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Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra $Cl_{3,0}$

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Abstract

First, the basic concept of the vector derivative in geometric algebra is introduced. Second, beginning with the Fourier transform on a scalar function we generalize to a real Fourier transform on Clifford multivector-valued functions ($f : \mathbb{R}^3 \to Cl_{3,0}$). Third, we show a set of important properties of the Clifford Fourier transform on $Cl_{3,0}$ such as differentiation properties, and the Plancherel theorem. Finally, we apply the Clifford Fourier transform properties for proving an uncertainty principle for $Cl_{3,0}$ multivector functions.

Keywords: vector derivative, multivector-valued function, Clifford (geometric) algebra, Clifford Fourier transform, uncertainty principle.

1 Introduction

In the field of applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. The Fourier transform provides also a technique for signal analysis where the signal from the original domain is transformed to the spectral or frequency domain. In the frequency domain many characteristics of the signal are revealed. With these facts in mind, we extend the Fourier transform in geometric algebra.

Brackx et al. [11] extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. They also showed some properties of this generalized Fourier transform. A related applied approach for hypercomplex Clifford Fourier Transformations in $Cl_{0,n}$ was

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followed by Bülow et. al. [7]. In [13], Li et. al. extended the Fourier Transform holomorphically to a function of m complex variables.

In this paper we adopt and expand the generalization of the Fourier transform in Clifford geometric algebra[1] $\mathcal{G}_3$ recently suggested by Ebling and Scheuermann [3]. We explicitly show detailed properties of the real[2] Clifford geometric algebra Fourier transform (CFT), which we subsequently use to define and prove the uncertainty principle for $\mathcal{G}_3$ multivector functions.

We start with a review of the vector derivative for a multivector valued function. We demonstrate that with a little modification it obeys rules which resemble the rules for a scalar partial derivative.

2 Clifford's geometric algebra

In this section we introduce the axioms and the vector derivative of geometric algebra. Fore more details we refer the reader to [2, 5].

2.1 Axioms of geometric algebra

For $\mathcal{G}_n$ to be a Clifford geometric algebra over the real n-dimensional Euclidean vector space $\mathbb{R}^n$, the *geometric product* of elements A, B, C $\in \mathcal{G}_n$ must satisfy the following axioms:

Axiom 1 *Addition is commutative:*

$$A + B = B + A.$$

Axiom 2 *Addition and the geometric product are associative:*

$$(A + B) + C = A + (B + C), \qquad A(BC) = (AB)C,$$

and distributive:

$$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC.$$

Axiom 3 *There exist unique additive and multiplicative identities 0 and 1 such that:*

$$A + 0 = A, \qquad 1A = A.$$

[1]In the geometric algebra literature [2] instead of the mathematical notation $Cl_{p,q}$ the notation $\mathcal{G}_{p,q}$ is widely in use. It is convention to abbreviate $\mathcal{G}_{n,0}$ to $\mathcal{G}_n$.

[2]The meaning of *real* in this context is, that we use the three dimensional volume element $i_3 = \boldsymbol{e}_{123}$ of the geometric algebra $\mathcal{G}_3$ over the field of the reals $\mathbb{R}$ to construct the kernel of the Clifford Fourier transformation of definition 3. This i_3 has a clear geometric interpretation.

Axiom 4 *Every A in $\mathcal{G}_n$ has an additive inverse:*

$$A + (-A) = 0.$$

Axiom 5 *For any nonzero vector $\boldsymbol{a}$ in $\mathcal{G}_n$ the square of $\boldsymbol{a}$ is equal to a unique positive scalar $|\boldsymbol{a}|^2$, that is*

$$\boldsymbol{a}\boldsymbol{a} = \boldsymbol{a}^{\mathbf{2}} = |\boldsymbol{a}|^2 > 0.$$

Axiom 6 *Every k-vector, $A_k = \boldsymbol{a}_1\boldsymbol{a}_2...\boldsymbol{a}_k$, can be factorized into pairwise orthogonal vector factors, which satisfy:*

$$\boldsymbol{a}_i\boldsymbol{a}_j = -\boldsymbol{a}_j\boldsymbol{a}_i, \qquad i,j = 1,2,...,k \quad \text{and} \quad i \neq j.$$

2.2 Clifford's geometric algebra $\mathcal{G}_3$ of $\mathbb{R}^3$

Let us consider an orthonormal vector basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the real 3D Euclidean vector space $\mathbb{R}^3$. The geometric algebra over $\mathbb{R}^3$ denoted by $\mathcal{G}_3$ then has the graded $2^3 = 8$-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \boldsymbol{e}_{123}\}, \tag{1}$$

where 1 is the real scalar identity element (grade 0) of Axiom 3, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the $\mathbb{R}^3$ basis vectors (grade 1), $\boldsymbol{e}_{12} = \boldsymbol{e}_1\boldsymbol{e}_2$, $\boldsymbol{e}_{31} = \boldsymbol{e}_3\boldsymbol{e}_1$, and $\boldsymbol{e}_{23} = \boldsymbol{e}_2\boldsymbol{e}_3$ are frequently used definitions for the basis bivectors (grade 2), and $\boldsymbol{e}_{123} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = i_3$ defines the unit oriented pseudoscalars[3] (grade 3), i.e. the highest grade blade element in $\mathcal{G}_3$.

The associative geometric multiplication of the basis vectors obeys according to the axioms

$$\begin{aligned} \boldsymbol{e}_k\,\boldsymbol{e}_l &= -\boldsymbol{e}_l\,\boldsymbol{e}_k \qquad \text{for} \quad k \neq l, \qquad k,l = 1,2,3, \\ \boldsymbol{e}_k^2 &= 1 \qquad\qquad \text{for} \qquad k = 1,2,3. \end{aligned}$$

Inner products obey therefore

$$\boldsymbol{e}_k \cdot \boldsymbol{e}_l = \frac{1}{2}(\boldsymbol{e}_k\boldsymbol{e}_l + \boldsymbol{e}_l\boldsymbol{e}_k) = \delta_{kl}, \qquad k,l = 1,2,3.$$

According to these rules the Clifford (geometric) product of two arbitrary grade 1 vectors $\boldsymbol{x}$, $\boldsymbol{y}$ comprises the *inner product* and the *outer product*, i.e. the symmetric scalar part and the antisymmetric bivector part:

$$\boldsymbol{x}\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \wedge \boldsymbol{y},$$

[3]Other names in use are *trivector* or *volume element.*

where in coordinates

$$\begin{aligned}\boldsymbol{x}\cdot\boldsymbol{y} &= \frac{1}{2}(\boldsymbol{x}\boldsymbol{y}+\boldsymbol{y}\boldsymbol{x}) \\ &= (x_1\boldsymbol{e}_1+x_2\boldsymbol{e}_2+x_3\boldsymbol{e}_3)\cdot(y_1\boldsymbol{e}_1+y_2\boldsymbol{e}_2+y_3\boldsymbol{e}_3) \\ &= x_1y_1+x_2y_2+x_3y_3,\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{x}\wedge\boldsymbol{y} &= \frac{1}{2}(\boldsymbol{x}\boldsymbol{y}-\boldsymbol{y}\boldsymbol{x}) \\ &= (x_1y_2-x_2y_1)\boldsymbol{e}_{12}+(x_3y_1-x_1y_3)\boldsymbol{e}_{31}+(x_2x_3-x_3x_2)\boldsymbol{e}_{23}.\end{aligned}$$

