Version of Proof of Morley's Trisector Theorem

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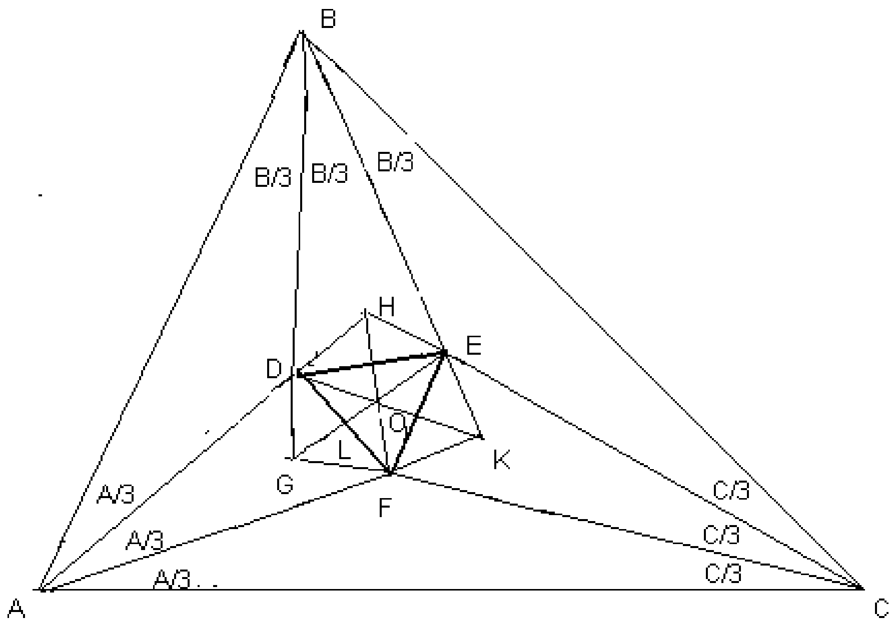
Abstract

In the following proof the properties of the figures created by trisection are utilized in the way different from other known proofs.



According to Morley's Theorem in any triangle ABC the three points of intersection of the trisectors of adjacent angles form after connection an equilateral triangle DEF.

The trisection creates triangles ABK, ACH, and BCG.
Each of them has point of intersection of two bisectors respectively D, F, and E. It means that KD, HF, and GE are bisectors too.
KD divides $\angle AKB = \pi - \frac{2}{3}(A + B)$;

HF divides $\angle AHC = \pi - \frac{2}{3}(A + C)$;

GE divides $\angle BGC = \pi - \frac{2}{3}(B + C)$.

There also are triangles
ΔABD with angles A/3, B/3, and $\angle ADB = \pi - \frac{1}{3}(A + B)$. Then $\angle ADG = \frac{1}{3}(A + B)$.

ΔACF with angles A/3, C/3, and $\angle AFC = \pi - \frac{1}{3}(A + C)$. Then $\angle AFG = \frac{1}{3}(A + C)$.

ΔBCE with angles B/3, C/3, and $\angle BEC = \pi - \frac{1}{3}(A + B)$. Then $\angle CEK = \frac{1}{3}(A + B)$.

In the ΔAFH: $\angle AHF = \frac{1}{2}\angle AHC = \frac{1}{2}[\pi - \frac{2}{3}(A + C)]$;

Then $\angle AFH = \pi - \frac{1}{3}A - \frac{1}{2}\left[\pi - \frac{2}{3}(A + C)\right] = \frac{1}{2}\pi + \frac{1}{3}C$;

Then $\angle GFH = \angle AFH - \angle AFG = \frac{1}{2}\pi + \frac{1}{3}C - \frac{1}{3}(A + C) = \frac{1}{2}\pi - \frac{1}{3}A$;

In the ΔADK: $\angle AKD = \frac{1}{2}\angle AKB = \frac{1}{2}[\pi - \frac{2}{3}(A + B)]$;

Then$\angle ADK = \pi - \frac{1}{3}A - \frac{1}{2}\left[\pi - \frac{2}{3}(A + B)\right] = \frac{1}{2}\pi + \frac{1}{3}B$;

Then $\angle GDK = \angle ADK - \angle ADG = \frac{1}{2}\pi + \frac{1}{3}B - \frac{1}{3}(A + B) = \frac{1}{2}\pi - \frac{1}{3}A$;

Hence $\angle GFH = \angle GDK$.

With two equal angles and common side ΔGDO = ΔGFO and GD = GF.
In the isosceles ΔDGF segment GL of the line GE becomes a bisector, a median, and a height.
Then right-angled ΔEDL and ΔEFL are equal and sides ED = EF.

Similar analyses of ΔCEG and ΔCFH will prove that ED = DF i.e. equality of all three sides of the ΔDEF.