Combined Gravitational Action (IV): *Exploitation*[1]

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Abstract: In previous papers relating to the concept of Combined Gravitational Action (CGA) we have established the CGA-theoretical foundations as an alternative gravity theory that already allowed us to resolve -in its context- some unexpected and defiant problems occurred inside and outside the Solar System like, *e.g*., the anomalous Pioneer 10's deceleration; the observed secular increase of the Astronomical Unit; the apsidal motion anomaly of the eclipsing binary star systems and the study of CGA-effects in the non-compact and compact stellar objects. All that has been done without exploiting fully the CGA-formalism, hence, the main purpose of the present paper is to exploit profoundly the CGA-equations in order to investigate, among other things, CGA-spin-orbit coupling precession and application of CGA to Large-Scale Structures with the aim of resolving the problem of galactic rotation curves.

Keywords: combined gravitational action; combined gravitational potential energy; Newton's law of gravitation; solar system; eclipsing binary stars; binary pulsars; dark matter

1. Introduction

Basing simply on the usual Euclidean geometry and the Galilean relativity principle, we were able to establish a coherent alternative gravity theory exclusively founded on the concept of Combined Gravitational Action. We have previously [1,2,3] shown that the theory (CGA) is very capable of predicting and explaining the anomalous Pioneer 10's deceleration; the secular perihelion precession of the inner planets; the angular deflection of light passing near the massive object and the observed secular increase of the Astronomical Unit [4]. These two last phenomena are known as the decisive tests support the general relativity theory (GRT). Here, our main motivation is the following: since in the previous papers [1,2,3] we did not exploit fully the CGA-formalism, hence, now it is time to do this in order to investigate, among other things, CGA-spin-orbit coupling precession and application of CGA to Large-Scale Structures with the aim of resolving the problem of galactic rotation curves.

Before the advent of the CGA as an alternative gravity theory, it was always stressed that the study of the compact stellar objects is exclusively belonging to GRT-domain because their strong compactness is enough to bend the local space-time in such a way that some observable GRT-effects should occur. However, as we shall see, the CGA is also able to investigate, predict and explain the same type of the compact stellar objects and all that in the context of the usual Euclidean geometry and the Galilean relativity principle. This reflects a tangible fact that the propagation of gravitational field and the action of gravitational force both are independent of the *topology* of space-time.

But why shall the CGA arrive at the same results as GRT or even better in some cases? Because if we take the concept of the curvature of space-time apart, we find that contrary to the Newton's

[1] This paper is dedicated to the memory of Prof. Thomas C.Van Flandern, 26 June 1940 – 9 January 2009

gravity theory, the CGA and GRT take in full consideration the relative motion of the test-body and the light speed in local vacuum which in CGA is playing the role of a specific kinematical parameter of normalization and in GRT it is considered as the speed of gravity propagation. The main consequence of the CGA-formalism [1,2,3] is the dynamic gravitational field (DGF), $\mathbf{\Lambda}$, which is in reality an induced field, it is more precisely a sort of gravitational induction due to the relative motion of material body in the vicinity of the principal gravitational source. Furthermore, in the present work, we will show in sections 5 and 6 that the existence of the dynamic gravitational acceleration at galactic scale should be attributed to the gravitodynamical evidence of the dark matter (DM) itself and the characteristic acceleration introduced by Milgrom as a universal constant in his theory of Modified Newtonian Dynamics (MOND), is in fact a special case of Eq.(32) *see* [3]. Consequently, MOND as an alternative theory to the DM 'hypothesis' *becomes* by means of the CGA an additional support for DM!

2. Combgravactional Kepler's third law and its Consequences

In this section, we will *combgravactionalize* the Kepler's third law, that is to say, we generalize it in the CGA-context for the binary system –system of two massive bodies linked gravitationally–. From this generalization, we will show the existence of an extra-term, $\Delta\omega$, added to the usual orbital angular velocity, $\omega = 2\pi\, p^{-1}$,of the orbiting spherical massive bodies around their common center of mass. Consequently these considerations allowing us to formulate the following physical quantities:(i) the generalized Kepler's third law; (ii) the CGA-orbital angular velocity Ω; (iii) the CGA-orbital angular momentum ℓ; (iv) the CGA-angular rate Ψ_{CGA} of the spin-orbit coupling precession.

2.1. The generalized Kepler's third law

Let us consider two spherical massive bodies A and B with masses m_A and m_B moving in orbits of radii a_A and a_B around a common center of mass C defined by

$$m_A\, a_A = m_B\, a_B \ . \tag{1}$$

Therefore, the ratio of the two masses is

$$\frac{m_A}{m_B} = \frac{a_B}{a_A}\,. \tag{2}$$

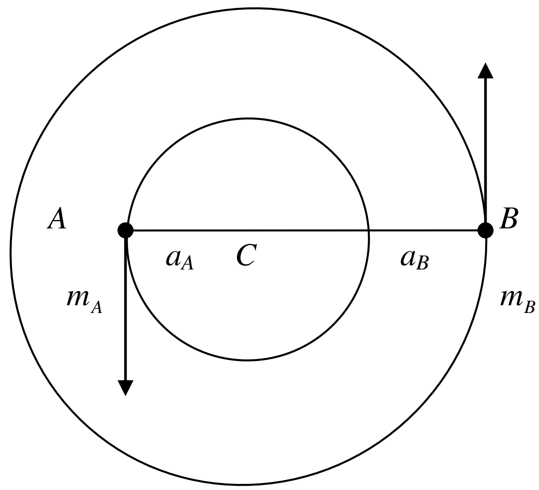

Figure 1. Two spherical massive bodies A and B moving in almost circular orbits of radii a_A and a_B about their common center of mass C.

Further, according to Eq.(27), see [3], the combined gravitational attraction (resultant force) between the two orbiting spherical massive bodies A and B is, for the present case, of the form $\mathbf{F} = -k\,r^{-3}\left(1 + Gm/c_0^2 r\right)\mathbf{r}$, where here $k = G\,m_A m_B$ and $m = m_A + m_B$. We have also the well-known usual expression of the Kepler's third law

$$\frac{P^2}{r^3} = \frac{4\pi^2}{Gm}, \tag{3}$$

or equivalently

$$\omega^2 = \frac{Gm}{r^3}. \tag{4}$$

Let us, for simplicity, consider temporary the massive body A playing the role of the main gravitational source and B as a test-body moving around A at the orbital velocity v, under such a condition (Fig.1) and by taking into account the expression of the combined gravitational attraction $\mathbf{F}$, the two massive bodies as a binary system $\{A, B\}$ must have the same orbital angular velocity vector $\mathbf{\Omega}$ defined by

$$\mathbf{\Omega} = \boldsymbol{\omega} + \Delta\boldsymbol{\omega}, \tag{5}$$

around their common center of mass C. Hence, using Eq.(27) that is the expression of $\mathbf{F}$ and the equating it to the centrifugal force, and after performing some algebraic calculations and neglecting the infinitesimal quantity, $2\|\boldsymbol{\omega}\|\cdot\|\Delta\boldsymbol{\omega}\|\cos\lambda$, resulting from the scalar product of $\boldsymbol{\omega}$ and $\Delta\boldsymbol{\omega}$, we get

$$\frac{Gm_A m_B}{r^2}\left[1 + \frac{Gm}{c_0^2 r}\right] = m_A\left(\boldsymbol{\omega}^2 + \Delta\boldsymbol{\omega}^2\right)a_A = m_B\left(\boldsymbol{\omega}^2 + \Delta\boldsymbol{\omega}^2\right)a_B\,. \tag{6}$$

Using the relation (2), we obtain $r = a_A + a_B = a_A\left(1 + m_A m_B^{-1}\right)$, thus (6) becomes after substitution and simplifications

$$\frac{Gm}{r^3}\left[1+\frac{Gm}{c_0^2 r}\right] = \left(\boldsymbol{\omega}^2 + \Delta\boldsymbol{\omega}^2\right), \tag{7}$$

Eq.(7) is exactly the expected generalized Kepler's third law applied to any two –spherical massive bodies in orbital motion. Also Eq.(7) is true for elliptical orbit in which case r becomes the semi-major axis of the orbit of one massive body relative to the other, which is at the focus of the ellipse. Now, let us determine the expression of the extra-term, $\Delta\boldsymbol{\omega}$, by substituting (4) in (7), and after performing some algebraic calculations, we obtain

$$\Delta\omega = \frac{Gm}{c_0\, r^2} = \frac{Gm_A(1+q)}{c_0\, r^2}. \tag{8}$$

Where $q = m_B / m_A$ is the mass ratio of the binary system $\{A, B\}$.