The general elements of a geometric algebra are called multivectors. Every multivector M can be represented as a linear combination of k-grade elements ($k = 0, 1, 2, 3$). It means that in $\mathcal{G}_3$ a multivector can be expressed as

$$\begin{aligned}M=\sum_A \alpha_A\boldsymbol{e}_A &= \underbrace{\alpha_0}_{\text{scalar part}} + \underbrace{\alpha_1\boldsymbol{e}_1+\alpha_2\boldsymbol{e}_2+\alpha_3\boldsymbol{e}_3}_{\text{vector part}} + \\ &\quad + \underbrace{\alpha_{12}\boldsymbol{e}_{12}+\alpha_{31}\boldsymbol{e}_{31}+\alpha_{23}\boldsymbol{e}_{23}}_{\text{bivector part}} + \underbrace{\alpha_{123}\boldsymbol{e}_{123}}_{\text{trivector part}}, \qquad (2)\end{aligned}$$

where $A \in \{0, 1, 2, 3, 12, 31, 23, 123\}$, and $\alpha_A \in \mathbb{R}$. Note that $i_3 = \boldsymbol{e}_{123}$ commutes with all other elements of $\mathcal{G}_3$ and squares to $i_3^2 = $ -1. The *grade selector* is defined as $\langle M\rangle_k$ for the k-vector part of M, especially $\langle M\rangle = \langle M\rangle_0$. Then equation (2) can be rewritten as

$$M = \langle M\rangle + \langle M\rangle_1 + \langle M\rangle_2 + \langle M\rangle_3. \qquad (3)$$

We introduce the *grade involution*:

$$\widehat{M} = \langle M\rangle - \langle M\rangle_1 + \langle M\rangle_2 - \langle M\rangle_3, \qquad (4)$$

which is an automorphism $\widehat{(MN)} = \widehat{M}\widehat{N}$ for every $M, N \in \mathcal{G}_3$. The *reverse* of M is defined by the anti-automorphism

$$\widetilde{M} = \langle M\rangle + \langle M\rangle_1 - \langle M\rangle_2 - \langle M\rangle_3, \qquad (5)$$

which fulfils $\widetilde{(MN)} = \widetilde{N}\widetilde{M}$ for every $M, N \in \mathcal{G}_3$. By combining the grade involution and the reverse, we obtain a second anti-automorphism, the *Clifford conjugation*

$$\bar{M} = \widehat{\widetilde{M}} = \langle M\rangle - \langle M\rangle_1 - \langle M\rangle_2 + \langle M\rangle_3, \qquad (6)$$

which fulfils $\overline{(MN)} = \bar{N}\bar{M}$ for every $M, N \in \mathcal{G}_3$. The *square norm* of M is defined by

$$\|M\|^2 = \langle M\widetilde{M}\rangle, \tag{7}$$

where

$$\langle M\widetilde{N}\rangle = M * \widetilde{N} = \sum_A \alpha_A \beta_A \tag{8}$$

is a real valued (inner) *scalar product* for any M, N in $\mathcal{G}_3$ with M of equation (2) and $N = \sum_A \beta_A \boldsymbol{e}_A$. Note that

$$\langle M\ N\rangle \ = \ \langle N\ M\rangle \ = \ \langle \widetilde{M}\ \widetilde{N}\rangle \ = \ \langle \widetilde{N}\ \widetilde{M}\rangle, \tag{9}$$

and that

$$\boldsymbol{x}^2\ \|M\|^2 = \|\boldsymbol{x}\|^2\|M\|^2 = \|\boldsymbol{x}M\|^2, \qquad \boldsymbol{x} \in \mathbb{R}^3 \tag{10}$$

For $N = M$ in (8) we can re-express (7) as

$$\|M\|^2 = \sum_A \alpha_A^2. \tag{11}$$

We can therefore show that the norm satisfies[4] the inequality

$$\langle M\widetilde{N}\rangle \ \leq \ \|M\|\ \|N\| \qquad \text{for all} \quad M, N \in \mathcal{G}_3. \tag{12}$$

As a consequence of equation (12) we obtain the *multivector Cauchy-Schwarz inequality*

$$|\langle M\widetilde{N}\rangle|^2 \ \leq \ \|M\|^2\ \|N\|^2 \qquad \text{for all} \quad M, N \in \mathcal{G}_3. \tag{13}$$

2.3 Multivector functions, vector differential and vector derivative

Let $f = f(\boldsymbol{x})$ be a multivector-valued function of a vector variable $\boldsymbol{x}$ in $\mathcal{G}_3$ (compare the expansion of f in the basis (1) as given in (24)). For an arbitrary vector $\boldsymbol{a} \in \mathbb{R}^3$ we define[5] the *vector differential* in the $\boldsymbol{a}$ direction as

$$\boldsymbol{a} \cdot \nabla f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{f(\boldsymbol{x} + \epsilon \boldsymbol{a}) - f(\boldsymbol{x})}{\epsilon} \tag{14}$$

[4] Compare appendix A for the proof of (11)

[5] Bracket convention: $A \cdot BC = (A \cdot B)C \neq A \cdot (BC)$ and $A \wedge BC = (A \wedge B)C \neq A \wedge (BC)$ for multivectors $A, B, C \in \mathcal{G}_{p,q}$. The vector variable index $\boldsymbol{x}$ of the vector derivative is dropped: $\boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{\nabla}$ and $\boldsymbol{a} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{a} \cdot \boldsymbol{\nabla}$, but not when differentiating with respect to a different vector variable (compare e.g. proposition 4).

Table 1: Multiplication table of $\mathcal{G}_3$ basis elements

	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_{12}$	$\boldsymbol{e}_{31}$	$\boldsymbol{e}_{23}$	$\boldsymbol{e}_{123}$
1	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_{12}$	$\boldsymbol{e}_{13}$	$\boldsymbol{e}_{23}$	$\boldsymbol{e}_{123}$
$\boldsymbol{e}_1$	$\boldsymbol{e}_1$	1	$\boldsymbol{e}_{12}$	$-\boldsymbol{e}_{31}$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	$\boldsymbol{e}_{123}$	$\boldsymbol{e}_{23}$
$\boldsymbol{e}_2$	$\boldsymbol{e}_2$	-$\boldsymbol{e}_{12}$	1	$\boldsymbol{e}_{23}$	-$\boldsymbol{e}_1$	$\boldsymbol{e}_{123}$	$\boldsymbol{e}_3$	$\boldsymbol{e}_{31}$
$\boldsymbol{e}_3$	$\boldsymbol{e}_3$	$\boldsymbol{e}_{31}$	$-\boldsymbol{e}_{23}$	1	$\boldsymbol{e}_{123}$	$\boldsymbol{e}_1$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_{12}$
$\boldsymbol{e}_{12}$	$\boldsymbol{e}_{12}$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	$\boldsymbol{e}_{123}$	-1	$\boldsymbol{e}_{23}$	$-\boldsymbol{e}_{31}$	$-\boldsymbol{e}_3$
$\boldsymbol{e}_{31}$	$\boldsymbol{e}_{31}$	$\boldsymbol{e}_3$	$\boldsymbol{e}_{123}$	$-\boldsymbol{e}_1$	$-\boldsymbol{e}_{23}$	-1	$\boldsymbol{e}_{12}$	$-\boldsymbol{e}_2$
$\boldsymbol{e}_{23}$	$\boldsymbol{e}_{23}$	$\boldsymbol{e}_{123}$	$-\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$\boldsymbol{e}_{31}$	$-\boldsymbol{e}_{12}$	-1	$-\boldsymbol{e}_1$
$\boldsymbol{e}_{123}$	$\boldsymbol{e}_{123}$	$\boldsymbol{e}_{23}$	$\boldsymbol{e}_{31}$	$\boldsymbol{e}_{12}$	$-\boldsymbol{e}_3$	$-\boldsymbol{e}_2$	$-\boldsymbol{e}_1$	-1

