui-u.ac.jp (Akihisa Hayashi), ashino@cc.osaka-kyoiku.ac.jp (Ryuichi Ashino).

quaternion-valued frequency domain signal. The four QFT components separate four cases of *symmetry* in real signals instead of only two in the complex FT [2,3].

The QFT was first proposed by Ell [4]. He proposed a two-sided QFT and demonstrated some important properties of this type of QFT. He also introduced the use of the QFT in the analysis of two dimensional linear time invariant dynamic systems. Later, Bülow [2] made a more extended investigation of important properties of the two-sided QFT mainly for *real signals* and applied it to signal and image processing. He obtained a local 2D *quaternionic phase*.

Pei et al. [5] discussed and *optimized* the implementation of different types of the QFT with applications to linear quaternion filters. Hitzer [6] described in detail properties of different types of QFT applied to *fully* quaternionic signals and then generalized the QFT to a *volume time*, as well as to a *space time* algebra Fourier transform. Sangwine and Ell [7] proposed the QFT application to *color* image analysis. Bas, Le Bihan and Chassery [8] used the QFT to design a *digital* color image watermarking scheme. Bayro et al. [9] applied the QFT in image pre-processing and neural computing techniques for *speech recognition*.

It is well known that the uncertainty principle for the FT relates the variances of a function and its Fourier transform which cannot both be simultaneously sharply localized [10,11]. In signal processing an uncertainty principle states that the product of the variances of the signal in the *time* and *frequency* domains has a lower bound. Yet Felsberg [3] notes for two dimensions: *In 2D however, the uncertainty relation is still an open problem. In [12] it is stated that there is no straightforward formulation for the 2D uncertainty relation.* A first straightforward *directional* 2D uncertainty principle was formulated by Hitzer and Mawardi [13], in Clifford algebras $Cl_{n,0}$ with $n = 2 \pmod 4$. Now we attempt another formulation for quaternions $\mathbb{H} \cong Cl_{0,2}$ using the *right-sided* QFT.

This paper briefly *reviews* the QFT and provides alternative proofs for some of its properties. The QFT considered in this paper enables us to extend the Heisenberg type uncertainty principle from the complex FT to the QFT.

The organization of the present paper is as follows. In section 2, we briefly establish our notation for quaternion algebra and its relationship with the Clifford geometric algebra $Cl_{3,0}$. In section 3, we demonstrate some important properties of the QFT, which are necessary to prove the uncertainty principle for the QFT. In section 4, the classical Heisenberg uncertainty principle is generalized for the QFT.

2 Quaternion Algebra

2.1 The Quaternion Algebra $\mathbb{H}$

The quaternion algebra [14] was first invented by *Sir W. R. Hamilton* in 1843 and is denoted by $\mathbb{H}$ in his honor. It is an extension of complex numbers to a four-dimensional algebra. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three orthogonal imaginary units (denoted $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$) with real coefficients

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{2.1}$$

where the elements $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ obey Hamilton's *multiplication rules*

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}. \tag{2.2}$$

Because $\mathbb{H}$ is according to (2.2) *non-commutative*, one cannot directly extend various results on complex numbers to quaternions. For simplicity, we express a quaternion q as sum of a scalar q_0 , and a *pure* 3D quaternion $\boldsymbol{q}$

$$q = q_0 + \boldsymbol{q} = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3, \tag{2.3}$$

where the scalar part is also denoted Sc(q) = q_0. The *conjugate* of a quaternion q is obtained by changing the sign of the pure part, i. e.

$$\bar{q} = q_0 - \boldsymbol{q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}. \tag{2.4}$$

The quaternion conjugation (2.4) is a *linear anti-involution*

$$\overline{\overline{p}} = p, \quad \overline{p+q} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q}\,\overline{p}, \qquad \forall p, q \in \mathbb{H}. \tag{2.5}$$

Given a quaternion q and its conjugate, we can easily check that the following properties are correct

$$q_0 = \frac{1}{2}(q + \bar{q}), \quad \boldsymbol{q} = \frac{1}{2}(q - \bar{q}), \quad q = -\bar{q} \Leftrightarrow q = \boldsymbol{q}. \tag{2.6}$$

Using (2.2) the multiplication of the two quaternions $q = q_0 + \boldsymbol{q}$ and $p = p_0 + \boldsymbol{p}$ can be expressed as

$$qp = q_0p_0 + \boldsymbol{q} \cdot \boldsymbol{p} + q_0\boldsymbol{p} + p_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p}, \tag{2.7}$$

where we recognize the scalar product $\boldsymbol{q} \cdot \boldsymbol{p} = -(q_1p_1 + q_2p_2 + q_3p_3)$ and the antisymmetric cross type product $\boldsymbol{q}\times\boldsymbol{p} = \boldsymbol{i}(q_2p_3 - q_3p_2) + \boldsymbol{j}(q_3p_1 - q_1p_3) + \boldsymbol{k}(q_1p_2 -$

$q_2p_1)$. The scalar part of the product is

$$\mathrm{Sc}(qp) = q_0p_0 + \boldsymbol{q} \cdot \boldsymbol{p}, \tag{2.8}$$

and the pure part is

$$q_0\boldsymbol{p} + p_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p}. \tag{2.9}$$

Especially, if both q and p are pure quaternions (2.7) reduces to

$$\boldsymbol{qp} = \boldsymbol{q} \cdot \boldsymbol{p} + \boldsymbol{q} \times \boldsymbol{p}. \tag{2.10}$$

According to (2.7) the multiplication of a quaternion q and its conjugate can be expressed as

$$\begin{aligned} q\bar{q} &= q_0q_0 - \boldsymbol{q} \cdot \boldsymbol{q} + q_0(-\boldsymbol{q}) + q_0\boldsymbol{q} + \boldsymbol{q} \times (-\boldsymbol{q}) \\ &= q_o^2 + q_1^2 + q_2^2 + q_3^2. \end{aligned} \tag{2.11}$$

Equation (2.11) leads to the *modulus* $|q|$ of a quaternion q defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_o^2 + q_1^2 + q_2^2 + q_3^2}. \tag{2.12}$$

It is straightforward to see that with (2.5) and (2.12) the following modulus properties hold

$$|pq| = |p||q|, \quad |p| = |\overline{p}|, \quad p, q \in \mathbb{H}. \tag{2.13}$$

Using the conjugate (2.4) and the modulus of a quaternion q, we can define the *inverse* of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2} \tag{2.14}$$

which shows that $\mathbb{H}$ is a *normed division algebra*. For unit quaternions with $|q| = 1$ equation (2.14) simplifies to

$$q^{-1} = \bar{q}, \tag{2.15}$$

and for pure unit quaternions equation (2.14) becomes

$$q^{-1} = -q. \tag{2.16}$$

It is important to note that with (2.6), we have for two quaternion-valued functions f, g (independent of their domain space)

$$\frac{1}{2}(g\overline{f} + f\overline{g}) = g_0f_0 - \boldsymbol{g} \cdot \boldsymbol{f} = \mathrm{Sc}(g\overline{f}). \tag{2.17}$$

2.2 *Quaternion Module*

According to (2.1) a quaternion-valued function $f : \mathbb{R}^2 \longrightarrow \mathbb{H}$ can be expressed as

$$f(\boldsymbol{x}) = f_0 + \boldsymbol{i}f_1(\boldsymbol{x}) + \boldsymbol{j}f_2(\boldsymbol{x}) + \boldsymbol{k}f_3(\boldsymbol{x}), \quad f_0, f_1, f_2, f_3 \in \mathbb{R}. \tag{2.18}$$

We introduce an *inner product* of functions f, g defined on $\mathbb{R}^2$ with values in $\mathbb{H}$ as follows

$$\langle f, g \rangle = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\; d^2\boldsymbol{x}, \tag{2.19}$$

and its associated scalar *norm* $\|f\|$ by defining

$$\|f\|^2 = \langle f, f \rangle = \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \; d^2\boldsymbol{x}. \tag{2.20}$$

