An introduction to an interesting naturally occurring mathematical pattern : PIITD series

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***Abstract*—This paper introduces a sequence of real number that models a very frequently occurring natural pattern which is observed by everyone. The pattern it tries to model is the angle subtended on the eye of the observer standing at a distance from a long line of equally spaced objects. There are some interesting properties that are captured by this series. It finds a connection with modelling certain aspects of human psychology and can create interesting study areas like stock market price prediction, resource allocations, efficient search algorithms, time versus effort modelling etc.**

I. Introduction

This paper is divided into seven parts. We start off part II by discussing the intuition and inspiration behind the series. We then discuss a mathematical formula for the series in part III. Part IV starts off by discussing some interesting mathematical properties of the series such as the behaviour of its first and second derivative. Part V discusses possible application and study areas. Part VI will conclude the discussion and you will be open to explore the series, interpret it in your own way. For part VII, we have an APPENDIX section which contains some relevant Mathematica code used to verify the mathematical results.

II. Intuition and Inspiration

When we observe the nature around us, we are surrounded by a structure of identical objects arranged in a very simple linear structure. For example, Street lights on the road are placed equidistant from each other, identical windows on a huge building spaced uniformly, and even if you look at your floor or the ceiling you will find identical tiles lined up. It is not just the physical manifestation of our observations, even our life has this structure, a good example would be time, and each day can be viewed as an equidistant tile in our life. Every day you watch some event/news say sports, stock market etc. which repeats although is not identical every day.

To make further discussion feasible, let us introduce some terminology. Let there be an observer **O**, who observes this linear structure. Let us describe the regularly spaced structure as events that **O** observes. So the regularly spaced tiles, or street lights, or in fact time (as in each day) represents an event **E**. The axis of these events might be distance (in case of tiles or street lights or windows) or time (in case of days) or anything else like the amount of resources utilized by a machine per unit time etc.

Let **y** denote the distance of the observer from the linear structure. In case of non-physical manifestations like observing time, news etc., **y** can be thought of as the sensitivity of observer to the event **E**. For example. If you follow news for a particular stock, say RELIANCE (in which you have an investment) then you are closer to the structure as compared to say TATASTEEL (in which you don't have an investment). Say you are studying for exam on two topics MUSIC and HISTORY, you might be closer to MUSIC than you are to HISTORY, and each day you might want to observe the amount of effort you put in. So, let us say **y** captures the sensitivity of the observer to the event in such a way that smaller value of **y** represents closeness to the event and more sensitivity to the occurring of the event. In the physical world, it just represents how close the observer is standing away from the linear structure of events.

Let **x** denote the distance between events. We want **x** to denote the actual importance of the occurrence of the event. In case of tiles the actual importance is the distance it measures. In case of time, it might denote the effect of occurrence of events for e.g. Price of a stock going up or down due to some news event etc.

Let **n** denote the number of the event observed, where **n** = 0 represents the closest event. For example, in case of tiles the closest tile represents **n** = 0. In case of time, the current time represents **n** = 0 while **n** $>$ 0 can represent either past or the future occurrence of events.

Although the events are equally spaced in distance or time, the projection of the event on the observer is different for all the events. So for the observer, the actual importance of the event **E**, depends on how far out in terms of **n**, the events is happening. So for e.g. the tile that is far out of view looks smaller as compared to the tile closest. Another example would be news for a stock loses its importance as time passes by, as far as its effect on the price of the stock (which is being observed) is concerned. This projection of an event can be captured by the angle subtended by that event on the observer, so a bigger angle represents more projection.

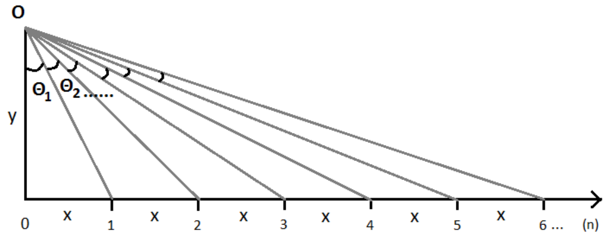

Fig. 1. Basic setup of observer and events

Fig. 2. College building of IIT Delhi, which inspired the series

It is this subtended angle on the observer **O**, denoted by $\boldsymbol{\theta_n}$ between the event **n** and **n-1** that we are trying to study.

Lastly let **k** denote the ration **x/y**. This is a simplification where we can assume **y** to be always equal to one and measure **x** in terms of **y**.

Here are some photographs from the college building of IIT Delhi that present this structure and were the inspiration for the series. You can see how the equally spaced windows on the building look smaller and smaller as we look farther away. They project a smaller angle on the eye of the observer as one looks farther away.