provided this limit exists and is well defined. The basis independent *vector derivative* ∇ defined in [2, 5] obeys equation (14) for all vectors $\boldsymbol{a}$ and can be expanded as

$$\nabla = \boldsymbol{e}_k \partial_k = \boldsymbol{e}_1\partial_1 + \boldsymbol{e}_2\partial_2 + \boldsymbol{e}_3\partial_3, \tag{15}$$

where

$$\partial_k = \boldsymbol{e}_k \cdot \nabla = \frac{\partial}{\partial x_k}, \qquad k = 1, 2, 3 \tag{16}$$

is the scalar partial derivative with respect to the k^{th} coordinate $x_k = \boldsymbol{x} \cdot \boldsymbol{e}_k$.

The properties of a vector differential applied to multivector functions resemble much that of one dimensional scalar differentiation sum, constant multiple, product, and chain rules. For example, if f and g are multivector functions of $\boldsymbol{x}$, then the *sum rule* gives

$$\boldsymbol{a} \cdot \nabla(f + g) = \boldsymbol{a} \cdot \nabla f + \boldsymbol{a} \cdot \nabla g, \tag{17}$$

and the *product rule* gives

$$\boldsymbol{a} \cdot \nabla(fg) = (\boldsymbol{a} \cdot \nabla f)g + f\boldsymbol{a} \cdot \nabla g. \tag{18}$$

If α is a real scalar constant, the *constant multiple rule* yields

$$\boldsymbol{a} \cdot \nabla(\alpha f) = \alpha(\boldsymbol{a} \cdot \nabla f). \tag{19}$$

Finally, if $f = f(\lambda(\boldsymbol{x}))$ where $\lambda = \lambda(\boldsymbol{x})$ is a scalar function of $\boldsymbol{x}$, then the *chain rule* leads to

$$\boldsymbol{a} \cdot \nabla f = (\boldsymbol{a} \cdot \nabla\lambda)\frac{\partial f}{\partial \lambda}. \tag{20}$$

By using (14) and definition 17 of [5] we can derive the general rules[6] for vector differentiation from the corresponding rules for the vector differential as follows:

Proposition 1 $\nabla(f+g) = \nabla f + \nabla g$.

Proposition 2 $\nabla(fg) = (\dot{\nabla}\dot{f})g + \dot{\nabla}f\dot{g} = (\dot{\nabla}\dot{f})g + \sum_{k=1}^{n} \boldsymbol{e}_k f(\partial_k g)$.
(Multivector functions f and g do not necessarily commute.)

Proposition 3 *For* $f(\boldsymbol{x}) = g(\lambda(\boldsymbol{x}))$, $\lambda(\boldsymbol{x}) \in \mathbb{R}$,

$$\boldsymbol{a} \cdot \nabla f = \{\boldsymbol{a} \cdot \nabla\lambda(\boldsymbol{x})\} \frac{\partial g}{\partial \lambda}$$

Proposition 4 $\nabla f = \nabla_{\boldsymbol{a}} (\boldsymbol{a} \cdot \nabla f)$ *(derivative from differential)*

Differentiating twice with the vector derivative, we get the differential Laplacian operator ∇^2. We can write $\nabla^2 = \nabla \cdot \nabla + \nabla \wedge \nabla$. But for integrable functions $\nabla \wedge \nabla = 0$. In this case we have $\nabla^2 = \nabla \cdot \nabla$.

Proposition 5 *(integration of parts)*

$$\int_{\mathbb{R}^3} g(\boldsymbol{x})[\boldsymbol{a}\cdot\nabla h(\boldsymbol{x})] d^3\boldsymbol{x} = \left[\int_{\mathbb{R}^2} g(\boldsymbol{x})h(\boldsymbol{x}) d^2\boldsymbol{x}\right]_{a\cdot x=-\infty}^{a\cdot x=\infty} - \int_{\mathbb{R}^3} [\boldsymbol{a}\cdot\nabla g(\boldsymbol{x})] h(\boldsymbol{x}) d^3\boldsymbol{x}$$

We illustrate proposition 5 by inserting $\boldsymbol{a} = \boldsymbol{e}_3$, i.e.

$$\int_{\mathbb{R}^3} g(\boldsymbol{x})[\partial_3 h(\boldsymbol{x})] d^3\boldsymbol{x} = \left[\int_{\mathbb{R}^2} g(\boldsymbol{x})h(\boldsymbol{x}) dx_1 dx_2\right]_{x_3=-\infty}^{x_3=\infty} - \int_{\mathbb{R}^3} [\partial_3 g(\boldsymbol{x})] h(\boldsymbol{x}) d^3\boldsymbol{x},$$

which is nothing but the usual integration of parts formula for the partial derivative $\partial_3 h(\boldsymbol{x})$.

3 Clifford Fourier transform

In this section we present the Fourier transform in $\mathbb{R}$ and generalize it to Clifford's geometric algebra $\mathcal{G}_3$.

[6] Compare [2, 5] for the frame (basis) independent proofs of these propositions.

Table 2: Properties of the traditional Fourier transform

Property	Function	Fourier Transform
Linearity	$\alpha f(x)+\beta g(x)$	$\alpha\mathcal{F}\{f\}(\omega)+\ \beta\mathcal{F}\{g\}(\omega)$
Delay	$f(x-a)$	$e^{-i\omega a}\mathcal{F}\{f\}(\omega)$
Shift	$e^{i\omega_0 x}f(x)$	$\mathcal{F}\{f\}(\omega-\omega_0)$
Scaling	$f(ax)$	$\frac{1}{\lvert a\rvert}\mathcal{F}\{f\}(\frac{\omega}{a})$
Convolution	$(f\star g)(x)$	$\mathcal{F}\{f\}(\omega)\,\mathcal{F}\{g\}(\omega)$
Derivative	$f^{(n)}(x)$	$(i\omega)^n\mathcal{F}\{f\}(\omega)$
Parseval theorem	$\int_{\mathbb{R}}\lvert f(x)\rvert^2\,dx$	$\frac{1}{2\pi}\int_{\mathbb{R}}\lvert\mathcal{F}\{f\}(\omega)\rvert^2\,d\omega$

3.1 Fourier transform in $\mathbb{R}$

Popoulis [1] defined the Fourier transform and its inverse as follows:

Definition 1 *For an integrable function $f\in L^2(\mathbb{R})$, the Fourier transform of f is the function $\mathcal{F}\{f\}$: $\mathbb{R}\to\mathbb{C}$ given by*

$$\mathcal{F}\{f\}(\omega)=\int_{\mathbb{R}} f(x)\,e^{-i\omega x}\,dx, \tag{21}$$

where i^2=-1 is the unit imaginary.

The function $\mathcal{F}\{f\}(\omega)$ has the general form

$$\mathcal{F}\{f\}(\omega)=A(\omega)+iB(\omega)=C(\omega)e^{i\phi(\omega)}. \tag{22}$$

$C(\omega)$ is called the Fourier spectrum of $f(t)$, $C^2(\omega)$ its energy spectrum, and $\phi(\omega)$ its phase angle.