The *quaternion module* $L^2(\mathbb{R}^2; \mathbb{H})$ is then defined as

$$L^2(\mathbb{R}^2; \mathbb{H}) = \{f | f : \mathbb{R}^2 \longrightarrow \mathbb{H}, \|f\| < \infty\}. \tag{2.21}$$

2.3 *Connection Between $\mathbb{H}$ and Clifford Algebras $Cl_{3,0}$ and $Cl_{0,2}$*

Quaternions are isomorphic to the even subalgebra $Cl_{3,0}^+$ of *scalars* and *bivectors* (see [15,16]) of the real associative 8-dimensional Clifford geometric algebra $Cl_{3,0}$. The latter has the basis of

$$\begin{array}{rlrl} 1 & \text{scalar}, & \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_3\boldsymbol{e}_1, \boldsymbol{e}_2\boldsymbol{e}_3 & \text{bivectors}, \\ \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 & \text{vectors}, & i_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 & \text{pseudoscalar}. \end{array} \tag{2.22}$$

In equation (2.22) the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthonormal vector basis of the real 3D Euclidean vector space $\mathbb{R}^3$. The isomorphism means that any quaternion q can be expanded in the form

$$q \in Cl_{3,0}^+ = Cl_{3,0}^0 + Cl_{3,0}^2, \qquad \text{i.e.} \qquad q = \alpha + i_3\boldsymbol{b}, \tag{2.23}$$

where $\alpha \in \mathbb{R}$ and vector $\boldsymbol{b} \in \mathbb{R}^3$. Equation (2.23) tells us that the elements of $Cl_{3,0}^+$ form a four-dimensional linear space with one scalar and three bivector dimensions.

Quaternions are also isomorphic to $Cl_{0,2}$. For this we identify $\boldsymbol{i}, \boldsymbol{j}$ with vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$ with square -1, respectively, and $\boldsymbol{k}$ as their product $\boldsymbol{e}_1\boldsymbol{e}_2$. This fact is helpful for defining quaternionic Fourier and wavelet transforms and to compare them with other Clifford Fourier and wavelet transformations [16,17].

3 Quaternionic Fourier Transform (QFT)

It is natural to extend the Fourier transform to quaternion algebra. These extensions are broadly called the *quaternionic Fourier transform* (QFT). Due to the non-commutative properties of quaternions, there are three different types of QFT: a left-sided QFT, a right-sided QFT and a two-sided QFT [5]. By reasons explained in more detail below we choose to apply the *right-sided* QFT of a 2D quaternion-valued signal. This version of the QFT defined here is also known as the 2D Clifford FT of Delanghe, Sommen and Brackx [15].

3.1 Definition of QFT

Definition 3.1. The QFT of $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is the function $\mathcal{F}_q\{f\}\colon \mathbb{R}^2 \to \mathbb{H}$ given by

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x}, \tag{3.1}$$

where $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$, and the quaternion exponential product $e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}$ is the quaternion Fourier *kernel*.

Remark 3.2. Apart from the convention used in Definition 3.1 with $\frac{1}{(2\pi)^2}$ in the inverse QFT (3.5), there are *two* other common conventions: One is obtained by substituting (3.1) $\boldsymbol{\omega} \to 2\pi\boldsymbol{\omega}$. The other is obtained by evenly distributing the 2π factors between the transformation and the inverse transformation $\mathcal{F}_q = \frac{1}{2\pi}\int \ldots d^2\boldsymbol{x}$, $\mathcal{F}_q^{-1} = \frac{1}{2\pi}\int \ldots d^2\boldsymbol{\omega}$. All calculations in this paper can *easily* be converted to these other conventions.

Using the Euler formula for the quaternion Fourier kernel we can rewrite (3.1) in the following form

$$\begin{aligned}
\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = &\int_{\mathbb{R}^2} f(\boldsymbol{x}) \cos(\omega_1 x_1)\cos(\omega_2 x_2)\, d^2\boldsymbol{x} \\
&- \int_{\mathbb{R}^2} f(\boldsymbol{x})\, \boldsymbol{i} \sin(\omega_1 x_1)\cos(\omega_2 x_2)\, d^2\boldsymbol{x} \\
&- \int_{\mathbb{R}^2} f(\boldsymbol{x})\, \boldsymbol{j} \cos(\omega_1 x_1)\sin(\omega_2 x_2)\, d^2\boldsymbol{x} \\
&+ \int_{\mathbb{R}^2} f(\boldsymbol{x})\, \boldsymbol{k} \sin(\omega_1 x_1)\sin(\omega_2 x_2)\, d^2\boldsymbol{x}.
\end{aligned} \tag{3.2}$$

Equation (3.2) clearly shows how the QFT separates real signals into four quaternion components, i. e. the even-even, odd-even, even-odd and odd-odd components of f. Let us now take an example to illustrate this expression.

Example 3.3. Consider the quaternionic *distribution* signal (see Fig. 1), i. e. the

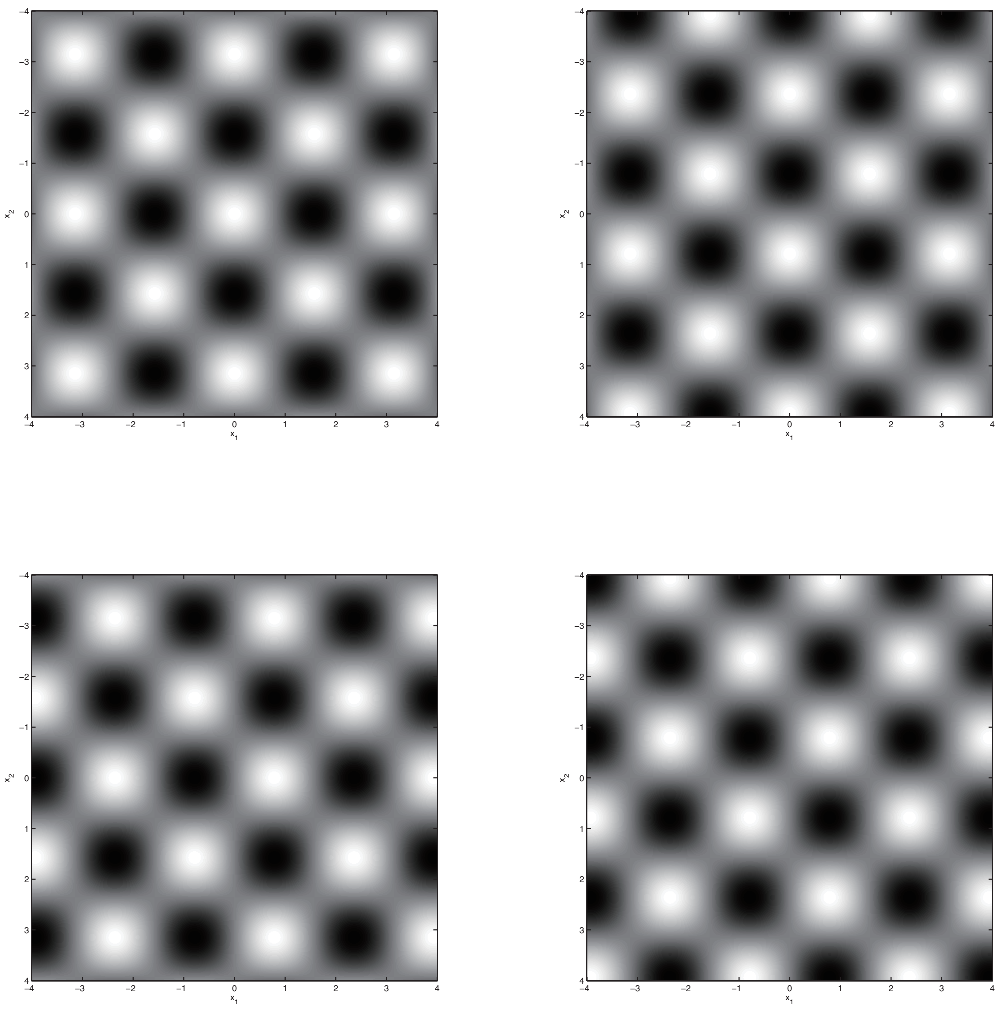

Fig. 1. Quaternionic signal of Example 3.3 in the spatial domain ($u_0 = v_0 = 2$). The resulting patterns are identical, apart from $\pi/4$ phase shifts along x_1 and x_2 [18].