2.2. CGA-orbital angular velocity

It follows from all above results that, in the context of CGA, the existence of the extra-term (8) implies, among other things, that the expression of CGA-orbital angular velocity, Ω, should be different from the usual expression defined in the framework of Newton's gravity theory, thus according to (7), we get

$$\Omega = \left(\boldsymbol{\omega}^2 + \Delta\boldsymbol{\omega}^2\right)^{1/2}. \tag{9}$$

Or more explicitly

$$\Omega \equiv \Omega(r) = \left(\frac{Gm}{r^3}\right)^{1/2}\left[1+\frac{Gm}{c_0^2 r}\right]^{1/2} = \frac{2\pi}{P}\left[1+\frac{Gm}{c_0^2 r}\right]^{1/2}, \tag{10}$$

which may be reduced to (4) for the case $\left(Gm/c_0^2 r\right) << 1$. As we can remark it, there is no any singularity in the expression (10), that is to say contrary to GRT in which the expression of orbital angular velocity containing a singularity. Moreover, it follows from (10) that the CGA-orbital velocity, $v \equiv v(r) = \Omega\, r$, takes the explicit form

$$v \equiv v(r) = \left(\frac{Gm}{r}\right)^{1/2}\left[1+\frac{Gm}{c_0^2 r}\right]^{1/2}. \tag{11}$$

Once again, when the term $\left(Gm/c_0^2 r\right)$ is very significantly less than unity, the CGA-orbital velocity (11) reduces to the usual one, that is why we have already used in [3] the usual expression $\left(Gm/r\right)^{1/2}$ when we have, *e.g.*, derived Eq.(27) because in general such an approach does not affect the results that may be found from the use of the new expression (11) since, here, we are exclusively dealing with the massive bodies in orbital motion. For instance, the system Earth-Moon is characterized by the value of $\left(Gm/c_0^2 r\right) = 1.168\times 10^{-11}$.

2.3. CGA-orbital angular momentum

As in Newton's gravity theory, the CGA-orbital angular momentum, ℓ, of the binary system (A, B) and the CGA-orbital angular velocity, Ω, are connected by

$$\ell \equiv \ell(r) = \mu \Omega r^2, \tag{12}$$

where $\mu = m_A m_B / m$ is the reduced mass of the binary system $\{A, B\}$. The expression (12) allows us to affirm in the context of CGA that the equality between the mutual combined gravitational attraction, $\mathbf{F}$, and the combined centrifugal force

$$\mathbf{F}_{\mathrm{CF}} = \frac{\ell^2}{\mu}\left(\frac{\mathbf{r}}{r^4}\right), \tag{13}$$

together ensures the orbital gravitational stability of the binary system $\{A, B\}$. Hence, such stability occurs according to the equation

$$\mathbf{F} - \mathbf{F}_{\mathrm{CF}} = 0. \tag{14}$$

Let us slightly focus our attention on the expression (13) which is called 'combined' centrifugal force because it is, in reality, a combination of two forces *viz.* the static centrifugal force $\mathbf{F}_{\mathrm{SCF}}$ and the dynamic centrifugal force $\mathbf{F}_{\mathrm{DCF}}$. So to be really sure of this combination, it is best to rewrite (13) in the following explicit form

$$\mathbf{F}_{\mathrm{CF}} = \mu\left(\frac{Gm}{r^2}\right)\frac{\mathbf{r}}{r} + \mu\left(\frac{Gm}{c_0 r}\right)^2 \frac{\mathbf{r}}{r^2}. \tag{15}$$

Therefore, the two components of $\mathbf{F}_{\mathrm{CF}}$ are of the form

$$\mathbf{F}_{\mathrm{SCF}} = \mu\left(\frac{Gm}{r^2}\right)\frac{\mathbf{r}}{r}, \quad \mathbf{F}_{\mathrm{DCF}} = \mu\left(\frac{Gm}{c_0 r}\right)^2 \frac{\mathbf{r}}{r^2}. \tag{16}$$

Thus that is why $\mathbf{F}_{\mathrm{CF}}$ is called 'combined' centrifugal force. As we know it, the extra-component force $\mathbf{F}_{\mathrm{DCF}}$ is induced by the motion of the binary system $\{A, B\}$ relative to the center of mass. Hence the existence of $\mathbf{F}_{\mathrm{DCF}}$ itself as an extra-component force implies that the mentioned orbital gravitational stability should not consider as an *absolute* fact since $\mathbf{F}_{\mathrm{DCF}}$ causes, among other effects, some very small secular gravitational perturbations which are, on the average, reflected in the precession of the elliptical orbit as we will see below.

2.4. CGA-spin-orbit coupling precession

In this subsection, we will study another post-Keplerian parameter, namely, the CGA-spin-orbit coupling precession rate, $\Psi_{\mathrm{CGA}}(\mathrm{rad/s})$, under the effect of CGA-spin-orbit coupling, which is originally caused by the couple $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$ *see* [3], and generally occurs in one component of system $\{A, B\}$ when the spin angular momentum vector $\mathbf{S}$ of that component is misaligned with the orbital angular momentum vector $\mathbf{J}$. The coupling of spin and orbital angular momenta causes the spin vector $\mathbf{S}$ to precess around $\mathbf{J}$ with the angular precession rate, Ψ_{CGA}, proportional to $\Delta\omega$ or equivalently

$$\Psi_{\mathrm{CGA}}(\mathrm{rad/s}) = K\,\Delta\omega\,. \tag{17}$$

Where K is a coefficient of proportionality to be determined later. In the framework of GRT, the spin-orbit coupling precession is called 'geodetic precession' or 'De Sitter precession', and is physically attributed to the curvature of space-time. However, for the CGA, this phenomenon is a pure consequence of the action of the couple $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$ and that is why is legitimately called '*CGA-spin-orbit coupling precession*' or simply '*CGA-orbital precession'* as we will see because as Einstein himself argued in 1912, "The gravitation acts more strongly on a moving body than on the same body in case it is at rest." But Einstein's claim has been stated in 1912, that is to say, before the publication of the final version of GRT in 1915 in which, as we know, the very realistic concept of the gravitational force is abandoned and replaced by the concept of the curvature of space-time, and at the same time, Einstein claimed that GRT may be reduced to Newtonian gravity theory for low-velocities and weak-gravitational fields! We now return to the coefficient of proportionality, K, contained in the relation (17), so as we are exclusively dealing with an orbital motion and for the purpose of our investigation, it seems more natural and very convenient to define it as a function of the form: $K \equiv (r, q) = \frac{3}{2}(1 + q/6)(1+q)^{-1} c_0^{-1}\, v(r)$, where $q = m_B / m_A$ is the mass ratio of system $\{A, B\}$ and $v(r) = \Omega\, r$ is of course the CGA-orbital velocity (11), we have

$$K \equiv K(r, q) = \frac{3(1 + q/6)}{2(1+q)} \frac{2\pi\, r}{c_0\, P} \left[1 + \frac{Gm}{c_0^2 r}\right]^{1/2} \cong \frac{(1 + q/6)}{(1+q)} \frac{3\pi\, r}{c_0\, P}\,, \tag{18}$$

because, as we know, for the orbital motion the term $(Gm / c_0^2 r)$ is generally very significantly less than unity. Consequently, by substituting (8) and (18) in (17), we get

$$\Psi_{\mathrm{CGA}}(\mathrm{rad/s}) = \frac{3\pi\, G m_A (1 + q/6)}{c_0^2\, r\, P}\,. \tag{19}$$

For an exact elliptical orbit, we have $r = a(1 - e^2)$, where a is the semi-major axis and e is the orbital eccentricity, hence after substitution, we obtain, the expected expression

$$\Psi_{\mathrm{CGA}} = \frac{3\pi\, G m_A (1 + \frac{1}{6} q)}{c_0^2\, a(1 - e^2)\, P}\,. \tag{20}$$

Eq.(20) is exclusively concerning the massive body B of mass m_B when playing the role of a test-body orbiting the main gravitational source A of mass m_A. However, when $q=1$, *i.e.*, $m_A = m_B$ in such a case the two massive bodies have the same CGA-orbital precession rate, because, as it was known [3], when $q=1$ the two massive bodies should play mutually the role of the main gravitational source. For the case of the test-body B; the causal origin of B's orbital precession rate (20) is of course the couple of the dynamic gravitational field-force $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$ induced by B during its motion in A's gravitational field. Moreover, as it said repetitively in this paper, the dynamically induced $\mathbf{\Lambda}$ and $\mathbf{F}_{\mathrm{D}}$ have an appreciable gravitational influence on the evolution and behavior of the massive bodies.