Definition 2 *If $\mathcal{F}\{f\}(\omega)\in L^2(\mathbb{R})$ and $f\in L^2(\mathbb{R})$, the inverse Fourier transform is given by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}(\omega)]=f(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}\{f\}(\omega)\,e^{i\omega x}\,d\omega. \tag{23}$$

The following table 2 summarizes some basic properties of the Fourier transform.

3.2 $\mathcal{G}_3$ Clifford Fourier transform

Consider a multivector valued function $f(\boldsymbol{x})$ in $\mathcal{G}_3$, i.e. f: $\mathbb{R}^3 \to \mathcal{G}_3$ where $\boldsymbol{x}$ is a vector variable. With the help of equation (2) $f(\boldsymbol{x})$ can be decomposed as

$$\begin{aligned} f(\boldsymbol{x}) = \sum_A f_A(\boldsymbol{x})\boldsymbol{e}_A &= f_0(\boldsymbol{x}) + f_1(\boldsymbol{x})\boldsymbol{e}_1 + f_2(\boldsymbol{x})\boldsymbol{e}_2 + f_3(\boldsymbol{x})\boldsymbol{e}_3 \\ &+ f_{12}(\boldsymbol{x})\,\boldsymbol{e}_{12} + f_{31}(\boldsymbol{x})e_{31} + f_{23}(\boldsymbol{x})\boldsymbol{e}_{23} + f_{123}(\boldsymbol{x})\boldsymbol{e}_{123}, \end{aligned} \tag{24}$$

where the f_A are eight real-valued functions. Equation (24) can also be written as (compare table 1)

$$\begin{aligned} f(\boldsymbol{x}) &= [f_0(\boldsymbol{x}) + f_{123}(\boldsymbol{x})i_3] + [f_1(\boldsymbol{x}) + f_{23}(\boldsymbol{x})i_3]\,\boldsymbol{e}_1 \\ &+ [f_2(\boldsymbol{x}) + f_{31}(\boldsymbol{x})i_3]\,\boldsymbol{e}_2 + [f_3(\boldsymbol{x}) + f_{12}(\boldsymbol{x})i_3]\,\boldsymbol{e}_3. \end{aligned} \tag{25}$$

Equation (25) can be regarded as a set of four complex functions. This motivates the extension of the Fourier transform to $\mathcal{G}_3$ multivector functions f. We will call this the Clifford Fourier transform (CFT).

Alternatively to (27), Bülow et. al. [7] extended the real Fourier transform to the n-dimensional geometric algebra $\mathcal{G}_{0,n}$. This variant of the Clifford Fourier transform of a multivector valued function in $\mathcal{G}_{0,n}$ is given by

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \prod_{k=1}^{n} e^{-\boldsymbol{e}_k 2\pi\omega_k x_k} d^n\boldsymbol{x}, \tag{26}$$

where

$$\boldsymbol{x} = \sum_{k=1}^{k=n} x_k\boldsymbol{e}_k, \quad \boldsymbol{\omega} = \sum_{k=1}^{k=n} \omega_k\boldsymbol{e}_k, \quad \text{and} \quad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = -\delta_{ij} \quad i,j = 1,2,...,n.$$

Yet in the following we will adopt (compare [3])

Definition 3 *The Clifford Fourier transform of $f(\boldsymbol{x})$ is the function $\mathcal{F}\{f\}$: $\mathbb{R}^3 \to \mathcal{G}_3$ given by*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}, \tag{27}$$

where we can write $\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3$, $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3$ with $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ the basis vectors of $\mathbb{R}^3$. Note that[7]

$$d^3\boldsymbol{x} = \frac{d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge d\boldsymbol{x_3}}{i_3} \tag{28}$$

is scalar valued ($d\boldsymbol{x_k} = dx_k \boldsymbol{e_k}$, $k = 1, 2, 3$, no summation). Because i_3 commutes with every element of $\mathcal{G}_3$, the Clifford Fourier kernel $e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}$ will also commute with every element of $\mathcal{G}_3$.

Theorem 1 *The Clifford Fourier transform* $\mathcal{F}\{f\}$ *of* $f \in L^2(\mathbb{R}^3, \mathcal{G}_3)$, $\int_{\mathbb{R}^3} \|f\|^2 d^3\boldsymbol{x} < \infty$ *is invertible and its inverse is calculated by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{\omega}. \tag{29}$$

Proof Substituting equation (27) in equation (29) gives

$$\begin{aligned}
\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(\boldsymbol{y})\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{y}}\, d^3\boldsymbol{y}\, e^{i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})\, \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i_3 (\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{\omega}}\, d^3\boldsymbol{\omega}\, d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})\, \boldsymbol{\delta}(\boldsymbol{x} - \boldsymbol{y})\, d^3\boldsymbol{y} \\
&= f(\boldsymbol{x}).
\end{aligned}$$

Equation (29) is called the Clifford Fourier integral theorem. It describes how to get from the transform back to the original function f.

4 Basic properties of Clifford Fourier transform

We summarize some important properties of the Clifford Fourier transform which are similar to the traditional scalar Fourier transform properties. Most of the properties can be proved by a change of variables.

[7] The division by the geometric algebra unit volume element i_3 in (28) to obtain a scalar infinitesimal volume is a matter of choice. Defining instead the pseudoscalar $d^3\boldsymbol{x}_p = d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge d\boldsymbol{x_3}$ would work equally well. It would simply mean, that all integrals using $d^3\boldsymbol{x}_p$ instead of $d^3\boldsymbol{x}$ in this paper would have to be multiplied by $-i_3 = \frac{1}{i_3}$, which commutes with every multivector.

4.1 Linearity

If $f(\boldsymbol{x}) = \alpha f_1(\boldsymbol{x}) + \beta f_2(\boldsymbol{x})$ for constants α and β, $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x}) \in \mathcal{G}_3$ then

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \alpha\mathcal{F}\{f_1\}(\boldsymbol{\omega}) + \beta\mathcal{F}\{f_2\}(\boldsymbol{\omega}). \tag{30}$$

Proof is trivial.

4.2 Delay property

If the argument of $f(\boldsymbol{x})$ is offset by a constant vector $\boldsymbol{a}$, i.e. $f_d(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{a})$, then

$$\mathcal{F}\{f_d\}(\boldsymbol{\omega}) = e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{31}$$

Proof Equation (27) gives

$$\mathcal{F}\{f_d\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x} - \boldsymbol{a})e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}.$$

We substitute $\boldsymbol{t}$ for $\boldsymbol{x} - \boldsymbol{a}$ in the above expression, and get with $d^3\boldsymbol{x}= d^3\boldsymbol{t}$

$$\begin{aligned}\mathcal{F}\{f_d\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(\boldsymbol{t})e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{t}}\, d^3\boldsymbol{t} \\ &= e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega}).\end{aligned}$$

This proves (31).