QFT kernel of (3.1)

$$f(\boldsymbol{x}) = e^{\boldsymbol{j} v_0 x_2} e^{\boldsymbol{i} u_0 x_1}. \tag{3.3}$$

It is easy to see that the QFT of f is a *Dirac* quaternion function, i. e.

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = (2\pi)^2 \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0), \quad \boldsymbol{\omega}_0 = u_0 \boldsymbol{e}_1 + v_0 \boldsymbol{e}_2. \tag{3.4}$$

The following theorem tells us that the QFT is *invertible*, that is, the original signal f can be recovered by simply taking the inverse of the quaternionic Fourier transform (3.1).

Theorem 3.4. *Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$. Then the QFT $\mathcal{F}_q\{f\}$ of f is an invertible transform and its inverse is given by*

$$\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) e^{\boldsymbol{j} \omega_2 x_2} e^{\boldsymbol{i} \omega_1 x_1} \, d^2\boldsymbol{\omega}. \tag{3.5}$$

This subsection describes important properties of the QFT which will be used to establish a new uncertainty principle for the QFT. For detailed discussions of the properties of the QFT and their proofs, see e.g. [6,5,18]. We now first establish a *Plancherel* theorem, specific to the right-sided QFT.

Theorem 3.5 (QFT Plancherel)**.** *The inner product (2.19) of two quaternion module functions $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$ and their QFT is related by*

$$\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{(2\pi)^2}\langle \mathcal{F}_q\{f\}, \mathcal{F}_q\{g\}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{3.6}$$

In particular, with $f = g$, we get *Parseval's* theorem, i. e.

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{(2\pi)^2}\|\mathcal{F}_q\{f\}\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{3.7}$$

This shows that the *total signal energy* computed in the spatial domain is equal to the total signal energy computed in the quaternionic domain. The Parseval theorem [1] allows the energy of a quaternion-valued signal to be considered in either the spatial domain or the quaternionic domain and the change of domains for convenience of computation.

In following we give an alternative proof of Plancherel's theorem (compare to Hitzer [6]).

$$\begin{aligned}
\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\mathcal{F}_q\{f\}(\boldsymbol{\omega})e^{\boldsymbol{j}\omega_2 x_2}e^{\boldsymbol{i}\omega_1 x_1}d^2\boldsymbol{\omega}\right.\\
&\quad\times\left.\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega'_1 x_1}e^{-\boldsymbol{j}\omega'_2 x_2}\overline{\mathcal{F}_q\{g\}(\boldsymbol{\omega}')}d^2\boldsymbol{\omega}'\right]d^2\boldsymbol{x}\\
&= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{F}_q\{f\}(\boldsymbol{\omega})e^{\boldsymbol{j}\omega_2 x_2}e^{\boldsymbol{i}x_1(\omega_1-\omega'_1)}e^{-\boldsymbol{j}\omega'_2 x_2}\overline{\mathcal{F}_q\{g\}(\boldsymbol{\omega}')}d^2\boldsymbol{x}d^2\boldsymbol{\omega}'d^2\boldsymbol{\omega}\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{F}_q\{f\}(\boldsymbol{\omega})\boldsymbol{\delta}(\boldsymbol{\omega}-\boldsymbol{\omega}')\overline{\mathcal{F}_q\{g\}(\boldsymbol{\omega}')}d^2\boldsymbol{\omega}d^2\boldsymbol{\omega}'\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathcal{F}_q\{f\}(\boldsymbol{\omega})\overline{\mathcal{F}_q\{g\}(\boldsymbol{\omega})}d^2\boldsymbol{\omega}.
\end{aligned}\tag{3.8}$$

This completes the proof of theorem 3.5. □

Due to the non-commutativity of the quaternion exponential product factors we only have a *left linearity* property for general linear combinations with quaternionic constants and a *special* shift property.

[1] Different from the QFT Plancherel Theorem 3.5, the Parseval theorem (3.7) can be established for all three variants of the QFT: right-sided, left-sided and two-sided.

Theorem 3.6 (Left linearity property)**.** *The QFT of two quaternion module functions $f, g \in L^1(\mathbb{R}^2;\mathbb{H})$ is a left linear operator* [2]*, i. e.*

$$\mathcal{F}_q\{\mu f + \lambda g\}(\boldsymbol{\omega}) = \mu\mathcal{F}_q\{f\}(\boldsymbol{\omega}) + \lambda\mathcal{F}_q\{g\}(\boldsymbol{\omega}), \tag{3.9}$$

where μ and $\lambda \in \mathbb{H}$ are quaternionic constants.

Theorem 3.7 (Shift property)**.** *If the argument of $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is offset by a constant vector $\boldsymbol{x}_0 = x_0\boldsymbol{e}_1 + y_0\boldsymbol{e}_2$, i. e. $f_{\boldsymbol{x}_0}(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{x}_0)$, then [6]*

$$\mathcal{F}_q\{f_{\boldsymbol{x}_0}\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f e^{-\boldsymbol{i}\omega_1 x_0}\}(\boldsymbol{\omega})\, e^{-\boldsymbol{j}\omega_2 y_0}. \tag{3.10}$$

Proof. Equation (3.1) gives

$$\mathcal{F}_q\{f_{\boldsymbol{x}_0}\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x} - \boldsymbol{x}_0)\, e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x}. \tag{3.11}$$

We substitute $\boldsymbol{t}$ for $\boldsymbol{x} - \boldsymbol{x}_0$ in the above expression, and get with $d^2\boldsymbol{x}= d^2\boldsymbol{t}$

$$\begin{aligned}\mathcal{F}_q\{f_{\boldsymbol{x}_0}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} f(\boldsymbol{t})\, e^{-\boldsymbol{i}\omega_1(t_1+x_0)} e^{-\boldsymbol{j}\omega_2(t_2+y_0)}\, d^2\boldsymbol{t} \\ &= \int_{\mathbb{R}^2} \left(f(\boldsymbol{t}) e^{-\boldsymbol{i}\omega_1 x_0}\right) e^{-\boldsymbol{i}\omega_1 t_1} e^{-\boldsymbol{j}\omega_2 t_2}\, d^2\boldsymbol{t}\, e^{-\boldsymbol{j}\omega_2 y_0}. \end{aligned} \tag{3.12}$$

This proves (3.10). ☐

Dual to Theorem 3.7 the following *modulation* type formula holds for the inverse QFT.

Theorem 3.8. *If the argument of $\mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2;\mathbb{H})$, $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$ is offset by a constant frequency vector $\boldsymbol{\omega}_0 = \omega_{01}\boldsymbol{e}_1 + \omega_{02}\boldsymbol{e}_2 \in \mathbb{R}^2$, then $f_0(\boldsymbol{x})$ and $\mathcal{F}_q\{f\}(\boldsymbol{\omega})$ are related by*

$$f_0(\boldsymbol{x}) = \mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)](\boldsymbol{x}) = \mathcal{F}_q^{-1}\{\mathcal{F}_q\{f\}(\boldsymbol{\omega})\, e^{-\boldsymbol{j}\omega_{02} x_2}\}(\boldsymbol{x})\, e^{-\boldsymbol{i}\omega_{01} x_1}. \tag{3.13}$$

Remark 3.9. Equation (3.10) and (3.13) are specific for the right-sided definition of Definition 3.1. The usual form of the modulation property of the complex FT does not hold for the QFT. It is obstructed by the non-commutativity of the exponential factors (eq. (38) of [6])

$$e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2} \neq e^{-\boldsymbol{j}\omega_2 x_2} e^{-\boldsymbol{i}\omega_1 x_1}. \tag{3.14}$$

Next we give an explicit proof of the *derivative properties* stated in Table 2 of [6].