Curiously, Lorentz has already arrived at some conclusion very comparable to that of Einstein, but more than one decade before him. In his very influential work entitled 'Considerations on Gravitation' published in 1900, Lorentz wrote *"Every theory of gravitation has to deal with the problem of the influence, exerted on this force by the motion of the heavenly bodies."* [5]. Again, Lorentz' claim clearly reinforcing the fact that $\mathbf{\Lambda}$ and $\mathbf{F}_{\mathrm{D}}$ are really induced by the motion of massive test-body B in the gravitational field of the central body A. In [2] we have already calculated the angular deflection of starlight, as a direct consequence of $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$, hence it goes without saying that the gravitational redshift and the gravitational lensing should be also caused by the same $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$. All that constitutes a counterexample to the concept of the curvature of space-time as an interpretation to the *gravitation*. Let us return to Eq.(20) which is also applicable to the eclipsing binary star systems and binary pulsars as we will see later on. But we begin its application to the Earth-Moon system as a whole. First, we shall investigate the secular Moon's orbital precession under the effect of CGA-spin-orbit coupling caused by the couple of the dynamic gravitational field-force $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$ induced by the Moon during its motion in the immediate local Earth's gravitational field (GF), and secondly we investigate the same phenomenon for the Earth-Moon system in the Sun's GF.

2.4.1. Moon's secular CGA-orbital precession in the immediate local Earth's GF

We have the following orbital and physical parameters of the Moon. Orbital eccentricity: $e = 0.0549$; orbital period: $P = 27.32$ days; semi-major axis: $a = 3.844\times10^{8}$ m; Moon's mass: $m_B = 7.3477\times10^{22}$ kg; Earth's mass: $m_A = 5.9722\times10^{24}$ kg; mass ratio: $q = m_B / m_A = 1.230317\times10^{-2}$. After substituting all these parameters in (20), we get the rate of the secular CGA-orbital precession:

$$\Psi_{\mathrm{CGA}} = 4.674804\times10^{-17}\ \text{rad/s} = 30.444760\ \text{mas/cy}\ . \tag{21}$$

Where 'mas/cy' is the abbreviation for 'milliarcsecond per century'. Kinematically, the value (21) means that the Moon's orbit itself rotates around the Earth at the velocity of $56.71\,\text{m/cy}$ and the CGA-orbital precession period, i.e., temporal interval for a complete rotation, 2π, is 4.26×10^{9} yr which is very comparable to the age of our Solar System! Now, let us evaluate the magnitude of the couple $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$ responsible for this secular orbital precession. First, we have according to Eqs.(42,43) in [3], the following expressions

$$\Lambda = \frac{1}{a}\left[\frac{GM}{c_0\, a}\right]^2 \quad \text{and} \quad F_{\mathrm{D}} = \frac{m}{a}\left[\frac{GM}{c_0\, a}\right]^2 .$$

Hence, after a direct substitution and calculation, we obtain

$$\Lambda = 3.112\times10^{-14}\ \mathrm{m/s^2} \ \text{ and } \ F_{\mathrm{D}} = 2.286547\times10^{9}\ \mathrm{N}\,.$$

2.4.2. Earth-Moon system secular CGA-orbital precession in the Sun's GF

Historically, De Sitter studied for the first time the so-called geodetic precession in 1916 as a consequence of GRT. He derived a formula similar to (20) and published it in his seminal article entitled "On Einstein's theory of gravitation and its astronomical consequences. Second paper."[6]. De Sitter applied his formula to the Earth-Moon system and found that the spin-orbit coupling contribution is reflected in the Earth-Moon orbital precession by 1.9194 arcsec/cy. But in the most complete investigation on the subject by Brumberg et al. [7,8]. As a result, their corrected value of the geodetic precession is 1.9199 arcsec/cy, which is quite comparable to our value as we will see below. De Sitter attributed the causal origin to the curvature of space-time.

However, the existence of the same phenomenon in the framework of CGA with the same amount found by [7,8] is considered as a counterexample to the concept of the curvature of space-time *itself* as an interpretation to the gravitation. Because of its distance from the Sun, the Earth-Moon system can be regarded as a single body which is rotating in the Sun's GF. further, since the Earth is physically dominated the system under consideration, thus on average, we take the orbital and physical parameters of the Earth for the all system and $m_B = m_\oplus + m_{\mathrm{M}}$ as a total mass for the system. Eccentricity: $e = 0.0167$; period: $P = 365.25$ days; semi-major axis: $a = \mathrm{AU} \approx 149.597870\times10^{9}$ m; Moon's mass: $m_{\mathrm{M}} = 7.3477\times10^{22}$ kg; Earth's mass: $m_\oplus = 5.9722\times10^{24}$ kg; Sun's mass: $m_A = m_\Theta = 1.9891\times10^{30}$ kg ; mass ratio: $q = m_B / m_A = 3.039403\times10^{-6}$. By substituting all these parameters in (20), we get

$$\Psi_{\mathrm{CGA}} = 2.948039\times10^{-15}\ \mathrm{rad/s} = 1.919917\ \mathrm{arc\,sec/cy}\,. \tag{22}$$

This is in excellent accord with the value found by Brumberg *et al.* [7,8]. Further, according to (22), an entire precession cycle would take 6.75030×10^{7} yr, that is 1.50% of the total age of our Solar System.

2.4.3. CGA-spin-orbit coupling Effect in Eclipsing Binary Star Systems

Finally, we now apply the formula (20) to investigate the CGA-orbital precession under the effect of CGA-spin-orbit coupling caused by the couple $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$ in the well-known eclipsing binary star systems: AS Camelopardalis and DI Herculis and also we study the same phenomenon in the binary pulsars PRS B 1913+16 and PRS B1534+12, and the double pulsar PSR J0737-3039. Since we have already studied the CGA-apsidal motion in AS Camelopardalis and DI Herculis, respectively, thus we can use the same orbital and stellar parameters according to [3], we have AS Cam: $e = 0.1695$; $P = 3.430$ days; $a = 17.20\,R_\Theta$; $m_A = 3.3\,m_\Theta$; $m_B = 2.5\,m_\Theta$ and DI Her: $e = 0.489$; $P = 10.55$ days; $a = 34.12\,R_\Theta$; $m_A = 3.3\,m_\Theta$; $m_B = 2.5\,m_\Theta$. Moreover, as the two systems are characterized by the mass ratio $q < 1$ this implies, among other things, that in each system the primary star A of mass m_A should play the role of main gravitational source and the secondary star B of mass m_B should be the orbiting test-body. Therefore, in the two systems {AS Cam, DI Her} the effect of CGA-spin-orbit coupling should concern exclusively the secondary star, *i.e.*, we are dealing with *B*'s CGA-orbital precession caused by the couple $(\mathbf{\Lambda},\mathbf{F}_{\mathrm{D}})$. Hence, after substituting all the necessary parameters in Eq.(20), we obtain the following CGA-orbital precession rates for AS Cam and DI Her, respectively:

$$\Psi_{\mathrm{CGA}}^{(1)} = 0.714363 \text{ deg/cy} \ , \tag{23}$$

and

$$\Psi_{\mathrm{CGA}}^{(2)} = 2.71681 \text{ deg/cy} \ . \tag{24}$$

2.4.4. CGA-spin-orbit coupling Effect in Binary Pulsars and Double Pulsars

Let us return to the compact stellar objects, dealt with in [3] and let us investigate the CGA-orbital precession under the effect of GCA-spin-orbit coupling. It is expected that the CGA-rate, Ψ_{CGA}, for the binary pulsars PSR B1913+16 and the double pulsar PRS J0737-3039 should be remarkably greater than that of the eclipsing binary star systems. This fact is mainly due to the *strength* of the dynamic gravitational force, $\mathbf{F}_{\mathrm{D}}$, in these compact stellar objects. For example in [3], we have already found for AS Cam and DI Her, respectively, the values of $F_{\mathrm{D}} = 6.1476\times10^{24}$ N and $F_{\mathrm{D}} = 1.730\times10^{24}$ N as well as for PSR B1913+16 and PRS J0737-3039, respectively, the values of $F_{\mathrm{D}} = 5.831340\times10^{26}$ N and $F_{\mathrm{D}} = 4.677426\times10^{27}$ N. Hence a simple comparison yields:

$$\frac{F_{\mathrm{D}}(\text{in compact binary pulsar sytem})}{F_{\mathrm{D}}(\text{in ordinary binary star system})} = 761$$