4.3 Scaling property

Let a be a positive scalar constant, then the Clifford Fourier transform of the function $f_a(\boldsymbol{x}) = f(a\boldsymbol{x})$ becomes

$$\mathcal{F}\{f_a\}(\boldsymbol{\omega}) = \frac{1}{a^3}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a}). \tag{32}$$

Proof. Equation (27) gives

$$\mathcal{F}\{f_a\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(a\boldsymbol{x})e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}.$$

We substitute $\boldsymbol{u}$ for $a\boldsymbol{x}$, and get

$$\begin{aligned}\mathcal{F}\{f_a\}(\boldsymbol{\omega}) &= \frac{1}{a^3}\int_{\mathbb{R}^3} f(\boldsymbol{u})e^{-i_3\left(\frac{\boldsymbol{\omega}}{a}\cdot\boldsymbol{u}\right)}\, d^3\boldsymbol{u} \\ &= \frac{1}{a^3}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a}).\end{aligned}$$

4.4 Shift property

If $\boldsymbol{\omega}_0 \in \mathbb{R}^3$ and $f_0(\boldsymbol{x}) = f(\boldsymbol{x})e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}}$, then

$$\mathcal{F}\{f_0\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0) \tag{33}$$

Proof Using equation (27) and simplifying it we obtain

$$\begin{aligned}\mathcal{F}\{f_0\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(\boldsymbol{x})e^{-i_3(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\ &= \mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0).\end{aligned}$$

The shift property shows that the multiplication by $e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}}$ shifts the CFT of the multivector function $f(\boldsymbol{x})$ so that it becomes centered on the point $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ in the frequency domain.

5 Differentiation of Clifford Fourier transform

The CFT differentiation properties also resemble that of the traditional scalar Fourier transform of table 2.

5.1 Vector differential and Partial differentiation

The Clifford Fourier transform of the vector differential of $f(\boldsymbol{x})$ is

$$\mathcal{F}\{\boldsymbol{a}\cdot\nabla f\}(\boldsymbol{\omega}) = i_3\, \boldsymbol{a}\cdot\boldsymbol{\omega}\, \mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{34}$$

The vector differential in the $\boldsymbol{a}$ direction is

$$\begin{aligned}\boldsymbol{a}\cdot\nabla f(\boldsymbol{x}) &= \boldsymbol{a}\cdot\nabla\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f\}(\boldsymbol{\omega})\, (\boldsymbol{a}\cdot\nabla e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}})\, d^3\boldsymbol{\omega} \\ &\overset{\text{Prop. 3}}{=} \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f\}(\boldsymbol{\omega})\, (i_3\, \boldsymbol{a}\cdot\boldsymbol{\omega})e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{\omega} \\ &= \mathcal{F}^{-1}[i_3\, \boldsymbol{a}\cdot\boldsymbol{\omega}\mathcal{F}\{f\}](\boldsymbol{x}).\end{aligned}$$

This proves (34). Setting $\boldsymbol{a} = \boldsymbol{e}_k$ we get for a partial derivative of $f(\boldsymbol{x})$

$$\mathcal{F}\{\partial_k f\}(\boldsymbol{\omega}) = i_3\omega_k\mathcal{F}\{f\}(\boldsymbol{\omega}), \qquad k = 1, 2, 3. \tag{35}$$

By a similar calculation we can find the derivatives of second order, i.e.

$$\mathcal{F}\{\boldsymbol{a}\cdot\nabla\ \boldsymbol{b}\cdot\nabla f\}(\boldsymbol{\omega}) = -\boldsymbol{a}\cdot\boldsymbol{\omega}\ \boldsymbol{b}\cdot\boldsymbol{\omega}\ \mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{36}$$

For $\boldsymbol{a} = \boldsymbol{e}_k, \boldsymbol{b} = \boldsymbol{e}_l$ we therefore get

$$\mathcal{F}\{\partial_k \partial_l\}(\boldsymbol{\omega}) = -\omega_k \omega_l \mathcal{F}\{f\}(\boldsymbol{\omega}) \qquad k, l = 1, 2, 3. \tag{37}$$

If $\boldsymbol{x}$ is a vector variable, then

$$\mathcal{F}\{\boldsymbol{x} f(\boldsymbol{x})\}(\boldsymbol{\omega}) = i_3 \nabla_{\boldsymbol{\omega}} \mathcal{F}\{f\}(\boldsymbol{\omega}) \tag{38}$$

Proof Direct calculation gives

$$\begin{aligned}
\mathcal{F}\{\boldsymbol{x} f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} \boldsymbol{x} f(\boldsymbol{x})\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \boldsymbol{x}\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}} f(\boldsymbol{x})\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} i_3\, \nabla_{\boldsymbol{\omega}}\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}} f(\boldsymbol{x})\, d^3\boldsymbol{x} \\
&= i_3\, \nabla_{\boldsymbol{\omega}} \int_{\mathbb{R}^3} f(\boldsymbol{x})\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= i_3 \nabla_{\boldsymbol{\omega}} \mathcal{F}\{f\}(\boldsymbol{\omega}),
\end{aligned}$$

because we get with propositions 3 and 4

$$\nabla_{\boldsymbol{\omega}}\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}} = -i_3 \left[\nabla_{\boldsymbol{\omega}} \left(\boldsymbol{\omega} \cdot \boldsymbol{x}\right)\right]\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}} = -i_3 \boldsymbol{x} e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}. \tag{39}$$

The Clifford Fourier transform of $\boldsymbol{a} \cdot \boldsymbol{x}\; f(\boldsymbol{x})$ gives

$$\begin{aligned}
\mathcal{F}\{\boldsymbol{a} \cdot \boldsymbol{x}\; f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} \boldsymbol{a} \cdot \boldsymbol{x}\; f(\boldsymbol{x})\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{x})\; \boldsymbol{a} \cdot \boldsymbol{x}\, e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{x} \\
&\overset{\text{Prop. 3}}{=} \int_{\mathbb{R}^3} f(\boldsymbol{x})\, i_3\; \boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}}\; e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= i_3\; \boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}} \int_{\mathbb{R}^3} f(\boldsymbol{x})\; e^{-i_3 \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= i_3\, \boldsymbol{a} \cdot \nabla_{\boldsymbol{\omega}} \mathcal{F}\{f\}(\boldsymbol{\omega}).
\end{aligned} \tag{40}$$

For $\boldsymbol{a} = e_k$ $(k = 1,\, 2,\, 3)$ we get

$$\mathcal{F}\{x_k\; f(\boldsymbol{x})\}(\boldsymbol{\omega}) = i_3\, \frac{\partial}{\partial \omega_k} \mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{41}$$

5.2 Vector derivative and Laplace operator

The Clifford Fourier transform of the vector derivative is

$$\mathcal{F}\{\nabla f\}(\boldsymbol{\omega}) = i_3\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega}) \tag{42}$$

and of the Laplace operator

$$\mathcal{F}\{\nabla^2 f\}(\boldsymbol{\omega}) = -\boldsymbol{\omega}^2\mathcal{F}\{f\}(\boldsymbol{\omega}) \tag{43}$$

Proof For $g(\boldsymbol{x})= e^{i_3\lambda(\boldsymbol{x})}$, $\lambda(\boldsymbol{x}) = \boldsymbol{\omega}\cdot\boldsymbol{x}$ reference [5] gives

$$\boldsymbol{a}\cdot\nabla g = \boldsymbol{a}\cdot\boldsymbol{\omega}\, i_3\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}},$$

where we used proposition 3 and $\boldsymbol{a}\cdot\nabla(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \boldsymbol{a}\cdot\boldsymbol{\omega}$. Applying proposition 4, we obtain

$$\begin{aligned}\nabla g &= \nabla_{\boldsymbol{a}}\,(\boldsymbol{a}\cdot\nabla g)\\ &= \nabla_{\boldsymbol{a}}\,\{\boldsymbol{a}\cdot\boldsymbol{\omega}\, i_3\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\}\\ &= \nabla_{\boldsymbol{a}}\,\{\boldsymbol{a}\cdot\boldsymbol{\omega}\}\, i_3\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\\ &= i_3\,\boldsymbol{\omega}\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}.\end{aligned} \tag{44}$$