[2] The QFT is also right linear for real constants $\mu, \lambda \in \mathbb{R}$.

Theorem 3.10. *If the QFT of the n-th partial derivative of $f \in L^1(\mathbb{R}^2;\mathbb{H})$ with respect to the variable x_1 exists and is in $\in L^1(\mathbb{R}^2;\mathbb{H})$, then the QFT of $\frac{\partial^n f}{\partial x_1^n}\boldsymbol{i}^{-n}$ is given by*

$$\mathcal{F}_q\{\frac{\partial^n f}{\partial x_1^n}\,\boldsymbol{i}^{-n}\}(\boldsymbol{\omega}) = \omega_1^n \mathcal{F}_q\{f\}(\boldsymbol{\omega}), \quad \forall n \in \mathbb{N}. \tag{3.15}$$

Proof. We first prove the theorem for $n = 1$. Applying integration by parts and using the fact that f tends to zero for $x \to \infty$ we immediately obtain

$$\begin{aligned}
\mathcal{F}_q\{\frac{\partial}{\partial x_1} f\,\boldsymbol{i}^{-1}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \left(\frac{\partial}{\partial x_1} f(\boldsymbol{x})\,\boldsymbol{i}^{-1}\right) e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left(\frac{\partial}{\partial x_1} f(\boldsymbol{x})\,\boldsymbol{i}^{-\mathbf{1}}\right) e^{-\boldsymbol{i}\omega_1 x_1}\, dx_1\right] e^{-\boldsymbol{j}\omega_2 x_2}\, dx_2 \\
&= \int_{\mathbb{R}} \Bigg[f(\boldsymbol{x})\,\boldsymbol{i}^{-\mathbf{1}} e^{-\boldsymbol{i}\omega_1 x_1}|_{x_1=-\infty}^{x_1=\infty} \\
&\qquad - \int_{\mathbb{R}} f(\boldsymbol{x})\,\boldsymbol{i}^{-\mathbf{1}} \frac{\partial}{\partial x_1} e^{-\boldsymbol{i}\omega_1 x_1} dx_1 \Bigg] e^{-\boldsymbol{j}\omega_2 x_2} dx_2 \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{x})\,\omega_1 e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x} \\
&= \omega_1 \mathcal{F}_q\{f\}(\boldsymbol{\omega}).
\end{aligned} \tag{3.16}$$

Using mathematical induction we can finish the proof of Theorem 3.10. □

Theorem 3.11. *If the QFT of the m-th partial derivative of a quaternion-valued function $f \in L^1(\mathbb{R}^2;\mathbb{H})$ with respect to the variable x_2 exists and is in $L^1(\mathbb{R}^2;\mathbb{H})$, then*

$$\mathcal{F}_q\{\frac{\partial^m f}{\partial x_2^m}\}\,(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega})(\boldsymbol{j}\omega_2)^m, \quad m \in \mathbb{N}. \tag{3.17}$$

Proof. Direct calculation gives

$$\begin{aligned}
\frac{\partial f(\boldsymbol{x})}{\partial x_2} &= \frac{\partial}{\partial x_2}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1}\, d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \left(\frac{\partial}{\partial x_2} e^{\boldsymbol{j}\omega_2 x_2}\right) e^{\boldsymbol{i}\omega_1 x_1}\, d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \left[\mathcal{F}_q\{f\}(\boldsymbol{\omega})\,\boldsymbol{j}\omega_2\right] e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1}\, d^2\boldsymbol{\omega} \\
&= \mathcal{F}_q^{-1}\left[\mathcal{F}_q\{f\}(\boldsymbol{\omega})\boldsymbol{j}\omega_2\right].
\end{aligned} \tag{3.18}$$

We therefore get

$$\mathcal{F}_q\{\frac{\partial f}{\partial x_2}\}\,(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega})\boldsymbol{j}\omega_2. \tag{3.19}$$

By successive differentiation with respect to the variable x_2 and with induction we easily obtain

$$\mathcal{F}_q\{\frac{\partial^m f}{\partial x_2^m}\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega})(\boldsymbol{j}\omega_2)^m, \quad \forall m \in \mathbb{N}. \tag{3.20}$$

This ends the proof of (3.17). □

As consequence of Theorem 3.10 we immediately obtain the following corollary.

Corollary 3.12. Suppose that the QFT of a partial derivative $\partial^{n+m} f/\partial x_1^n \partial x_2^m$ of a quaternion-valued function $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is in $L^1(\mathbb{R}^2;\mathbb{H})$, and that $f = f_0 + \boldsymbol{i} f_1$, then

$$\mathcal{F}_q\{\frac{\partial^{n+m} f}{\partial x_1^n \partial x_2^m}\}(\boldsymbol{\omega}) = (\boldsymbol{i}\omega_1)^n \mathcal{F}_q\{f\}(\boldsymbol{\omega})(\boldsymbol{j}\omega_2)^m, \quad m, n \in \mathbb{N}. \tag{3.21}$$

Proof. For $n \in \mathbb{N}$ and $m = 0$ multiplication of (3.15) with $\boldsymbol{i}^n$ from the left gives

$$\mathcal{F}_q\{(\frac{\partial}{\partial x_1})^n f\}(\boldsymbol{\omega}) = (\boldsymbol{i}\omega_1)^n \mathcal{F}_q\{f\}(\boldsymbol{\omega}), \quad n \in \mathbb{N}. \tag{3.22}$$

The combination of (3.22) and (3.20) for $f = f_0 + \boldsymbol{i} f_1$ gives (3.21). □

Another important consequence of Theorems 3.10 and 3.11 is formulated in the following lemma.

Lemma 3.13. If the QFT of the 1st partial derivative of $f \in L^2(\mathbb{R}^2;\mathbb{H})$ with respect to the variable x_k, $k \in \{1, 2\}$ exists and is in $\in L^2(\mathbb{R}^2;\mathbb{H})$, then

$$(2\pi)^2 \int_{\mathbb{R}^2} |\frac{\partial}{\partial x_k} f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x} = \int_{\mathbb{R}^2} \omega_k^2 |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 d^2\boldsymbol{\omega}, \quad k \in \{1, 2\}. \tag{3.23}$$

Proof. For $k = 1$ straightforward calculation using Parseval's theorem (3.7) and Theorem 3.10 gives

$$\begin{aligned}(2\pi)^2 \int_{\mathbb{R}^2} |\frac{\partial}{\partial x_1} f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x} &\overset{(2.13)}{=} (2\pi)^2 \int_{\mathbb{R}^2} |\frac{\partial}{\partial x_1} f(\boldsymbol{x}) \, \boldsymbol{i^{-1}}|^2 \, d^2\boldsymbol{x} \\ &\overset{(3.7)}{=} \int_{\mathbb{R}^2} |\mathcal{F}_q\{\frac{\partial}{\partial x_1} f \, \boldsymbol{i^{-1}}\}(\boldsymbol{\omega})|^2 \, d^2\boldsymbol{\omega} \\ &\overset{(3.15)}{=} \int_{\mathbb{R}^2} \omega_1^2 |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 d^2\boldsymbol{\omega}. \end{aligned} \tag{3.24}$$

For $k = 2$ we similarly use Theorem 3.11 to get

Table 1
Properties of quaternionic Fourier transform of quaternion functions $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$, the constants are $\alpha, \beta \in \mathbb{H}, a \in \mathbb{R}\setminus\{0\}$, $\boldsymbol{x}_0 = x_0\boldsymbol{e}_1 + y_0\boldsymbol{e}_2 \in \mathbb{R}^2$ and $n \in \mathbb{N}$.