In the context of GRT, Weisberg and Taylor [9] found a theoretical geodetic precession rate of 1.213 deg/yr for PRS B1913+16 and Manchester *et al.* [10] predicted geodetic precessional periods of 75 yr and 71 yr for PRS J0737-3039A and PRS J0737-3039B, respectively, which correspond to the following geodetic precession rates: $\Psi_{\mathrm{GR}}^{(\mathrm{A})} = 4.8$ deg/cy and $\Psi_{\mathrm{GR}}^{(\mathrm{B})} = 5.070$ deg/cy.

Since we have already studied the CGA-apsidal motion in the above mentioned pulsars, thus we can use the same orbital and stellar parameters according to [3], we have PRS B1913+16: $e = 0.6171$; $P = 0.322997$ day; $a = 1.950100\times10^{9}$ m; $m_A = 1.4414\,m_{\Theta}$; $m_B = 1.3867\,m_{\Theta}$ and PRS J0737-3039: $e = 0.0877$; $P = 0.102251$ day; $a = 8.8\times10^{8}$ m; $m_A = 1.338\,m_{\Theta}$; $m_B = 1.249\,m_{\Theta}$. Moreover, as the two compact systems are characterized by the mass ratio $q \approx 1$ this implies that in each system the two compact neutron stars should play mutually the role of the main gravitational source. For this case the effect of CGA-spin-orbit coupling should concern each system as a whole, *i.e.*, we are dealing with {*A,B*}'s CGA-orbital precession caused by the couple $(\mathbf{\Lambda},\mathbf{F}_{\mathrm{D}})$ in each system. Therefore, after substituting all the necessary parameters in Eq.(20), we obtain the following CGA-orbital precession rates for PSR B1913+16 and PRS J0737-3039, respectively:

$$\Psi_{\mathrm{CGA}}^{(1913)} = 1.248659 \text{ deg/cy} \ , \tag{25}$$

and

$$\Psi_{\mathrm{CGA}}^{(0737)} = 5.044625 \text{ deg/cy} \ . \tag{26}$$

As we can remark it repeatedly, that is to say as before for the other investigated gravitational phenomena, also our calculated values are in good agreement with those theoretically predicted by [9,10]. Once again, the CGA as a post-Newtonian gravity theory is very able to study and predict some old and new gravitational phenomena inside and outside the Solar System, both in weak and strong (combined) gravitational field.

3. Consequence of the Potential Equations

Let us focus our attention on Eqs.(5,6,7) *see* [3]. The first one, *i.e.*, Eq.(5) is in fact the well-known Laplace equation in radial coordinates, which defining us the *potential* outside the gravitational source *A* of mass *M* from where the test-body *B* of mass *m* evolving. Besides that, we have also the two other equations, *viz.*, Eqs.(6) and (7), respectively:

$$\frac{\partial^2 U}{\partial v^2} - \frac{1}{v}\frac{\partial U}{\partial v} = 0, \qquad \frac{\partial^2 U}{\partial r \partial v} + \frac{1}{r}\frac{\partial U}{\partial v} = 0.$$

With

$$U \equiv U(r,v) = -\frac{k}{r}\left(1 + \frac{v^2}{w^2}\right),$$

which is the velocity-dependent CGA-potential energy function Eq.(3) in [3] and w is a specific kinematical parameter having the dimensions of a constant velocity defined in [3] by

$$w = \begin{cases} c_0, \text{ if } B \text{ is in relative motion inside the vicinity of } A \\ v_{\mathrm{esc}}, \text{ if } B \text{ is in relative motion outside the vicinity of } A \end{cases},$$

where c_0 is the light speed in local vacuum and v_{esc} is the escape velocity at the surface of the gravitational source *A*. However, despite their different expressions, the above two equations define us in the context of CGA, the same new physical quantity, namely, the *gravitational momentum*. This new concept is derived from the following fact: we have from the above equations

$$-v\frac{\partial^2 U}{\partial v^2} = -\frac{\partial U}{\partial v}, \qquad r\frac{\partial^2 U}{\partial r \partial v} = -\frac{\partial U}{\partial v}.$$

Further, we have

$$-\frac{\partial U}{\partial v} = \frac{2k\,v}{w^2 r} = \varepsilon\left(m v\right). \tag{27}$$

Where $\varepsilon = \left(2GM / w^2 r\right)$. Since ε is a dimensionless physical quantity and $\left(m v\right)$ is the magnitude of the classical linear momentum vector, $\mathbf{P}$, thus (27) may be written as

$$P_{\mathrm{G}} \equiv P_{\mathrm{G}}\left(r, v\right) = \varepsilon P, \tag{28}$$

where $P = m v$ and P_{G} is the magnitude of the *gravitational momentum vector* $\mathbf{P}_{\mathrm{G}}$ which is defined below as follows

$$\mathbf{P}_{\mathrm{G}} : \begin{cases} P_{\mathrm{G}}^{(1)} = \varepsilon\, m\, v_x \\ P_{\mathrm{G}}^{(2)} = \varepsilon\, m\, v_y \\ P_{\mathrm{G}}^{(2)} = \varepsilon\, m\, v_z \end{cases} . \tag{29}$$

The quantity ε plays the role of the factor of proportionality between the magnitude of the gravitational momentum and the magnitude of the classical linear momentum. Therefore the existence of the gravitational momentum as an additional physical quantity should boost the principal momentum of the moving test-body. Note that the rate of change of the gravitational momentum vector (29) should be defined as the derivative of $\mathbf{P}_{\mathrm{G}}$ with respect to time, that is:

$$\frac{d\mathbf{P}_{\mathrm{G}}}{dt} = \varepsilon \frac{d\mathbf{P}}{dt}, \tag{30}$$

or

$$m^{-1} \frac{d\mathbf{P}_{\mathrm{G}}}{dt} = \varepsilon \frac{d\mathbf{v}}{dt}. \tag{31}$$

Remark, since $\varepsilon = \left(2GM / w^2 r\right)$ therefore the term on the right hand side of Eq.(31) coincides perfectly with the second term on the right hand side of Eqs.(15,18), *see* [3]. Consequently Eqs.(30) and (31) play the role of additional perturbating force and acceleration respectively. Returning to Eq.(29) and considering the case when the test-body *B* orbiting the gravitational source *A* at average distance r inside *A*'s vicinity thus in this case the factor of proportionality, ε, becomes $\left(2GM / c_0^2 r\right)$ and with the help of (28), the magnitude of the gravitational angular momentum of the orbiting test-body should be of the form

$$\ell_{\mathrm{G}} = P_{\mathrm{G}} r , \tag{32}$$

or explicitly

$$\ell_{\mathrm{G}} = \left(\frac{2GM}{c_0^2} \right) P . \tag{33}$$

It follows from above equations that any material body in state of orbital motion in combined gravitational field is characterized by a gravitational angular momentum vector whose magnitude is given by Eq.(32) or (33). Moreover, since the quantity $\left(2GM / c_0^2\right)$ is called, in the framework of GRT, 'Schwarzschild radius' hence we arrive at the following operational definition. 'The magnitude of the gravitational angular momentum for an orbiting test-body inside the vicinity of the gravitational source is the scalar product of the Schwarzschild radius and the magnitude of the classical linear momentum.' And from (33), we arrive at the following result

$$\frac{\ell_{\mathrm{G}}}{P} = \frac{2GM}{c_0^2} . \tag{34}$$

Eq.(34) means that the Schwarzschild radius is in fact a *gravitational radius* that characterizing any material body in state of orbital motion in combined-GF.