According to proposition 72 of [5], the application of (44) leads to

$$\begin{aligned}\nabla f(\boldsymbol{x}) &= \nabla\,\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}\{f\}(\boldsymbol{\omega})e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}d^3\boldsymbol{\omega}\\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\nabla\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\mathcal{F}\{f\}(\boldsymbol{\omega})\,d^3\boldsymbol{\omega}\\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} i_3\boldsymbol{\omega}\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\mathcal{F}\{f\}(\boldsymbol{\omega})\,d^3\boldsymbol{\omega}\\ &= \mathcal{F}^{-1}[i_3\boldsymbol{\omega}\mathcal{F}\{f\}](\boldsymbol{x}),\end{aligned} \tag{45}$$

and therefore

$$\mathcal{F}\{\nabla f\}(\boldsymbol{\omega}) = i_3\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega}).$$

Vector differentiating equation (45) once more we get

$$\begin{aligned}\mathcal{F}\{\nabla^2 f\} &= \mathcal{F}\{\nabla(\nabla f)\}\\ &= i_3\boldsymbol{\omega}\mathcal{F}\{\nabla f\}(\boldsymbol{\omega})\\ &= -\boldsymbol{\omega}^2\mathcal{F}\{f\}(\boldsymbol{\omega}).\end{aligned} \tag{46}$$

In general[8] we get

$$\mathcal{F}\{\nabla^m f\} = (i_3\boldsymbol{\omega})^m\mathcal{F}\{f\}(\boldsymbol{\omega}), \qquad m\in\mathbb{N}. \tag{47}$$

[8] This general formula should prove very useful for transforming partial differential equations (more precisely: vector derivative equations) into algebraic equations.

6 Convolution

The most important property of the Clifford Fourier tansform for signal processing applications is the convolution theorem. Because of the non-Abelian geometric product we have the following definition:

Definition 4 *Let f and g be multivector valued functions and both have Clifford Fourier transforms, then the convolution of f and g is denoted $f{\star}g$, and defined by*

$$(f{\star}g)(\boldsymbol{x}) = \int_{\mathbb{R}^3} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})\, d^3\boldsymbol{y}, \tag{48}$$

Theorem 2 *The Clifford Fourier transform of the convolution of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ is equal to the product of the Clifford Fourier transforms of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, i.e*

$$\mathcal{F}\{f{\star}g\}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega})\mathcal{F}\{g\}(\boldsymbol{\omega}), \tag{49}$$

Proof Let $\mathcal{F}\{f\}(\boldsymbol{\omega})$ and $\mathcal{F}\{g\}(\boldsymbol{\omega})$ denote the Clifford Fourier transforms of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ respectively. Transforming equation (48), we get

$$\begin{aligned}
\mathcal{F}\{f{\star}g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3}[\int_{\mathbb{R}^3} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})d^3\boldsymbol{y}]\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})[\int_{\mathbb{R}^3} g(\boldsymbol{x}-\boldsymbol{y})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^3\boldsymbol{x}]\, d^3\boldsymbol{y}.
\end{aligned}$$

By introducing the vector $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$, the transform can be reexpressed as

$$\begin{aligned}
\mathcal{F}\{f{\star}g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(\boldsymbol{y})[\int_{\mathbb{R}^3} g(\boldsymbol{z})\, e^{-i_3[\boldsymbol{\omega}\cdot(\boldsymbol{y}+\boldsymbol{z})]}\, d^3\boldsymbol{z}]\, d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})[\int_{\mathbb{R}^3} g(\boldsymbol{z})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d^3\boldsymbol{z}]\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d^3\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} f(\boldsymbol{y})e^{-i_3(\boldsymbol{\omega}\cdot\boldsymbol{y})}\, d^3\boldsymbol{y}\mathcal{F}\{g\}(\boldsymbol{\omega}) \\
&= \mathcal{F}\{f\}(\boldsymbol{\omega})\mathcal{F}\{g\}(\boldsymbol{\omega}).
\end{aligned}$$

7 Plancherel and Parseval theorems

Just as in the case of the traditional scalar Fourier transform, the **Plancherel theorem** in the geometric algebra $\mathcal{G}_3$ relates two multivector functions with their Clifford Fourier transforms.

Theorem 3 *Assume that* $f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) \in \mathcal{G}_3$ *with Clifford Fourier transform* $\mathcal{F}\{f_1\}(\boldsymbol{\omega})$ *and* $\mathcal{F}\{f_2\}(\boldsymbol{\omega})$ *respectively, then*

$$\langle f_1(\boldsymbol{x})\ \widetilde{f_2(\boldsymbol{x})}\rangle_V \ = \frac{1}{(2\pi)^3}\langle \mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V, \tag{50}$$

where we define the volume integral

$$\langle f_1(\boldsymbol{x})\ \widetilde{f_2(\boldsymbol{x})}\rangle_V \ = \int_{\mathbb{R}^3} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}d^3\boldsymbol{x} \tag{51}$$

Proof Direct calculation yields

$$\begin{aligned}
\langle f_1(\boldsymbol{x})\ \widetilde{f_2(\boldsymbol{x})}\rangle_V &= \int_{\mathbb{R}^3} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}d^3\boldsymbol{x} \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}[\int_{\mathbb{R}^3} \mathcal{F}\{f_1\}(\boldsymbol{\omega})\, e^{i_3\boldsymbol{\omega}\cdot\boldsymbol{x}} d^3\boldsymbol{\omega}]\,\widetilde{f_2(\boldsymbol{x})}d^3\boldsymbol{x} \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathcal{F}\{f_1\}(\boldsymbol{\omega})\left[\int_{\mathbb{R}^3} \widetilde{f_2(\boldsymbol{x})\, e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{x}}}\, d^3\boldsymbol{x}\right] d^3\boldsymbol{\omega}. \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}d^3\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^3}\langle \mathcal{F}\{f_1\}(\boldsymbol{\omega})\ \widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V.
\end{aligned}$$

In particular, with $f_1(\boldsymbol{x}) = f_2(\boldsymbol{x}) = f(\boldsymbol{x})$, we get the (multivector) **Parseval theorem**, i.e.

$$\langle f(\boldsymbol{x})\ \widetilde{f(\boldsymbol{x})}\rangle_V = \frac{1}{(2\pi)^3}\langle \mathcal{F}\{f\}(\boldsymbol{\omega})\ \widetilde{\mathcal{F}\{f\}(\boldsymbol{\omega})}\rangle_V, \tag{52}$$

Note that equation (50) is multivector valued. This theorem holds for each grade k of the multivectors on both sides of equation (50)

$$\langle\langle f_1(\boldsymbol{x})\ \widetilde{f_2(\boldsymbol{x})}\rangle_V\rangle_k = \frac{1}{(2\pi)^3}\langle\langle \mathcal{F}\{f_1\}(\boldsymbol{\omega})\ \widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V\rangle_k, \quad k = 0, 1, 2, 3. \tag{53}$$

For $k = 0$ and according to equations (51) and (7), the (scalar) Parseval theorem becomes

$$\int_{\mathbb{R}^3} \|f(\boldsymbol{x})\|^2\ d^3\boldsymbol{x} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\ d^3\boldsymbol{\omega}. \tag{54}$$

Because of the similarity with equation (22) we call $\int_{\mathbb{R}^3} \|f(\boldsymbol{x})\|^2\ d^3\boldsymbol{x}$ the energy of f. Finally, we summarize the properties of the Clifford Fourier transform (CFT) in table 3.