Fig. 3. College building of IIT Delhi another view

Fig. 4. College building of IIT Delhi front view

Fig. 5. Example in everyday life, look at how the distance between street lights looks smaller as we view farther away in distance

III. Mathematical Formula for the Series

The mathematical formula for the series is given by the following equation:-

$$\theta_n = \tan^{-1}(nk) - \tan^{-1}((n-1)k)$$

Here **n** represents how far out in distance the interval subtends an angle θ_n on the observer. **k** here represents the ration **x/y**

This can be simplified to:-

$$\theta_n = \tan^{-1}\left(\frac{nk-(n-1)k}{1+nk^2(n-1)}\right)$$

In order to study some mathematical properties we will derive the first and second order derivative of θ_n with respect to **n**. We will find an interesting structure which will be discussed in the next section. Refer to the Appendix section for the relevant Mathematica codes. We will omit the subscript **n** from θ_n in the following derivations.

$$\frac{d\theta}{dn} = \frac{k}{1+k^2n^2} - \frac{k}{1+(n-1)^2k^2}$$

$$\frac{d^2\theta}{dn^2} = -\frac{2k^3n}{(1+k^2n^2)^2} - \frac{2k^3(n-1)}{(1+k^2(n-1)^2)^2}$$

Now putting

$$\frac{d^2\theta}{dn^2} = 0$$

We get the following quartic equation:-

$$3k^4n^4 - 6k^4n^3 + (4k^4+2k^2)n^2 - (2k^2+k^4)n - 1 = 0$$

Using Mathematica(Tool) we find the four roots of this equation:-

$$n = \frac{3k^2 - \sqrt{3}\sqrt{-4k^2+k^4-2k^2\sqrt{16+4k^2+k^4}}}{6k^2}$$

$$n = \frac{3k^2 + \sqrt{3}\sqrt{-4k^2+k^4-2k^2\sqrt{16+4k^2+k^4}}}{6k^2}$$

$$n = \frac{3k^2 - \sqrt{3}\sqrt{-4k^2+k^4+2k^2\sqrt{16+4k^2+k^4}}}{6k^2}$$

$$n = \frac{3k^2 + \sqrt{3}\sqrt{-4k^2+k^4+2k^2\sqrt{16+4k^2+k^4}}}{6k^2}$$

IV. Some interesting mathematical properties

For different values of **k** we see that the second derivative is zero when **n** takes the following values :-

$$For k = 1, n = \frac{3+\sqrt{3(-3+2\sqrt{21})}}{6}$$

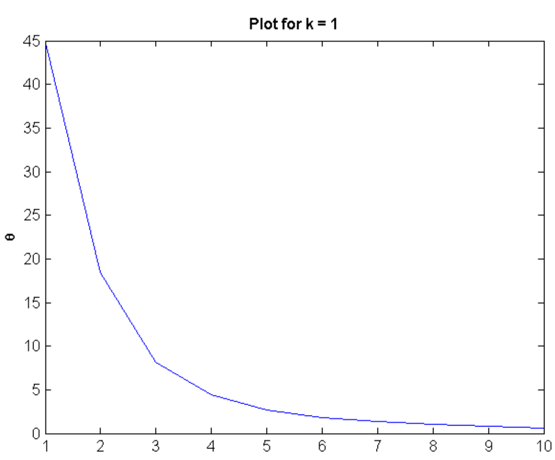

Fig. 6. Plot for k = 1

So the only real valued saddle point is a fraction with value less than one as seen in the graph below.

As we reduce to k = 0.1, i.e. as we move farther away from the linear structure, we get a real saddle point around