4. CGA-Binet's orbital Equation

Through the present work we have seen that the CGA as an alternative post-Newtonian gravity theory is very capable of predicting some old and new gravitational phenomena. For example, in the second paper [2] we have derived two important formulae one for the perihelion advance of Mercury and the other for the angular deflection of starlight. Indeed, the two formulae had been deduced from the CGA-Binet's orbital equation, which has *exactly* the same physico-mathematical structure as the general relativistic Binet's orbital equation developed in the framework of curved space-time and Schwarzschild metric [11,12,13]. The fact seemed a pure coincidence at first sight, but when one analyzes the paper [3] with fully open mind, he/she will find that in spit of the concept of curved space-time there is a certain *compatibility* of CGA with GRT reflected by Eq.(25) deduced from Eq.(24) which itself is an expression of the gravitational force derived by Ridgely [3] in the context of GRT. Also, from Eq.(25) we can easily deduced the basic result of CGA, namely, the dynamic gravitational field-force $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$. Therefore, from all that, we can logically assert that the concept of curved space-time is nothing but only a mathematical artifact and the existence of such compatibility signifies, among other things, the CGA is a counterexample to GRT. Furthermore, in order to make this compatibility more lucid, more localizable and more understandable, we shall derive once again the above mentioned CGA-Binet's orbital equation as follows. Let us consider the test-body *B* of mass *m* orbiting the gravitational source *A* of mass *M*. Therefore *B* evolving in the combined gravitational field. Moreover, supposing that the orbital motion takes place in the polar plan $(0, \mathbf{e}_r, \mathbf{e}_\phi)$ inside the vicinity of *A*; so that in such case we can replace the orbital velocity v with $r\dot{\phi}$ $\left(\dot{\phi} = d\phi/dt\right)$ and the kinematical parameter w with c_0 in CGA-potential energy function, *see* Section 3, we get

$$U = -\frac{k}{r}\left(1 + \frac{r^2\dot{\phi}^2}{c_0^2}\right). \tag{35}$$

Moreover, since the test-body *B* is in orbital motion, we obtain the following relations regarding classical angular momentum

$$h = m\, r^2 \dot{\phi}, \tag{36}$$

from which we get

$$r\dot{\phi} = \frac{h}{m\,r}. \tag{37}$$

After substituting (37) in (35), we have

$$U = -\frac{k}{r}\left(1 + \frac{h}{m^2\, c_0^2\, r^2}\right). \tag{38}$$

We can now write directly the force due to the combined gravitational potential energy

$$F = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) - \frac{\partial U}{\partial r} = -\frac{k}{r^2}\left(1 + \frac{3h^2}{m^2\, c_0^2\, r^2}\right). \tag{39}$$

We have also for a central force

$$\mathbf{F} = \mathrm{f}(r)\mathbf{r}, \tag{40}$$

and according to Newton's second law

$$\mathbf{F} = m\mathbf{a}, \tag{41}$$

or more explicitly

$$\mathbf{F} = \mathrm{f}(r)\mathbf{r} = m\left(\ddot{r} - r\dot{\phi}^2\right)\mathbf{e}_r + m\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\mathbf{e}_\phi \ . \tag{42}$$

Since we are dealing with elliptical orbits, therefore, by taking into account the Kepler's second law, we get from (42) the following differential equations relative to the directions $\mathbf{e}_r$ and $\mathbf{e}_\phi$:

$$\frac{F}{m} = \frac{\mathrm{f}(r)}{m} = \ddot{r} - r\dot{\phi}^2 . \tag{43}$$

$$m\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 . \tag{44}$$

That is

$$r^2\dot{\phi} = \kappa = \mathrm{constant} . \tag{45}$$

Let us put

$$r = \frac{1}{u} . \tag{46}$$

Thus

$$\dot{\phi} = \frac{\kappa}{r^2} = \kappa u^2 . \tag{47}$$

By differentiating relation (46), with respect to time, we get

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\left(\frac{du}{d\phi}\right)\frac{d\phi}{dt} ,$$

thus

$$\dot{r} = -\frac{1}{u^2}\dot{\phi}\,\frac{du}{d\phi} . \tag{48}$$

By substituting (47) in (48), we obtain

$$\dot{r} = -\kappa\,\frac{du}{d\phi} . \tag{49}$$

From (49), the second time derivative is

$$\ddot{r} = -\kappa\,\frac{d}{dt}\left(\frac{du}{d\phi}\right) = -\kappa\frac{d}{dt}\left(\frac{du}{d\phi}\right)\frac{d\phi}{d\phi} = -\kappa\frac{d}{d\phi}\left(\frac{du}{d\phi}\right)\frac{d\phi}{dt} ,$$

that is

$$\ddot{r} = -\kappa\,\frac{d^2u}{d\phi^2}\dot{\phi} . \tag{50}$$

Taking account of (47), Eq.(50) becomes

$$\ddot{r} = -\,\kappa^2 \frac{d^2u}{d\phi^2}\,u^2\,, \tag{51}$$

Again, by substituting (46), (47) and (51) in (43), we get

$$\frac{F}{m} = \frac{\mathrm{f}(r)}{m} = -\,\kappa^2 \frac{d^2u}{d\phi^2}\,u^2 - \frac{1}{u}\kappa^2\,u^4 = -\,\kappa^2 u^2 \frac{d^2u}{d\phi^2} - \kappa^2\,u^3\,. \tag{52}$$

Since $k = GMm$, thus from (39), (46) and (52), we find

$$\frac{d^2u}{d\phi^2} + u - \left[\frac{3GM\,h^2}{m^2\,c_0^2\,\kappa^2}\right]u^2 = \frac{GM}{\kappa^2}\,. \tag{53}$$

By taking into account the relations (37) and (47), the quantity included in square brackets, on the right hand side of Eq.(53), becomes

$$\left[\frac{3GM\,h^2}{m^2\,c_0^2\,\kappa^2}\right] = \left[\frac{3GM}{c_0^2}\right]. \tag{54}$$

Finally, after substitution in (53), we get the expected CGA-Binet's orbital equation

$$\frac{d^2u}{d\phi^2} + u - \left[\frac{3GM}{c_0^2}\right]u^2 = \frac{GM}{\kappa^2}\,. \tag{55}$$

Eq.(55) has exactly the physico-mathematical structure of the general relativistic Binet's orbital equation developed in the context of curved space-time and Schwarzschild metric [11,12,13] .

5. Application of CGA *to* Large-Scale Structures

Preamble: After we have applied the CGA-as post Newtonian gravity theory- to the Solar System, eclipsing binary star systems and binary pulsars, we now undertake to apply it to galactic scales. We are here interested in the 'hypothetical' dark matter (DM) without entering in the full details since there are a lot of research articles and books together provide authoritative coverage of the literature on the subject as well as derivations of the most important results. Nevertheless, in the present section, we will show that the existence of the dynamic gravitational acceleration - defined by Eq.(32) in [3] - at galactic scale should be attributed to the gravitodynamical evidence of the DM itself and the characteristic acceleration

$$a_0 \approx 2\times10^{-10}\,\mathrm{m\,s^{-2}}\,, \tag{56}$$

introduced by Milgrom as a universal constant [14,15,16] in his theory of Modified Newtonian Dynamics (MOND) [17,18,19,20], is in fact a special case of Eq.(32). Consequently, MOND as an alternative theory to the DM 'hypothesis' *becomes* by means of the CGA an additional support for DM!

Brief History: in most cases Newtonian gravitational inverse-square law and its well-known relativistic generalization have passed several critical tests on very different spatial and temporal scales. However, the first incongruous seems to show up only on galactic scales with the observed discrepancy between the gravitodynamical mass and the directly observable luminous mass. To resolve that discrepancy, two obvious explanations have been proposed: (i) either large quantities of invisible DM dominate the gravitodynamics of large systems [21,22,23,24]; (ii) or gravity itself is not correctly described by Newtonian theory on every scale [14,25,26,27].