Property	Quaternion Function	Quaternionic Fourier Transform
Left linearity	$\alpha f(\boldsymbol{x})+\beta g(\boldsymbol{x})$	$\alpha\mathcal{F}_q\{f\}(\boldsymbol{\omega})+\beta\mathcal{F}_q\{g\}(\boldsymbol{\omega})$
Scaling	$f(a\boldsymbol{x})$	$\frac{1}{\lvert a\rvert^2}\mathcal{F}_q\{f\}(\frac{\boldsymbol{\omega}}{a})$
$\boldsymbol{x}$-Shift	$f(\boldsymbol{x}-\boldsymbol{x}_0)$	$\mathcal{F}_q\{fe^{-\boldsymbol{i}\omega_1 x_0}\}(\boldsymbol{\omega})\, e^{-\boldsymbol{j}\omega_2 y_0}$
Part. deriv.	$\left(\frac{\partial}{\partial x_1}\right)^n f(\boldsymbol{x})\,\boldsymbol{i}^{-n}$	$\omega_1^n\mathcal{F}_q\{f\}(\boldsymbol{\omega}),\quad f\in L^2(\mathbb{R}^2;\mathbb{H})$
	$\left(\frac{\partial}{\partial x_1}\right)^n f(\boldsymbol{x})$	$(\boldsymbol{i}\omega_1)^n\mathcal{F}_q\{f\}(\boldsymbol{\omega}),\quad f=f_0+\boldsymbol{i}f_1$
	$\left(\frac{\partial}{\partial x_2}\right)^n f(\boldsymbol{x})$	$\mathcal{F}_q\{f\}(\boldsymbol{\omega})(\boldsymbol{j}\omega_2)^n,\quad f\in L^2(\mathbb{R}^2;\mathbb{H})$
Plancherel	$\langle f_1, f_2\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} =$	$\frac{1}{(2\pi)^2}\langle\mathcal{F}_q\{f_1\},\mathcal{F}_q\{f_2\}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}$
Parseval	$\lVert f\rVert_{L^2(\mathbb{R}^2;\mathbb{H})} =$	$\frac{1}{2\pi}\lVert\mathcal{F}_q\{f\}\rVert_{L^2(\mathbb{R}^2;\mathbb{H})}$

$$
\begin{aligned}
(2\pi)^2\int_{\mathbb{R}^2}\lvert\frac{\partial}{\partial x_2}f(\boldsymbol{x})\rvert^2\,d^2\boldsymbol{x} &\overset{(3.7)}{=}\int_{\mathbb{R}^2}\lvert\mathcal{F}_q\{\frac{\partial}{\partial x_2}f\}(\boldsymbol{\omega})\rvert^2\,d^2\boldsymbol{\omega}\\
&\overset{(3.17)}{=}\int_{\mathbb{R}^2}\lvert\mathcal{F}_q\{f\}(\boldsymbol{\omega})\boldsymbol{j}\omega_2\rvert^2 d^2\boldsymbol{\omega}.\\
&\overset{(2.13)}{=}\int_{\mathbb{R}^2}\omega_2^2\lvert\mathcal{F}_q\{f\}(\boldsymbol{\omega})\rvert^2 d^2\boldsymbol{\omega}.
\end{aligned}\tag{3.25}
$$

□

Some important properties of the QFT are summarized in Table 1. For more details we refer to [6].

Example 3.14. Consider a two-dimensional Gaussian quaternion function (Fig. 2) of the form

$$f(\boldsymbol{x}) = C_0\, e^{-(a_1x_1^2+a_2x_2^2)}, \tag{3.26}$$

where $C_0 = C_{00}+\boldsymbol{i}C_{01}+\boldsymbol{j}C_{02}+\boldsymbol{k}C_{03} \in \mathbb{H}$ is a quaternion constant and $a_1, a_2 \in \mathbb{R}$ are positive real constants. Then the QFT of f as shown Fig. 3 is given by

$$
\begin{aligned}
\mathcal{F}_q\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}}\int_{\mathbb{R}} C_0\, e^{-a_1x_1^2}\, e^{-a_2x_2^2}e^{-\boldsymbol{i}\omega_1x_1}\, e^{-\boldsymbol{j}\omega_2x_2}\, dx_1dx_2\\
&= C_0\int_{\mathbb{R}} e^{-a_2x_2^2}\left(\int_{\mathbb{R}} e^{-a_1x_1^2}e^{-\boldsymbol{i}\omega_1x_1}dx_1\right)e^{-\boldsymbol{j}\omega_2x_2}dx_2\\
&= C_0\int_{\mathbb{R}} e^{-a_2x_2^2}\left(\sqrt{\frac{\pi}{a_1}}e^{-\omega_1^2/(4a_1)}\right)e^{-\boldsymbol{j}\omega_2x_2}dx_2.\\
&= C_0\sqrt{\frac{\pi}{a_1}}e^{-\omega_1^2/(4a_1)}\sqrt{\frac{\pi}{a_2}}e^{-\omega_2^2/(4a_2)}\\
&= C_0\,\frac{\pi}{\sqrt{a_1a_2}}\,e^{-(\frac{\omega_1^2}{4a_1}+\frac{\omega_2^2}{4a_2})}.
\end{aligned}\tag{3.27}
$$

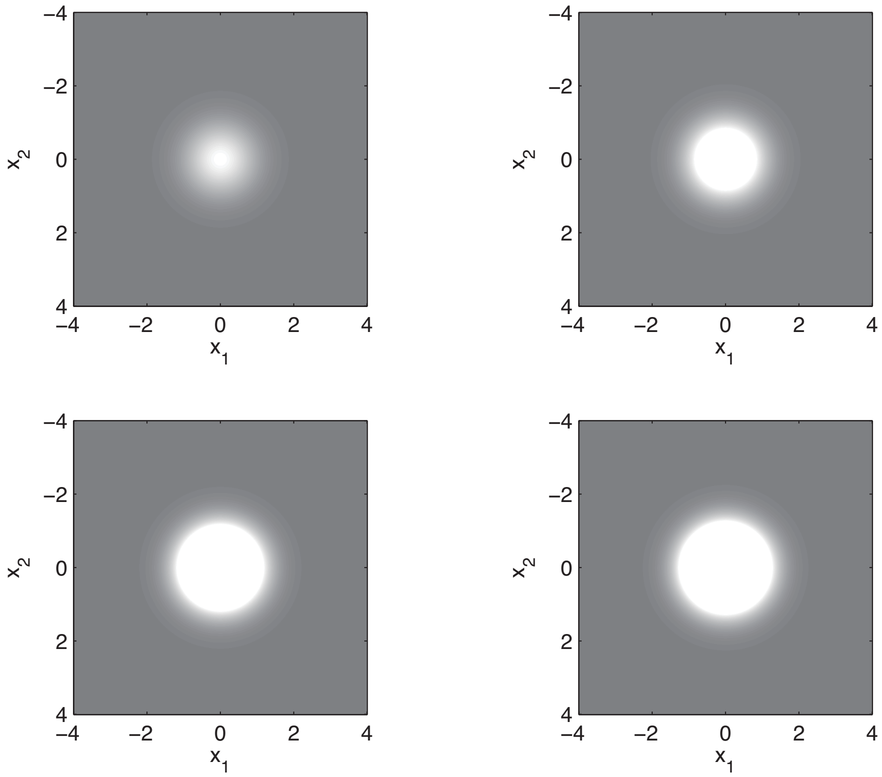

Fig. 2. Quaternion Gaussian function for $a_1 = a_2 = 1, C_{00} = 1, C_{01} = 2, C_{02} = 4$, and $C_{03} = 5$ in the spatial domain. **Top row**: real part and imaginary part $\boldsymbol{i}$. **Bottom row**: imaginary parts $\boldsymbol{j}$ and $\boldsymbol{k}$.

This shows that the QFT of the Gaussian quaternion function is another Gaussian quaternion function.

