How Dirac and Majorana equations are related

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ABSTRACT

Majorana and Dirac equations are usually considered as two different and mutually exclusive equations. In this paper we demonstrate that both of them can be considered as a special cases of the more general equation.

Keywords: Dirac equation, Majorana equation, Majorana neutrino

Dirac and Majorana equations: Definitions

Majorana and Dirac equations are usually considered as two different and mutually exclusive equations. However, both of them can be considered as a special cases of the more general equation.

Let's start with Dirac equation written in terms of the "left" (ξ) and "right" ($\dot{\eta}$) spinor components:

$$\begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} \begin{bmatrix} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix} = -im \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$$

$$\begin{bmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = -im \begin{bmatrix} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix} \tag{1}$$

The Majorana equation has the same form as Dirac equation, but with additional Lorentz invariant condition (known as *Majorana condition*, or *Neutrality condition*):

$$\eta_{\dot{1}} = +\,\overline{\xi^2} \qquad \xi^1 = -\,\overline{\eta}_{\dot{2}}$$

$$\eta_{\dot{2}} = -\,\overline{\xi^1} \qquad \xi^2 = +\,\overline{\eta}_{\dot{1}} \tag{2}$$

If we will put (2) into Dirac equation (1), we will obtain:

$$\begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} \begin{bmatrix} +\,\overline{\xi^2} \\ -\,\overline{\xi^1} \end{bmatrix} = -im \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$$

$$\begin{bmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = -im \begin{bmatrix} +\,\overline{\xi^2} \\ -\,\overline{\xi^1} \end{bmatrix} \tag{3}$$

Hence, Majorana condition makes both pairs of Dirac equation equivalent, leaving only one independent pair.

Dirac and Majorana equations: Generalization

Let us now introduce the more general equation by replacing the mass terms in Dirac equation with the "mass matrix" M

$$M = \begin{bmatrix} M^1_1 & M^1_2 \\ \\ M^2_1 & M^2_2 \end{bmatrix} \tag{4}$$

and it's complex conjugated matrix $\dot{M}$

$$\dot{M} = \begin{bmatrix} \dot{M}^{\dot{1}}_{\dot{1}} & \dot{M}^{\dot{1}}_{\dot{2}} \\ \\ \dot{M}^{\dot{2}}_{\dot{1}} & \dot{M}^{\dot{2}}_{\dot{2}} \end{bmatrix} \tag{5}$$

The modified equation will have the form:

$$\begin{gathered}
\begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} \begin{bmatrix} \eta_{\dot{1}} \\ \\ \eta_{\dot{2}} \end{bmatrix} = \begin{bmatrix} M^1_1 & M^1_2 \\ \\ M^2_1 & M^2_2 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \\ \xi^2 \end{bmatrix} \\
\begin{bmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \\ \xi^2 \end{bmatrix} = \begin{bmatrix} \dot{M}^{\dot{1}}_{\dot{1}} & \dot{M}^{\dot{1}}_{\dot{2}} \\ \\ \dot{M}^{\dot{2}}_{\dot{1}} & \dot{M}^{\dot{2}}_{\dot{2}} \end{bmatrix} \begin{bmatrix} \eta_{\dot{1}} \\ \\ \eta_{\dot{2}} \end{bmatrix}
\end{gathered} \tag{6}$$

If we require that "left" spinor ξ is an eigenvector of matrix M, and "right" spinor $\dot{\eta}$ is an eigenvector of matrix $\dot{M}$, both corresponding to the same eigenvalue $(-im)$

$$\begin{gathered} M\xi = -im\ \xi \\ \\ \dot{M}\dot{\eta} = -im\ \dot{\eta} \end{gathered} \tag{7}$$

we again reproduce the structure of Dirac equation (1).

Now the "type" of equation (i.e. Dirac, Majorana or Weyl) will only depend on the special choice of matrix M.

For instance, if we choose M as

$$\begin{gathered} M = \begin{bmatrix} 0 & m \\ -m & 0 \end{bmatrix} \\ \\ \dot{M} = \begin{bmatrix} 0 & m \\ -m & 0 \end{bmatrix} \end{gathered} \tag{8}$$

the eigenvectors corresponding to the eigenvalue $(-im)$ will be:

$$\xi_D = \begin{bmatrix} 1 \\ \\ -i \end{bmatrix} \phi(x) \quad \dot{\eta}_D = \begin{bmatrix} 1 \\ \\ -i \end{bmatrix} \phi(x) \tag{9}$$

as it should be in the case of Dirac fermions (see, for instance, Peskin & Schroeder, Chapter 3.3).

Alternatively, we can choose M as

$$\begin{aligned} M &= \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix} \\ \dot{M} &= \begin{bmatrix} -im & 0 \\ 0 & im \end{bmatrix} \end{aligned} \tag{10}$$

and the eigenvectors corresponding to the eigenvalue $(-im)$ will be:

$$\xi_M = \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} \phi(x) \quad \dot{\eta}_M = \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \phi(x) \tag{11}$$

It is easy to check that spinors ξ_M and $\dot{\eta}_M$ *automatically* satisfy Majorana condition (3).

The most general form of the "mass matrix" M in the generalized equation (6) is as follows:

$$M = \begin{bmatrix} M_1^1 & M_2^1 \\ \\ M_1^2 & -M_1^1 \end{bmatrix} = F^k \sigma_k = \begin{bmatrix} F^3 & F^1 - iF^2 \\ \\ F^1 + iF^2 & -F^3 \end{bmatrix}, \qquad k = 1, 2, 3 \tag{12}$$

and it's eigenvalues are

$$\lambda_\pm = \pm\sqrt{(F^1)^2 + (F^2)^2 + (F^3)^2} \tag{13}$$

The matrix M belongs to the Lie algebra of the group $SL(2, C)$.

In order to preserve Lorentz invariance of the equation (6), the components F^k of the mass matrix M are required to transform like vector $E^k - iB^k$, where E^k and B^k are spatial components of the electric and magnetic field strengths. In that case the eigenvalues (13) of matrix M will be invariant w.r.t. Lorentz transformations.

Further generalization of the equation (by allowing M to be not constant, but *variable* matrix) lead to the model that explains the origin of mass and charge in electrodynamics (see [1]).

References

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