5.1. Evidence *for* Dark Matter

Let us begin by the fundamental question arises whether the observed luminous matter distribution is really compatible with the rotation curve, without need for additional DM. Forthrightly, to the best of our knowledge, no galaxies are known with an extended rotation curve for which luminous matter is sufficient to explain the gravitational field without recourse to DM. The main evidence for large amounts of matter in the Universe which are not associated with the luminous components. There are several observations which are usually interpreted as providing evidence for (cold) DM [28,29,30,31,32,33] including:

1- The rotation curves of galaxies compared with their luminous matter distribution;
2- The gas content of clusters compared with velocity, x-ray or lensing mass estimates;
3- The normalization of galaxy clustering compared with microwave anisotropies;
4- The shape of the large-scale galaxy correlations;
5- Cosmic flows and redshift space distortions;
6- The amplitude of weak lensing by large scale structure.

5.2. MOND

As already mentioned, to avoid the need for DM, the best known suggested modification to Newtonian gravity theory [14,15,16,17] is usually referred to as MOND. The basic idea of MOND is that there exists a fundamental acceleration (56) *below* which the real acceleration is larger than the Newtonian one $\|\mathbf{a}_{\mathrm{N}}\| = a_{\mathrm{N}}$. This is essentially formulated by the real observed acceleration, a, through the relation

$$\mathbf{a}_{\mathrm{N}} = \mu(a/a_0)\mathbf{a}, \tag{57}$$

where $\mu(x)$ is an interpolating function with limits

$$\mu(x) = \begin{cases} x & \text{if } x << 1 \\ 1 & \text{if } x >> 1 \end{cases} . \tag{58}$$

MOND should reach its *regime* only when $\|\mathbf{a}\| = a << a_0$ and in this limit

$$a = \sqrt{a_0 \frac{F}{m}}, \tag{59}$$

which is often given as the fundamental equation of MOND. Further, Milgrom [14,15,16] suggested the following expression

$$\mu(x)=\frac{x}{\sqrt{1+x^2}} \quad . \tag{60}$$

Phenomenologically speaking, MOND works well for the observed phenomena at the level of galaxies and clusters of galaxies almost exactly like DM paradigm. For example, MOND has reproduced the (flat) galactic rotation curves and explained the Tully-Fisher law $(v^4 \propto L)$ without of course evoking the DM hypothesis. But the advocates of DM paradigm claimed that MOND works well for the cited phenomena because the low argument from that of $\mu(x)$ and the value of a_0 have been rigged to obtain these remarkable results.

5.3. MOND's conceptual difficulties

Before evoking MOND's ambiguities, we must keep in mind that the Newton's laws of motion and the law of gravitation are very closely *woven* together in such a way that any simple modifications without profound and serious intellectual reflection reinforced by an extreme caution rapidly lead to incalculable consequences. Therefore, one urgent basic question is whether MOND applies equally to *decelerations* as to *accelerations*, or whether the motion needs to be just a change in the vector direction of acceleration in order to show MOND effects. Since the usual interpretation is that all changes in *velocity* are subject to MOND. Consequently, we can immediately see a fundamental difference with ordinary dynamics when we consider a test-body moving away from a central gravitational source. In the MOND's vision, the test-body's deceleration never drops below the value of a_0. Thus it cannot escape to infinity that is in MOMD there are no unbound orbits.

Several critical difficulties with Milgrom's original scheme as stated were firstly identified by Felten [34] soon after the introduction of the MOND in 1983. One pertinent example is that, in MOND, when the accelerations a given to a test-body by two or more attracting bodies acting mutually do not add linearly; however in Newtonian gravity theory, accelerations a_{N} do add linearly, thus their square roots cannot do so. Or equivalently, this is that because acceleration is not inversely proportional to mass, momentum is not in general conserved for an isolated system. Furthermore, as the gravitational force is no longer linear, hence in MONDian framework the resultant gravitational force not being equal to the sum of partial forces. In particular, as it has been pointed out by Felten [34], the motion of the center of mass of a body no longer obeys the classical mechanics.

5.4. MOND is an adjustable phenomenological theory

MOND is exceptionally successful in explaining the shapes of galactic rotation curves because it is an *adjustable* theory. More precisely, MOND has three principal parameters that need to be *adjusted*, *viz.,* a_0; $\mu(x)$ and the stellar mass to luminosity ratio Υ. Indeed, originally Milgrom [14,15,16] used the forms (56) and (60); but several authors employed the following values: $1\times10^{-10}\,\mathrm{m\,s^{-2}}$, $1.2\times10^{-10}\,\mathrm{m\,s^{-2}}$, $3.4\times10^{-10}\,\mathrm{m\,s^{-2}}$ and $3.9\times10^{-10}\,\mathrm{m\,s^{-2}}$ instead of (56), this is clearly uncomfortable for MOND since a_0 is basically supposed to be a universal constant! Accordingly, it follows from the above values that a_0 is not strictly speaking a universal constant as speculated by Milgrom [14,15,16], but *adjustable* parameter having the dimensions of acceleration defined by

$$1\times10^{-10}\ \mathrm{m\,s^{-2}} \le a_0 \le 4\times10^{-10}\ \mathrm{m\,s^{-2}}. \tag{61}$$

Concerning the interpolating function, Bekenstein [35] proposed $\mu(x)=\left[\sqrt{1+4x}-1\right]\left[\sqrt{1+4x}+1\right]^{-1}$ in preference to (60) also Famaey and Binney [36] suggested $\mu(x)=x(1+x)^{-1}$ rather than (60).

5.5. Some empirical difficulties *with* MOND

Without mentioning the *adjustability*, some authors have claimed that MOND has been very successful in explaining observations of rotation curves for a variety of objects over a wide range of scales (see e.g., Milgrom [37] ; Bekenstein and Sanders [38]). But huge number of investigations have indicated difficulties in reconciling MOND with data under its main postulation that there is *no* DM, and that the critical acceleration parameter a_0 is has well fixed value. For example, concerning the astronomical evidence for large amounts of DM, Faber and Gallagher [39] and Davis *et al*., [40] have already revealed that in and around galaxies comes almost exclusively from applications of Newton's second law to galaxies and clusters of galaxies. Certainly, the accelerations in these large cosmological systems are much smaller than those for which the law has been well tested in the laboratory or in the Solar System. Kent [41] pointed out that while MOND could fit his H I rotation curve data there was a factor of 5 required in the value of a_0 and also no clear evidence for the slightly falling rotation curves that MOND would still predict. Hernquist and Quinn [42] examined simulations of shell galaxies within MOND, and arrived at the conclusion that the observed value and radial distribution of shells in NGC 3923 could not be rigorously explained without a DM halo. The and White [43] found that a MONDian fit to the coma cluster requires a *higher* value of a_0 than for galaxies and also does not predicted the correct temperature profile for the x-ray gas. Lake [44] identified inconsistencies between MOND and observations of group of seven dwarf galaxies: DDO 125, IC 1613, VCC 381, NGC 3109, DDO 154, IC 3522 and NGC 3198. Furthermore, Lake and Skillman [45] found that MONDian fits to Local Group dwarf IC 1613 would need values of a_0 at least an order of magnitude *below* the preferential values. Gerhard and Spergel [46] studied dwarf spheroidal galaxies in the Local Group and concluded that a number of the dwarfs need to contain some DM even under MOND paradigm. This *conclusion* is exactly the main purpose of our present section as we will see. We now return to CGA with the following typical galaxy scenario.