Table 3: Properties of the Clifford Fourier transform (CFT)

Property	Multivector Function	CFT
Linearity	$\alpha f(\boldsymbol{x})+\beta\, g(\boldsymbol{x})$	$\alpha\mathcal{F}\{f\}(\boldsymbol{\omega})+\, \beta\mathcal{F}\{g\}(\boldsymbol{\omega})$
Delay	$f(\boldsymbol{x}-\boldsymbol{a})$	$e^{-i_3\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega})$
Shift	$e^{i_3\boldsymbol{\omega}_0\cdot\boldsymbol{x}}f(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$
Scaling	$f(a\boldsymbol{x})$	$\frac{1}{a^3}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$
Convolution	$(f\star g)(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\,\mathcal{F}\{g\}(\boldsymbol{\omega})$
Vec. diff.	$\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})$	$i_3\;\boldsymbol{a}\cdot\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega})$
	$\boldsymbol{a}\cdot\boldsymbol{x}\; f(\boldsymbol{x})$	$i_3\;\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\;\mathcal{F}\{f\}(\boldsymbol{\omega})$
	$\boldsymbol{x}f(\boldsymbol{x})$	$i_3\;\nabla_{\boldsymbol{\omega}}\;\mathcal{F}\{f\}(\boldsymbol{\omega})$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$(i_3\;\boldsymbol{\omega})^m\mathcal{F}\{f\}(\boldsymbol{\omega})$
Plancherel T.	$\langle f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}\rangle_V$	$\frac{1}{(2\pi)^3}\langle\mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\rangle_V$
sc. Parseval T.	$\int_{\mathbb{R}^3}\;\|f(\boldsymbol{x})\|^2\,d^3\boldsymbol{x}$	$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\;\|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\,d^3\boldsymbol{\omega}$

8 The uncertainty principle

The uncertainty principle plays an important role in the development and understanding of quantum physics. It is also central for information processing [9]. In quantum physics it states e.g. that particle momentum and position cannot be simultaneously known. In Fourier analysis such conjugate entities correspond to a function and its Fourier transform which cannot both be simultaneously sharply localized. Futhermore much work (e.g. [9, 12]) has been devoted to extending the uncertainty principle to a function and its Fourier transform. From the view point of geometric algebra an uncertainty principle gives us information about how a multivector valued function and its Clifford Fourier transform are related.

Theorem 4 *Let f be a multivector valued function in $\mathcal{G}_3$ which has the Clifford Fourier transform $\mathcal{F}\{f\}(\boldsymbol{\omega})$. Assume $\int_{\mathbb{R}^3}\|f(\boldsymbol{x})\|^2\;d^3\boldsymbol{x}=F<\infty$, then the following inequality holds for arbitrary constant vectors $\boldsymbol{a}$, $\boldsymbol{b}$:*

$$\int_{\mathbb{R}^3}(\boldsymbol{a}\cdot\boldsymbol{x})^2\|f(\boldsymbol{x})\|^2\;d^3\boldsymbol{x}\int_{\mathbb{R}^3}(\boldsymbol{b}\cdot\boldsymbol{\omega})^2\;\|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega}\geq(\boldsymbol{a}\cdot\boldsymbol{b})^2\;\frac{(2\pi)^3}{4}F^2 \quad (55)$$

Proof Applying previous results we have[9]

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \ \|f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x} \ \int_{\mathbb{R}^3} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \ \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \\
&\overset{(34)}{=} \quad \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \ \|f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x} \int_{\mathbb{R}^3} \ \|\mathcal{F}\{\boldsymbol{b}\cdot\nabla f\}(\boldsymbol{\omega})\|^2 \ d^3\boldsymbol{\omega} \\
&\overset{(54)}{=} \quad (2\pi)^3 \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \ \|f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x} \int_{\mathbb{R}^3} \ \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x} \\
&\overset{\text{footnote 9}}{\geq} \quad (2\pi)^3 \left(\int_{\mathbb{R}^3} \ \boldsymbol{a}\cdot\boldsymbol{x} \ \|f(\boldsymbol{x})\| \ \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\| \ d^3\boldsymbol{x} \right)^2 \\
&\overset{(13)}{\geq} \quad (2\pi)^3 \left(\int_{\mathbb{R}^3} \ \boldsymbol{a}\cdot\boldsymbol{x} |\langle \widetilde{f(\boldsymbol{x})} \ \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle| \ d^3\boldsymbol{x} \right)^2 \\
&\geq \quad (2\pi)^3 \left(\int_{\mathbb{R}^3} \ \boldsymbol{a}\cdot\boldsymbol{x} \langle \widetilde{f(\boldsymbol{x})} \ \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle \ d^3\boldsymbol{x} \right)^2 .
\end{aligned}$$

Because of

$$(\boldsymbol{b}\cdot\nabla)\|f\|^2 = 2\langle \widetilde{f} \ (\boldsymbol{b}\cdot\nabla) f\rangle, \tag{56}$$

we furthermore obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \ \|f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x} \ \int_{\mathbb{R}^3} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \ \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \\
&\geq \quad (2\pi)^3 \left(\int_{\mathbb{R}^3} \ \boldsymbol{a}\cdot\boldsymbol{x} \ \frac{1}{2} \ (\boldsymbol{b}\cdot\nabla\|f\|^2) \ d^3\boldsymbol{x} \right)^2 \\
&\overset{\text{Prop. 5}}{=} \quad \frac{(2\pi)^3}{4} \Bigg(\left[\int_{\mathbb{R}^2} \boldsymbol{a}\cdot\boldsymbol{x} \|f(\boldsymbol{x})\|^2 d^2\boldsymbol{x} \right]_{b\cdot x=-\infty}^{b\cdot x=\infty} \\
&\qquad - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)(\boldsymbol{a}\cdot\boldsymbol{x})] \ \|f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x} \Bigg)^2 \\
&= \quad \frac{(2\pi)^3}{4} \left(0 - \boldsymbol{a}\cdot\boldsymbol{b} \int_{\mathbb{R}^3} \|f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x}) \right)^2 \\
&= \quad (\boldsymbol{a}\cdot\boldsymbol{b})^2 \ \frac{(2\pi)^3}{4} \ F^2.
\end{aligned}$$

Choosing $\boldsymbol{b} = \pm\boldsymbol{a}$, with $\boldsymbol{a}^2 = 1$ we get the following **uncertainty principle**, i.e.

$$\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \ \|f(\boldsymbol{x})\|^2 \ d^3\boldsymbol{x} \ \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{\omega})^2 \ \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \geq \frac{(2\pi)^3}{4} \ F^2. \tag{57}$$

[9] $\phi, \psi : \mathbb{R}^n \to \mathbb{C}, \quad \int_{\mathbb{R}^n} |\phi(x)|^2 d^n x \int_{\mathbb{R}^n} |\psi(x)|^2 d^n x \geq (\int_{\mathbb{R}^n} \phi(x)\psi\bar{(}x)\, d^n x)^2$

In (57) equality holds for Gaussian multivector valued functions (See appendix B)

$$f(\boldsymbol{x}) = C_0\, e^{-k\, \boldsymbol{x}^2} \tag{58}$$

where $C_0 \in \mathcal{G}_3$ is a constant multivector, $0 < k \in \mathbb{R}$.