$$For \mathbf{k} = 0.1, \mathbf{n} = 6.2879$$

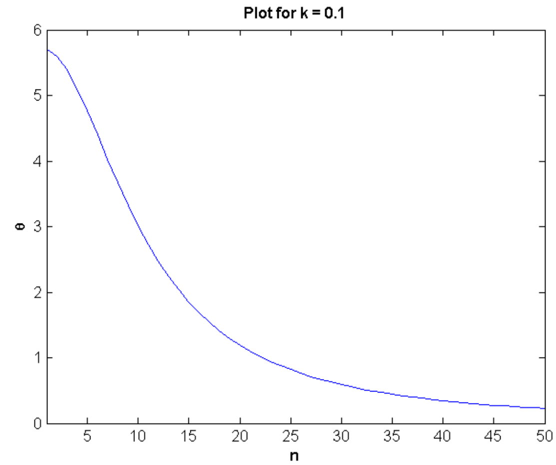

Fig. 7. Plot for k = 0.1

As we further move away, at k = 0.01, we get the saddle point at

$$For \mathbf{k} = 0.01, \mathbf{n} = 58.2365$$

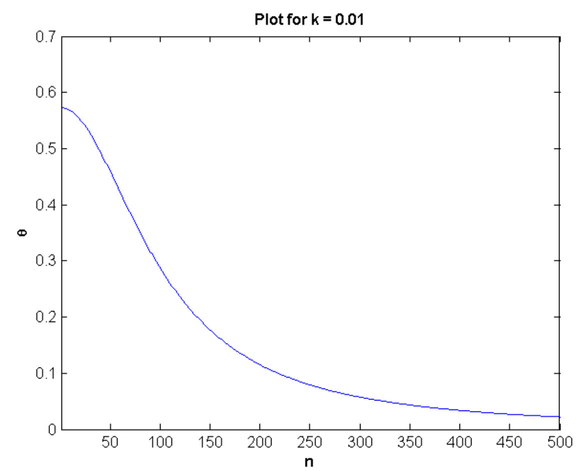

Fig. 8. Plot for k = 0.01

This saddle point reveals a very interesting phenomena about how we observe around surroundings. The angle subtended θ reduces with a pattern as we look farther along the structure. The rate at which the angle reduces is slow at first, after a point i.e. the saddle point the rate of decrease gradually increases. If we are standing too close to the structure we won't be able to see the saddle point, but as we move farther away from this structure, we can actually pin point a saddle point behaviour for some value of **n**.

If the angle subtended represents the perceived distance between consecutive events, we see how it has a pattern i.e.

this perception of distance reduces slowly at first and after the saddle point decreases rapidly.

It is a very interesting pattern and this physical observation might have a connection with our psychology and how our brain (neurons) is trained in general about the surrounding, and how we respond or feel about observing such phenomena. We will discuss some possible areas of study in the next section.

V. Possible Applications and Study Areas

The pattern discussed above might find an interesting connection with human psychology. The abundance of such linear structures in life might be responsible for human brain to be trained to respond in this pattern.

Lets take stock market price for e.g. You can model the stock market as the following, suppose you have invested some cash in a stock ABC. You will be very sensitive to its information and daily news. Relating to the pattern that we discussed the value of **y** that represents the distance from the structure will be low, representing high sensitivity towards the event. As compared to some other stock XYZ in which you don't have an investment, so there, the value of **y** is large, i.e. lower sensitivity to its daily news.

Lets take another example, if you are working on a project or an exam, the amount of effort you might put in daily, until the final exam day might be similar to the pattern that we discussed, essentially because that is the way our brains were trained to respond to such events. You might identify a saddle point and plan accordingly if you can successfully model psychological behaviour using this pattern.

There are numerous other areas that can be explored, one possible starting point can be to compare the behaviour of this pattern with the Fibonacci sequence. For e.g its performance as compared to Fibonacci search techniques, rate of increase/decrease etc.

VI. Conclusion

We introduced a mathematical pattern of real numbers that exists all around us. This pattern is very common around us and still goes totally ignored. It occurs more frequently than the famous Fibonacci sequence. Its mathematical properties might find connection with human psychology and how our brain is trained to respond to events with such structure. For e.g. time versus effort problems, stock market prediction, preparing for an exam, resource queuing etc. It might provide a crucial insight into how our brain perceives events in life.

We call it **PIITD series** after the name of the college IITD(Indian Institute of Technology Delhi) that inspired the discovery of this series. This series might open up new areas of studies and help in modelling some areas in human psychology.

VII. Appendix

The following Mathematica code was used to verify all derivations above. Original equation:-

$$\theta = ArcTan[n * k] - ArcTan[(n-1) * k]$$

First order derivative:-

$$D[\theta, \{n, 1\}]$$

$$\frac{k}{1+k^2n^2} - \frac{k}{1+(n-1)^2k^2}$$

$$D[\theta, \{n, 2\}]$$

$$-\frac{2k^3n}{(1+k^2n^2)^2} - \frac{2k^3(n-1)}{(1+k^2(n-1)^2)^2}$$

$$Solve\left[-\frac{2k^3n}{(1+k^2n^2)^2} - \frac{2k^3(n-1)}{(1+k^2(n-1)^2)^2} == 0, n\right]$$

$$n = \frac{3k^2 - \sqrt{3}\sqrt{-4k^2 + k^4 - 2k^2\sqrt{16+4k^2+k^4}}}{6k^2}$$

$$n = \frac{3k^2 + \sqrt{3}\sqrt{-4k^2 + k^4 - 2k^2\sqrt{16+4k^2+k^4}}}{6k^2}$$

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