6. Typical Galaxy Scenario

Let us consider a test-body *B* as a star of mass m in rotational motion with velocity $v(r)$ at the radial distance r sufficiently far from the galactic center of the main gravitational source *A* which is in the present *scenario* a typical galaxy of mass $M(r)$ inside the radius r. With these considerations and by taking into account the definition of the kinematical parameter w in Section 3, Eq.(32) of the dynamic gravitational acceleration [3] becomes

$$\Lambda=\frac{GM(r)}{r^2}\frac{v^2(r)}{v_{\text{esc}}^2}. \tag{62}$$

Since $m<<M(r)$ we get, respectively, for rotational and escape velocity *at* the radius r the following expressions

$$v^2(r)=GM(r)r^{-1}, \tag{63}$$

$$v_{\text{esc}}^2 = 2GM(r)r^{-1}\,, \tag{64}$$

and after substitution in (62), we get the very important expression for the dynamic gravitational acceleration at the galactic scale

$$\Lambda = \frac{1}{2}\frac{v^2(r)}{r}\,. \tag{65}$$

Furthermore, in view of the fact that

$$\mathbf{a} = \frac{v^{\,2}(r)}{r^2}\mathbf{r}\,, \tag{66}$$

which is, in terms of vector, the classical centrifugal acceleration, hence according to (66), we have

$$\boldsymbol{\Lambda} = \frac{1}{2}\mathbf{a}\,. \tag{67}$$

Or in terms of force vector

$$\mathbf{F}_{\text{D}} = \frac{1}{2}(m\mathbf{a})\,. \tag{68}$$

As we can remark it easily, the second term in brackets in Eq.(68) represents the well-known Newton's second law that governing the classical dynamics thus Eq.(68) may be obviously written as $\mathbf{F}_{\text{D}} = \frac{1}{2}\mathbf{f}$. But what does Eq.(68) mean? Firstly, it means that at the *galactic level*, the dynamic gravitational force $\mathbf{F}_{\text{D}}$ is always equal to the half of the inertial force $\mathbf{f} = m\mathbf{a}$; secondly since $\mathbf{F}_{\text{D}}$ is in fact an additional force thus this signifies, among other things, that at large-scale structures the force defined by Newton's second law is not really a single force *as* in the common classical sense, but a resultant $\mathbf{F}$ of two forces $\mathbf{f}$ and $\mathbf{F}_{\text{D}}$, that is

$$\mathbf{F} = \mathbf{f} + \mathbf{F}_{\text{D}}\,. \tag{69}$$

Consequently, if the baryonic (luminous) matter is evidently the main responsible for $\mathbf{f}$ this immediately implies that the other component $\mathbf{F}_{\text{D}}$ is causally due to the permanent presence of some invisible matter which should be, of course, the DM. Therefore, according to Eq.(68), the DM is not strictly speaking *inert*, on the contrary, it is gravitationally very *active* and this dynamicity is largely reflected in the manifestation of $\mathbf{F}_{\text{D}}$ itself as an additional force. Therefore, if we take into account the universal equivalence between inertial mass and gravitational mass, we obtain from Eq.(69) the *net* force applied by DM on the *moving* ordinary matter: $\mathbf{F}_{\text{D}} = \mathbf{F} - \mathbf{f}$ and precisely for that reason the dynamic gravitational field-force $(\boldsymbol{\Lambda}, \mathbf{F}_{\text{D}})$ should be *induced* by the motion of ordinary matter under the permanent gravitodynamical influence of DM and in such situation Eq.(68) tells us that there is a direct *gravitodynamical link* between moving ordinary matter and DM. Accordingly, the gravitodynamical study of DM's effects on the moving ordinary matter should depending *exclusively* on the couple $(\boldsymbol{\Lambda}, \mathbf{F}_{\text{D}})$, *i.e.*, Eqs.(67) and (68). It seems to us that the *incomprehension* of the mentioned process of DM's gravitodynamical influence has urged some researchers to conclude from this incomprehension that the Newton's second law is not applicable to galactic scales. For example, in his very interesting pedagogical article entitled 'Does Dark Matter Really Exist?' published in

Scientific American, August 2002, Milgrom wrote "When the acceleration is much large than a_0, Newton's second law applies as usual: force is proportional to acceleration. But when the acceleration is small compared with a_0, Newton's second law is altered: force becomes proportional to the square of the acceleration. By this scheme, the force needed to impart a given acceleration is always smaller than Newtonian dynamics requires. To account for the observed accelerations in galaxies, MOND predicts a smaller force–hence, less gravity-producing mass–than Newtonian dynamics does (…). In this way, it can eliminate the need for dark matter." In one sense, Milgrom's claim was/is correct since his a_0 is exceptionally very comparable to the magnitude of (67), *i.e.*, $\Lambda = \frac{1}{2} a$ as we will see more explicitly. However, the exclusion of DM from the *existence* is a mistake mainly caused by the above mentioned incomprehension because, here, Milgrom –as a father of MOND– has consciously or unconsciously omitted to think of the causal origin of a_0 at large-scale structures and the universality of the equivalence between inertial and gravitational mass!

As an additional clarification, let us show that *conceptually* the Milgram's law for $a << a_0$ is a particular case of (65). To this end, substituting Λ for Λ_0 in (65), and after multiplying the two sides by the quantity $2a_{\mathrm{N}}$, where a_{N} is the Newtonian acceleration in MONDian sense, we get $2\Lambda_0\, a_{\mathrm{N}} = v^2(r) r^{-1} a_{\mathrm{N}}$; finally if we consider the particular situation $a^2 \approx v^2(r) r^{-1} a_{\mathrm{N}}$, we obtain the expected formula

$$a = \sqrt{2\Lambda_0\, a_{\mathrm{N}}} \quad . \tag{70}$$

This is remarkably very similar to Milgrom's law (59). We can also deduce an expression more general than Milgram's interpolating function. In the CGA's context, we call such a function: *functional relation*. To this end, let us rewrite the vectorial Eq.(69) in terms of acceleration vector fields as follows

$$\mathbf{g} = \mathbf{a} + \boldsymbol{\Lambda}. \tag{71}$$

By applying the well-known definition of the scalar product of two vectors $\mathbf{A} \cdot \mathbf{B} = \| \mathbf{A} \| \cdot \| \mathbf{B} \| \cos\theta$, we get

$$\mathbf{g}^2 = \mathbf{a}^2 + 2\| \mathbf{a} \| \cdot \| \boldsymbol{\Lambda} \| \cos\theta + \boldsymbol{\Lambda}^2 \quad . \tag{72}$$

Where θ is between $\mathbf{a}$ and $\boldsymbol{\Lambda}$, hence from (72) we get in terms of magnitude

$$\| \mathbf{g} \| = \| \mathbf{a} \| \cdot \sqrt{1 + 2\| \boldsymbol{\Lambda} \| \cdot \| \mathbf{a} \|^{-1} \cos\theta + \boldsymbol{\Lambda}^2 \cdot \mathbf{a}^{-2}} \quad . \tag{73}$$

Since according to (67), we have always $\| \boldsymbol{\Lambda} \| < \| \mathbf{a} \|$ thus by dividing the two sides of (73) by $\| \boldsymbol{\Lambda} \|$ and putting, respectively, $x = \| \boldsymbol{\Lambda} \| \cdot \| \mathbf{a} \|^{-1}$ and $\eta^{-1} = \| \mathbf{g} \| \cdot \| \boldsymbol{\Lambda} \|^{-1}$ we obtain the very expected functional relation

$$\eta \equiv \eta_\theta(x) = \frac{x}{\sqrt{1 + 2x\cos\theta + x^2}}, \quad \theta \equiv (\mathbf{a}, \boldsymbol{\Lambda}). \tag{74}$$

The functional relation $\eta_\theta(x)$ is important because containing some physical and geometrical information about acceleration vector fields. To be precise, the modulus x defines the magnitude ratio of acceleration vector fields and the argument θ defines the angular position of $\mathbf{\Lambda}$ with respect to $\mathbf{a}$ in the reference frame of the galaxy under consideration. Moreover, the expression of the functional relation (74) is more general than that of Milgram's interpolating function (60). Concretely, we can recover the expression $\mu(x) = x\left(1+x^2\right)^{-1/2}$ for the case $\theta = \pi/2$, *i.e.*, when $\mathbf{\Lambda} \perp \mathbf{a}$, and also we can recover the expression $\mu(x) = x(1+x)^{-1}$ of Famaey-Binney [37] for the case $\theta = 0$, *i.e.*, when $\mathbf{\Lambda} // \mathbf{a}$. It follows from this that, in fact, the interpolating functions proposed by Milgram and Famaey-Binney are not fortuitously suggested or needed as a mathematical artifact but have quantitatively and qualitatively a deep role and meaning. Hence MOND itself is *theoretically* incorporated in CGA.