Theorem 5 *For* $\boldsymbol{a} \cdot \boldsymbol{b} = 0$*, we get*

$$\int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \;\geq 0. \tag{59}$$

Proof The right side of equation (55) is 0 for $(\boldsymbol{a}\cdot\boldsymbol{b}) = 0$.
Note that with

$$\boldsymbol{x}^2 = \sum_{k=1}^{3} x_k^2 = \sum_{k=1}^{3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2, \qquad \boldsymbol{\omega}^2 = \sum_{l=1}^{3} \omega_l^2 = \sum_{l=1}^{3} (\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \tag{60}$$

we can extend the formula of the uncertainty principle to

Theorem 6 *Under the same assumptions as in theorem 4, we obtain*

$$\int_{\mathbb{R}^3} \boldsymbol{x^2}\, \|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \boldsymbol{\omega^2}\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \geq 3\, \frac{(2\pi)^3}{4} F^2. \tag{61}$$

Proof Direct calculation gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \boldsymbol{x^2}\, \|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \boldsymbol{\omega^2}\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega} \\
&\overset{(60)}{=} \sum_{k,\,l=1}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_l\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\, d^3\boldsymbol{\omega} \\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_k\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\, d^3\boldsymbol{\omega} \\
&\quad + \underbrace{\sum_{k\neq l}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_l\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\, d^3\boldsymbol{\omega}}_{\geq 0} \\
&\overset{\text{Theor. 5}}{\geq} \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\boldsymbol{e}_k\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} (\boldsymbol{e}_k\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2\, d^3\boldsymbol{\omega} \\
&\overset{\text{Theor. 4}}{=} 3\, \frac{(2\pi)^3}{4}\, F^2.
\end{aligned}$$

In the last step we used theorem 4 with $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{e}_k$, $k = 1, 2, 3$.

9 Conclusions

We showed how the (real) Clifford Fourier transform extends the traditional Fourier transform on scalar functions to multivector functions. Basic properties and rules for differentiation, convolution, the Plancherel and Parseval theorems were demonstrated. We then presented an uncertainty principle in the geometric algebra $\mathcal{G}_3$ which describes how a multivector-valued function and its Clifford Fourier transform relate. The formula of the uncertainty principle in $\mathcal{G}_3$ can be extended to $\mathcal{G}_n$ using properties of the Clifford Fourier transform for geometric algebras with unit pseudoscalars squaring to -1.

It is known that the Fourier transform is successfully applied to solving physical equations such as the heat equation, wave equations, etc. Therefore in the future, we can apply geometric algebra and the Clifford Fourier transform to solve such problems involving scalar, vector, bivector and pseudoscalar fields.

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A Multivector Cauchy-Schwartz inequality

We will show that

$$|\langle M\tilde{N}\rangle| \leq \|M\|\,\|N\| \tag{A1}$$

Proof Note that for any $t \in \mathbb{R}$ holds

$$\begin{aligned} 0 &\leq \|M+tN\|^2 = (M+tN)*(\widetilde{M+tN}) \\ &= M*\tilde{M} + t(M*\tilde{N} + N*\tilde{M}) + t^2 N*\tilde{N} \\ &= \|M\|^2 + 2t\langle M\tilde{N}\rangle + t^2\|N\|^2. \end{aligned} \tag{A2}$$

The negative discriminant of this quadratic polynomial implies

$$\langle M\tilde{N}\rangle^2 - \|M\|^2\|N\|^2 \leq 0. \tag{A3}$$

This proves (A1) and (13):

$$\langle M\widetilde{N}\rangle = M*\widetilde{N} \leq |\langle M\widetilde{N}\rangle| \leq \|M\|\,\|N\|. \tag{A4}$$

Inserting into (8) and (11) into the multivector Cauchy-Schwartz inequality (A4) we can express it in a basis (1) of the geometric algebra as

$$|\sum_A \alpha_A\beta_A| \leq \left(\sum_A \alpha_A^2\right)^{\frac{1}{2}} \left(\sum_B \beta_B^2\right)^{\frac{1}{2}}. \tag{A5}$$

B Uncertainty equality for Gaussian multivector functions

Note that according to line 3 of the proof for theorem 4 the uncertainty principle (57) can be rewritten as

$$(2\pi)^3 \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2\|f(\boldsymbol{x})\|^2\, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \|\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})\|^2 d^3\boldsymbol{x} \geq \frac{(2\pi)^3}{4}F^2. \tag{B1}$$

Now we have for Gaussian multivector functions (58)

$$\begin{aligned} \boldsymbol{a}\cdot\nabla f &= \boldsymbol{a}\cdot\nabla\, C_0\, e^{-k\boldsymbol{x}^2} \\ &= -2k\,\boldsymbol{a}\cdot\boldsymbol{x}\, C_0\, e^{-k\,\boldsymbol{x}^2} \\ &= -2k\,\boldsymbol{a}\cdot\boldsymbol{x}\, f. \end{aligned} \tag{B2}$$

so, we get

$$\boldsymbol{a}\cdot\boldsymbol{x}\, f = \frac{-1}{2k}\,\boldsymbol{a}\cdot\nabla f, \tag{B3}$$

and

$$\|\boldsymbol{a}\cdot\nabla f\|^2 = 4k^2\|(\boldsymbol{a}\cdot\boldsymbol{x})f\|^2 = 4k^2\,(\boldsymbol{a}\cdot\boldsymbol{x})^2\|f\|^2 \tag{B4}$$

Substituting (B4) and (B3) in the left side of (B1) we get for $\boldsymbol{a}^2 = 1$

$$
\begin{aligned}
&(2\pi)^3 \int_{\mathbb{R}^3} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x} \int_{\mathbb{R}^3} \|\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})\|^2 d^3\boldsymbol{x} \\
&\overset{(B4)}{=} 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x}\ \boldsymbol{a}\cdot\boldsymbol{x} \|f\|^2 d^3\boldsymbol{x} \right)^2 \\
&= 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x}\boldsymbol{a}\cdot\boldsymbol{x} \langle f\tilde{f}\rangle d^3\boldsymbol{x} \right)^2 \\
&= 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x} \langle \boldsymbol{a}\cdot\boldsymbol{x} f\tilde{f}\rangle d^3\boldsymbol{x} \right)^2 \\
&\overset{(B3)}{=} 4k^2(2\pi)^3 \left(\int_{\mathbb{R}^3} \frac{\boldsymbol{a}\cdot\boldsymbol{x}}{-2k} \langle (\boldsymbol{a}\cdot\nabla f)\tilde{f}\rangle d^3\boldsymbol{x} \right)^2 \\
&\overset{(56)}{=} (2\pi)^3 \left(\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\boldsymbol{x} \frac{1}{2} \boldsymbol{a}\cdot\boldsymbol{\nabla} \|f\|^2 d^3\boldsymbol{x} \right)^2 \\
&\overset{\text{P. 5}}{=} \frac{(2\pi)^3}{4} \left(\int_{\mathbb{R}^3} \underbrace{(\boldsymbol{a}\cdot\boldsymbol{\nabla}\ \boldsymbol{a}\cdot\boldsymbol{x})}_{=\boldsymbol{a}^2=1} \|f\|^2 d^3\boldsymbol{x} \right)^2 \\
&= \frac{(2\pi)^3}{4} F^2.
\end{aligned}
$$

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