4 Uncertainty principle for QFT

In physics the uncertainty principle [10] was introduced for the first time 80 years ago by Heisenberg who demonstrated the impossibility of simultaneous precise measurements of a particle's *momentum* and its *position*. In a communication theory setting, an uncertainty principle states that a signal cannot be arbitrarily confined in both the *spatial* and *frequency* domains. Many efforts have been devoted to extend the uncertainty principle to various types of functions and Fourier transforms. Shinde et al. [19] established an uncertainty principle for *fractional* Fourier transforms which provides a lower bound on the uncertainty product of signal representations in both time and frequency domains for real signals. Korn [20] proposed Heisenberg type uncertainty principles for *Cohen* transforms which describe lower limits for the time-frequency concentration. In our previous papers [13,16,17,21], we established a new *directional* uncertainty principle for the Clifford Fourier transform which describes how the variances (in arbitrary but fixed

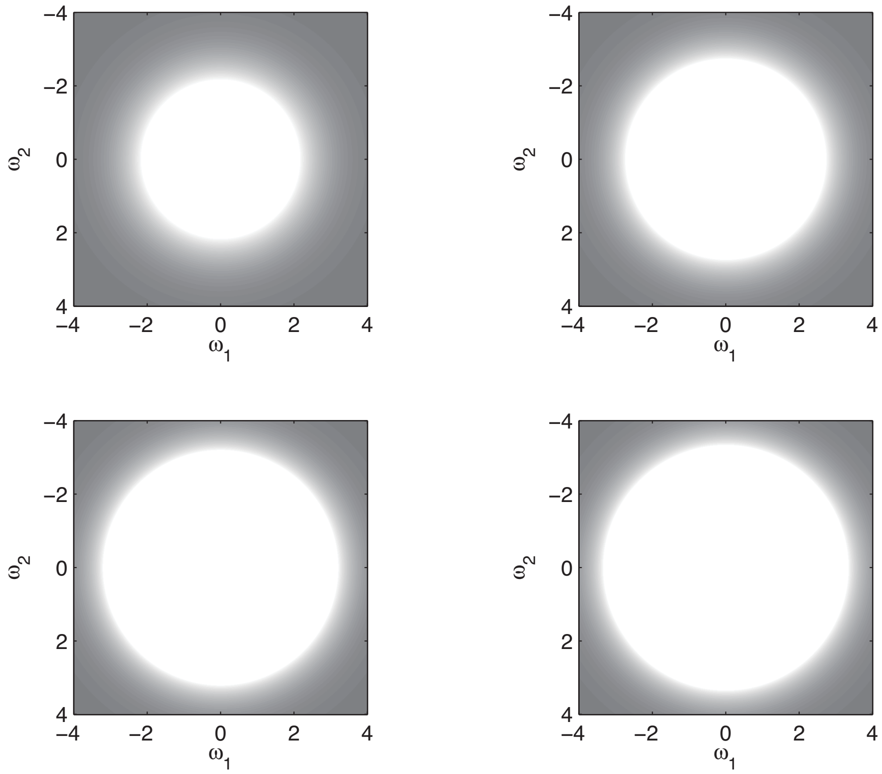

Fig. 3. Quaternion Gaussian function in the quaternionic frequency domain. **Top row**: real part and imaginary part $\boldsymbol{i}$. **Bottom row**: imaginary parts $\boldsymbol{j}$ and $\boldsymbol{k}$.

directions) of a multivector-valued function and its Clifford Fourier transform are related.

Bülow [2] showed a quaternion uncertainty principle, for quaternion valued signals according to which

$$\triangle x_1 \triangle x_2 \triangle \omega_1 \triangle \omega_2 \geq \frac{1}{16\pi^2}, \tag{4.1}$$

where $\triangle x_k$, $k = 1, 2$ is the effective width and $\triangle \omega_k$, $k = 1, 2$ is its effective bandwidth, defined as in Definition 4.1, only replacing $\mathcal{F}_q$ by a two-sided version of the QFT. [The factor $\frac{1}{4\pi^2}$ results from Bülow's use of the linear substitution $\boldsymbol{\omega} \to 2\pi\boldsymbol{\omega}$ in (3.1), compare Remark 3.2.] He showed that a Gabor filter can lead to equality in (4.1). It must be remembered that he applied the *two-sided* QFT for his uncertainty principle. His uncertainty principle is similar to the uncertainty principle for the conventional two-dimensional Fourier transform.

In the following we explicitly *generalize* and *prove* the classical uncertainty principle to quaternion module functions. We also give an explicit proof for *Gaussian* quaternion functions (Gabor filters) to be indeed the *only* functions that minimize the uncertainty. We further emphasize that our generalization is non-trivial because the multiplication of quaternions and the quaternion Fourier kernel are non-commutative. For this purpose we introduce the following definition.

Definition 4.1. Let $f \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion-valued signal such that $x_k f \in L^2(\mathbb{R}^2;\mathbb{H})$, $k = 1,2$, and let $\mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2;\mathbb{H})$ be its QFT such that $\omega_k \mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2;\mathbb{H})$, $k = 1,2$. The effective spatial width or spatial uncertainty $\triangle x_k$ of f is evaluated by

$$\triangle x_k = \sqrt{Var_k\{f\}}, \quad k \in \{1,2\}, \tag{4.2}$$

where $Var_k\{f\}$ is the variance of the energy distribution of f along the x_k axis defined by

$$Var_k\{f\} = \frac{\|x_k f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}}{\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}} = \frac{\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 x_k^2 \, d^2\boldsymbol{x}}{\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x}}, \quad k \in \{1,2\}. \tag{4.3}$$

Similarly, in the quaternionic domain we define the effective spectral width as

$$\triangle \omega_k = \sqrt{Var_k\{\mathcal{F}_q\{f\}\}}, \quad k \in \{1,2\}, \tag{4.4}$$

where $Var_k\{\mathcal{F}_q\{f\}\}$ is the variance of the frequency spectrum of f along the ω_k frequency axis given by

$$Var_k\{\mathcal{F}_q\{f\}\} = \frac{\|\omega_k \mathcal{F}_q\{f\}\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}}{\|\mathcal{F}_q\{f\}\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}} = \frac{\int_{\mathbb{R}^2} |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 \omega_k^2 \, d^2\boldsymbol{\omega}}{\int_{\mathbb{R}^2} |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 \, d^2\boldsymbol{\omega}}. \tag{4.5}$$

Theorem 4.2. *Let $f \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion-valued signal such that both $(1 + |x_k|)f(\boldsymbol{x}) \in L^2(\mathbb{R}^2;\mathbb{H})$ and $\frac{\partial}{\partial x_k} f(\boldsymbol{x}) \in L^2(\mathbb{R}^2;\mathbb{H})$ for $k = 1,2$. Then two uncertainty relations are fulfilled*

$$\triangle x_1 \triangle \omega_1 \geq \frac{1}{2}, \quad \textit{and} \quad \triangle x_2 \triangle \omega_2 \geq \frac{1}{2}. \tag{4.6}$$

The combination of the two spatial uncertainty principles above leads to the uncertainty principle for the two-dimensional quaternion signal f($\boldsymbol{x}$) of the form

$$\triangle x_1 \triangle x_2 \triangle \omega_1 \triangle \omega_2 \geq \frac{1}{4}. \tag{4.7}$$

Equality holds in (4.7) if and only if f is a Gaussian quaternion function, i. e.

$$f(\boldsymbol{x}) = K_0 \, e^{-(\frac{x_1^2}{2a_1} + \frac{x_2^2}{2a_2})}, \tag{4.8}$$

where K_0 is a quaternion constant, and a_1, $a_2 \in \mathbb{R}$ are positive real constants.

Analogous to complex numbers, we will use equation (2.17) to derive the following lemma which will be necessary to prove Theorem 4.2.

Lemma 4.3. For two quaternion-valued functions $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$, the *Schwarz inequality* takes the form

$$\left[\int_{\mathbb{R}^2} (g\bar{f} + f\bar{g}) d^2\boldsymbol{x}\right]^2 \leq 4 \int_{\mathbb{R}^2} f\bar{f} d^2\boldsymbol{x} \int_{\mathbb{R}^2} g\bar{g} d^2\boldsymbol{x} = 4 \int_{\mathbb{R}^2} |f|^2 d^2\boldsymbol{x} \int_{\mathbb{R}^2} |g|^2 d^2\boldsymbol{x} \tag{4.9}$$

Remark 4.4. An alternative form of Lemma 4.3 is given by equation (4.14).

To prove Schwarz's inequality, let $\epsilon \in \mathbb{R}$ be a real constant. Then

$$0 \leq \int_{\mathbb{R}^2} (f + \epsilon g)\overline{(f + \epsilon g)}\, d^2\boldsymbol{x}. \tag{4.10}$$

Applying the second part of (2.5) and then expanding the above inequality we can rewrite it in the following form

$$\begin{aligned} 0 \leq& \int_{\mathbb{R}^2} (f + \epsilon g)(\bar{f} + \epsilon \bar{g})\, d^2\boldsymbol{x} \\ =& \int_{\mathbb{R}^2} f\bar{f}\, d^2\boldsymbol{x} + \epsilon \int_{\mathbb{R}^2} (g\bar{f} + f\bar{g})\, d^2\boldsymbol{x} + \epsilon^2 \int_{\mathbb{R}^2} g\bar{g}\, d^2\boldsymbol{x}. \end{aligned} \tag{4.11}$$

The right-hand side of equation (4.11) is a quadratic expression in ϵ. The discriminant of this quadratic polynomial must be negative or zero and gives therefore

$$\left[\int_{\mathbb{R}^2} (g\bar{f} + f\bar{g}) d^2\boldsymbol{x}\right]^2 - 4 \int_{\mathbb{R}^2} f\bar{f} d^2\boldsymbol{x} \int_{\mathbb{R}^2} g\bar{g} d^2\boldsymbol{x} \leq 0, \tag{4.12}$$

which is equivalent to (4.9). This finishes the proof of Lemma 4.3. □

Now let us begin the proof of Theorem 4.2.

Proof. We prove Theorem 4.2 for $k \in 1, 2$. *First*, by applying Lemma 3.13 and the Parseval theorem (3.7) we immediately obtain

$$\begin{aligned} \triangle x_k^2 \triangle \omega_k^2 \quad &= \quad \frac{\int_{\mathbb{R}^2} x_k^2 |f(\boldsymbol{x})|^2\ d^2\boldsymbol{x}\ \int_{\mathbb{R}^2} \omega_k^2 |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2 d^2\boldsymbol{\omega}}{\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\ \int_{\mathbb{R}^2} |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2\, d^2\boldsymbol{\omega}} \\ &\overset{\text{Lemma 3.13}}{=} \frac{\int_{\mathbb{R}^2} x_k^2\, |f(\boldsymbol{x})|^2\ d^2\boldsymbol{x}\ \ (2\pi)^2 \int_{\mathbb{R}^2} |\frac{\partial}{\partial x_k} f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}}{\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\ \int_{\mathbb{R}^2} |\mathcal{F}_q\{f\}(\boldsymbol{\omega})|^2\, d^2\boldsymbol{\omega}} \\ &\overset{(3.7)}{=} \quad \frac{\int_{\mathbb{R}^2} |x_k f(\boldsymbol{x})|^2\ d^2\boldsymbol{x} \int_{\mathbb{R}^2} |\frac{\partial}{\partial x_k} f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}}{\left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\right)^2} \\ &\overset{\text{Lemma 4.3}}{\geq} \frac{\left[\int_{\mathbb{R}^2} \left(\frac{\partial}{\partial x_k} f(\boldsymbol{x})\, x_k \bar{f}(\boldsymbol{x}) + x_k f(\boldsymbol{x})\, \frac{\partial}{\partial x_k} \bar{f}(\boldsymbol{x})\right) d^2\boldsymbol{x}\right]^2}{4\, \|f\|^4_{L^2(\mathbb{R}^2;\mathbb{H})}} \\ &= \quad \frac{\left(\int_{\mathbb{R}^2} x_k \frac{\partial}{\partial x_k} \left[f(\boldsymbol{x})\bar{f}(\boldsymbol{x})\right]\ d^2\boldsymbol{x}\right)^2}{4\, \|f\|^4_{L^2(\mathbb{R}^2;\mathbb{H})}}. \end{aligned} \tag{4.13}$$

Second, using integration by parts we further get

$$\begin{aligned}\triangle x_k^2 \triangle\omega_k^2 &\geq \frac{\left([\int_{\mathbb{R}} x_k |f(\boldsymbol{x})|^2 dx_l]_{x_k=-\infty}^{x_k=\infty} - \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x}\right)^2}{4\,\|f\|^4_{L^2(\mathbb{R}^2;\mathbb{H})}} \\ &= \frac{(0 - \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x})^2}{4\,\|f\|^4_{L^2(\mathbb{R}^2;\mathbb{H})}} = \frac{1}{4},\end{aligned}$$

where $l \in \{1,2\}, l \neq k$. This proves (4.6). □

Remark 4.5. Consequently replacing the right-sided QFT (3.1) by the left-sided QFT $\mathcal{F}_q^{\text{left}}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\boldsymbol{j}\omega_2 x_2} e^{-\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{x})\, d^2\boldsymbol{x}$, allows to establish a corresponding Parseval theorem, left-sided QFT formulas for the partial derivatives (analogous to Theorems 3.10 and 3.11), and formulas for the norms $\|\partial_k f\|, k \in \{1,2\}$ (corresponding to Lemma 3.13). Theorem 4.2 applies therefore also to the left-sided QFT.

We finally show that the equality in (4.6) is satisfied if and only if f is a Gaussian quaternion function.

Using (2.17) we can rewrite Lemma 4.3 in the following form (compare to Chui [11])

$$\left[\int_{\mathbb{R}^2} \mathrm{Sc}(g\overline{h}) d^2\boldsymbol{x}\right]^2 \leq \int_{\mathbb{R}^2} |h|^2 d^2\boldsymbol{x} \int_{\mathbb{R}^2} |g|^2 d^2\boldsymbol{x}. \tag{4.14}$$

For $k = 1,2$ we first take $g = x_k f(\boldsymbol{x}) \in L^2(\mathbb{R}^2;\mathbb{H})$ and $h = \frac{\partial}{\partial x_k} f(\boldsymbol{x}) \in L^2(\mathbb{R}^2;\mathbb{H})$. Equation (4.14) can then be expressed as

$$\left[\int_{\mathbb{R}^2} \mathrm{Sc}\left(x_k f(\boldsymbol{x}) \overline{\frac{\partial}{\partial x_k} f(\boldsymbol{x})}\right) d^2\boldsymbol{x}\right]^2 \leq \int_{\mathbb{R}^2} |x_k f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x} \int_{\mathbb{R}^2} |\frac{\partial}{\partial x_k} f(\boldsymbol{x})|^2 \, d^2\boldsymbol{x}. \tag{4.15}$$

Equality in (4.15) implies that

$$-\mathrm{Sc}\left(x_k f(\boldsymbol{x}) \overline{\frac{\partial}{\partial x_k} f(\boldsymbol{x})}\right) = |x_k f(\boldsymbol{x}) \overline{\frac{\partial}{\partial x_k} f(\boldsymbol{x})}|, \tag{4.16}$$

and

$$|x_k f(\boldsymbol{x})| = a_k |\frac{\partial}{\partial x_k} f(\boldsymbol{x})|, \tag{4.17}$$

where the a_k are positive real constants. From equation (4.17) we obtain

$$x_k f(\boldsymbol{x}) = q\, a_k \frac{\partial}{\partial x_k} f(\boldsymbol{x}), \quad k = 1,2, \tag{4.18}$$

where q is a *unit* quaternion. Equation (4.16) implies that

$$-x_k f(\boldsymbol{x}) \overline{\frac{\partial}{\partial x_k} f(\boldsymbol{x})} \geq 0. \tag{4.19}$$

Multiplying both sides of (4.18) by $-\overline{\frac{\partial}{\partial x_k} f(\boldsymbol{x})}$ we get

$$-x_k f(\boldsymbol{x}) \overline{\frac{\partial}{\partial x_k} f(\boldsymbol{x})} = -q\, a_k |\frac{\partial}{\partial x_k} f(\boldsymbol{x})|^2, \quad k = 1, 2. \tag{4.20}$$

Applying (4.19) we get $q = -1$. Hence, we conclude that

$$\frac{\partial}{\partial x_k} f(\boldsymbol{x}) = -\frac{1}{a_k} x_k f(\boldsymbol{x}), \quad k = 1, 2. \tag{4.21}$$

Solving the equations (4.21) we further obtain that f must be a *Gaussian* quaternion function

$$f(\boldsymbol{x}) = K_0\, e^{-(\frac{x_1^2}{2a_1} + \frac{x_2^2}{2a_2})}, \quad k = 1, 2, \tag{4.22}$$

where $K_0 \in \mathbb{H}$ is a quaternion constant. $\square$

Since the Gaussian quaternion function $f(\boldsymbol{x})$ of (4.22) achieves the *minimum* width-bandwidth product, it is theoretically a very good prototype *wave form*. One can therefore construct a basic wave form using spatially or frequency scaled versions of $f(\boldsymbol{x})$ to provide *multiscale* spectral resolution. Such a wavelet basis construction from a Gaussian quaternion function prototype waveform has for example been realized in the quaternion *wavelet* transforms of [22]. The optimal localization in space and frequency is also the reason why (algebraically related to quaternions) two-dimensional Clifford *Gabor* bandpass filters (with Gaussian impulse response) were suggested in [23].

5 Conclusion

Using the basic concepts of quaternion algebra $\mathbb{H}$ we introduced the two-dimensional quaternionic Fourier transform (QFT). Important properties of the QFT such as partial derivative, Plancherel and Parseval theorems, specific shift- and modulation properties, and the quaternion function Schwarz inequality were demonstrated. We finally proposed a new uncertainty principle for the right-sided QFT.

So far no such uncertainty principle for a one-sided QFT (left or right) had been established. In our previous works on Clifford FT uncertainty principles, we always had the benefit of invariant vector derivatives. As far as we know, in quaternion calculus no suitable analogue to such a vector derivative has been established.

Before introducing the QFT Plancherel theorem 3.5, we pointed out that this theorem is specific for the right-sided QFT. With the definition of the inner product (2.19) it seems not possible to establish a similar QFT Plancherel theorem for the left-sided QFT. But as explained in Remark 4.5, the uncertainty principle for the right-sided QFT can be shown to apply to the left-sided QFT as well.

The case of the two-sided QFT is solved. A Parseval theorem can be shown [2,6] and equation (3.21) holds for general $f \in L^1(\mathbb{R}^2; \mathbb{H})$.

Acknowledgements

We do thank O. Yasukura for helpful comments and T. Khairuman for assistance in producing the figures.

References

[1] R. Bracewell, *The Fourier Transform and its Applications*, third ed., McGraw-Hill Book Company, New York, 2000.

[2] T. Bülow, Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images, Ph.D. Thesis, Institut für Informatik und Praktische Mathematik, University of Kiel, Germany, 1999.

[3] M. Felsberg, Low-Level Image Processing with the Structure Multivector, Ph.D. Thesis, Institut für Informatik und Praktische Mathematik, University of Kiel, Germany, 2002.

[4] T. A. Ell, Quaternion-Fourier Transforms for Analysis of Two-dimensional Linear Time-Invariant Partial Differential Systems, in: Proceeding of the 32nd Conference on Decision and Control, San Antonio, Texas, 1993, pp. 1830-1841.

[5] S. C. Pei, J. J. Ding and J. H. Chang, Efficient Implementation of Quaternion Fourier Transform, Convolution, and Correlation by 2-D Complex FFT, *IEEE Transactions on Signal Processing* **49**(11) (2001) 2783–2797.

[6] E. Hitzer, Quaternion Fourier Transform on Quaternion Fields and Generalizations, *Advances in Applied Clifford Algebras* **17**(3) (2007) 497–517.

[7] S. J. Sangwine and T. A. Ell, Hypercomplex Fourier Transforms of Color Images, *IEEE Transactions on Image Processing* **16**(1) (2007) 22–35.

[8] P. Bas, N. Le Bihan and J.M. Chassery, Color Image Watermarking using Quaternion Fourier Transform, in: Proceedings of the IEEE International Conference on Acoustics Speech and Signal Processing (ICASSP), Hong-Kong, 2003, pp. 521–524.

[9] E. Bayro-Corrochano, N. Trujillo and M. Naranjo, Quaternion Fourier Descriptors for Preprocessing and Recognition of Spoken Words Using Images of Spatiotemporal Representations, *Journal of Mathematical Imaging and Vision* **28**(2) (2007) 179–190.

[10] H. Weyl, *The Theory of Groups and Quantum Mechanics*, second ed., Dover, New York, 1950.

[11] C. K. Chui, *An Introduction to Wavelets*, Academic Press, New York, 1992.

[12] G. H. Granlund and H. Knutsson, *Signal Processing for Computer Vision*, Kluwer, Dordrecht, 1995.

[13] E. Hitzer and B. Mawardi, Clifford Fourier Transform on Multivector Fields and Uncertainty Principle for Dimensions $Cl_{n,0}$, $n = 2 \pmod 4$ and $n = 3 \pmod 4$, accepted for P. Angeles (ed.), Proceedings of the Seventh International Conference on Clifford Algebra (ICCA7), Toulouse, France, May 19-29, 2005.

[14] J. B. Kuipers, *Quaternions and Rotation Sequences*, Princeton University Press, New Jersey, 1999.

[15] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Vol. 76 of Research Notes in Mathematics, Pitman Advanced Publishing Program, Boston, 1982.

[16] B. Mawardi and E. Hitzer, Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra $Cl_{3,0}$, *Advances in Applied Clifford Algebras*, **16**(1) (2006) 41–61.

[17] B. Mawardi and E. Hitzer, Clifford Algebra $Cl_{3,0}$-valued Wavelet Transformation, Clifford Wavelet Uncertainty Inequality and Clifford Gabor Wavelets, *International Journal of Wavelet, Multiresolution and Information Processing* **5**(6) (2007) 997–1019.

[18] T. Bülow, M. Felsberg and G. Sommer, Non-commutative Hypercomplex Fourier Transforms of Multidimensional Signals, in: G. Sommer (ed.), *Geometric Computing with Clifford Algebras*, (Chapter 8). Springer, Heidelberg, 2001, pp. 187–207.

[19] S. Shinde and V. M. Gadre, An Uncertainty Principle for Real Signals in the Fractional Fourier Transform Domain, *IEEE Transactions on Signal Processing* **49**(11) (2001) 2545–2548.

[20] P. Korn, Some Uncertainty Principle for Time-Frequency Transforms for the Cohen Class, *IEEE Transactions on Signal Processing* **53**(12) (2005) 523–527.

[21] E. Hitzer and B. Mawardi, Uncertainty Principle for the Clifford Geometric Algebra $Cl_{n,0}$, $n = 3 \pmod 4$ Based on Clifford Fourier transform, in: T. Qian, M. I. Vai, and Y. Xu (eds.), the Springer (SCI) book series "Applied and Numerical Harmonic Analysis", 2006, pp. 45–54.

[22] E. Bayro-Corrochano, The Theory and Use of the Quaternion Wavelet Transform, *Journal of Mathematical Imaging and Vision* **24**(1) (2006) 19–35.

[23] F. Brackx, N. De Schepper and F. Sommmen, The Two-dimensional Clifford-Fourier Transform, *Journal of mathematical Imaging and Vision* **26**(1) (2006) 5–18.

B. Mawardi, E. Hitzer, A. Hayashi, R. Ashino, An Uncertainty Principle for Quaternion Fourier Transform, Computer & Mathematics with Applications, 56,