Now, returning to our scenario. In order to make it heuristically and adequately close to reality and without specifying the shape, we attribute to our typical galaxy an average total mass $M(r) = 2\times 10^{11} m_\Theta$, average radius $r = 2R_0$, and the galactic (rotation) constants of $R_0 = 8.5\,\mathrm{kpc}$ and $V_0 = 229\,\mathrm{km\,s^{-1}}$. Thus our typical galaxy becomes realistically very comparable to the Milky Way.

6.1. Third typical galactic constant

Since the dynamic gravitational acceleration at the galactic scale (65) has mathematical structure of function, *i.e.*, $\Lambda \equiv \Lambda(r, v)$, therefore, in addition to the above adopted standard (rotation) constants, we have another, which has the dimensions of a constant acceleration and defined for $r = R_0$ and $v = V_0$, respectively, as follows

$$\Lambda_0 \equiv \Lambda_0(R_0, V_0) = 1\times 10^{-10}\,\mathrm{m\,s^{-2}}. \tag{75}$$

According to (61), the numerical value of third typical galactic constant Λ_0 is exactly equal to the minimal value of a_0, *viz*., 1×10^{-10} $\mathrm{m\,s^{-2}}$. Moreover, we can evaluate the minimal value of the dynamic gravitational acceleration at the average radius $r = 2R_0$ with the help of average total mass $M(r)$ of the typical galaxy and by taking into account the expression (63) of rotational velocity at $r = R$, we get after substitution in (65)

$$\Lambda_{\min} = 4.828920\times 10^{-11}\,\mathrm{m\,s^{-2}}. \tag{76}$$

Form (75) and (76), we obtain the following relation

$$\Lambda_{\min} \cong \frac{1}{2}\Lambda_0 \quad . \tag{77}$$

It follows from (77) that Λ_0 may be used as an *acceleration-scale* at galactic level exactly like a_0 in MOND's framework. Hence, from all that we arrive at the following result: according to CGA, MOND is the natural *sister* of DM with only different family name! This result coincides perfectly with Milgrom's conclusion that ending the above mentioned article "But it is possible that MOND follows from the dark matter paradigm in a different way. Time will tell."

6.2. Galactic rotation curves

In galaxies, particularly spiral ones, the presence of large quantities of invisible matter with a distribution different from baryonic (luminous) matter is now to be very well established. It seems the primary observational evidence for the existence of DM comes from optical and 21 cm rotation curves of spiral galaxies which do not show the expected Keplerian drop-off at large radii but remains flat or even rise over their entire observed range Faber [39]; Bosma [47]; Rubin *et al.*, [48] . Theoretically, this DM is supposed to be in the form of a spherical (halo) component in order to stabilize the spiral disks against bar instabilities Ostriker and Peebles [49]. But the physical and chemical structure and propriety of DM are still completely unknown.

To exemplify the practical applicability of the formalism developed here, we first determine the shape and behavior of rotation curve of our typical galaxy by deriving from (65) an expression for the rotational velocity $v(r)$ as a function of the radial distance $r\ (r \le 2R_0)$

$$v(r) = \sqrt{2\Lambda r}\ \ ,\ r \le 2R_0 . \tag{78}$$

It seems heuristically more convenient for the purpose of our scenario to rewrite (78) for the special case $\Lambda = \Lambda_0$ to obtain the following expected function

$$v(r) = \sqrt{2\Lambda_0 r}\ \ ,\ r \le 2R_0 . \tag{79}$$

Now, with the help of Mathematica5, we plot the function (79) for $\Lambda_0 = 1\times10^{-10}\ \mathrm{m\,s^{-2}}$. With this aim, we have conveniently converted $[0;\ 17\,\mathrm{kpc}]$ to $[0;\ 5.25\times10^{20}\ \mathrm{m}]$ and $[0;\ 300\ \mathrm{km\,s^{-1}}]$ to $[0;\ 3\times10^{5}\ \mathrm{m\,s^{-1}}]$

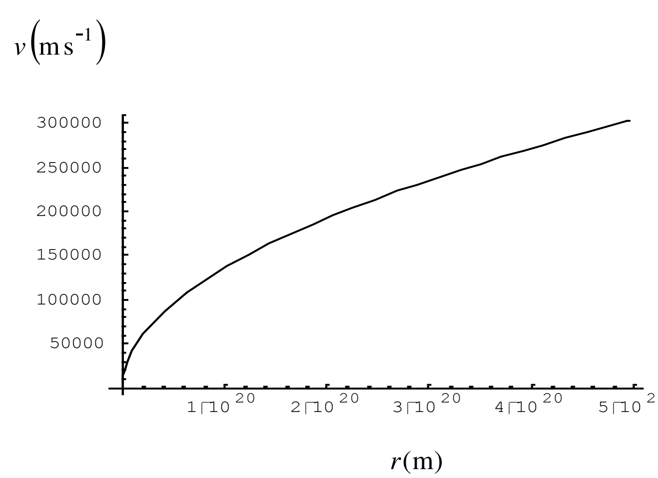

Figure 2. Rotation curve of a typical galaxy of average total mass $M(r) = 2\times10^{11} m_{\Theta}$ and average radius $r = 17\,\mathrm{kpc}$ using the function (79) for $\Lambda_0 = 1\times10^{-10}\ \mathrm{m\,s^{-2}}$.

In Fig.2 we have constructed the rotation curve of our typical galaxy. This rotation curve obtained through CGA's formalism illustrates the general shape and behavior of the majority of rotation curves of the well-observed galaxies.

7. Tully-Fisher relation

Historically, the Cepheid variable stars were the primary means by which distances are measured over the local volume of space. However, beyond about $20\,\mathrm{Mpc}$ Cepheids become too faints, even for Hubble Space Telescope, and so astronomers thought about an alternative means of measuring distances are needed.

Fortunately, one solution came from the several astronomical observations that show more conclusively that for disc galaxies the fourth power of the rotational velocity of stars moving around the core of the galaxy is proportional to the total luminosity of the galaxy $\left(v^4 \propto L\right)$, this is well-known as the (empirical) Tully-Fisher relation [50]. Since L itself is proportional to the mass M of the galaxy, therefore we will find $v^4 \propto M$. Like Milgrom's law under the expression (70), let us show that an equivalent expression to Tully-Fisher relation may be naturally occurred from CGA's formalism as follows: we have from (63) $r = GM/v^2$ and after direct substitution in (65), we get immediately the desired relation

$$v^4 = 2G\Lambda M\,. \tag{80}$$

This is exactly the Tully-Fisher relation under another expression, which here may be called *'mass-rotational velocity relation'* Furthermore, it is worthwhile to note that according to the above law, there are two types of *dependence*, namely implicit and explicit dependence. More precisely, in (81), the quantity is depending implicitly on the radial distance r, and also *it* is depending explicitly on the mass $M \equiv M(r)$ which itself is inside the radius r. Consequently, the rotational velocity is not strictly speaking independent of the radial distance. A very analogous relation has been already found in MOND's context for the special case $\Lambda = \Lambda_0$, namely

$$v^4 = 2G\Lambda_0 M\,. \tag{81}$$

Since the proportionality coefficient $2G\Lambda_0$ is constant thus we have really $v^4 \propto M$ and accordingly we deduce from (81) the two following significant relations

$$v = \left(2G\Lambda_0 M\right)^{1/4}, \tag{82}$$

$$M = \left(2G\Lambda_0\right)^{-1} v^4\,. \tag{83}$$

Now, let us illustrate graphically the double importance of the relation (81), *i.e.*, when v^4 and v are, respectively, considered as functions of the same (distributed) average total mass, M, of the typical galaxy in question. The first graph should have the same usual aspect and behavior of that defined by the original expression of Tully-Fisher relation [50] and the second graph should have, in general, the same standard aspect and behavior of the observed rotation curves. With this aim, we have conveniently converted $[0;\, 2\times10^{11} m_\Theta]$ to $[0;\, 4\times10^{41}\,\mathrm{kg}]$ and $[0;\, 300\,\mathrm{km\,s^{-1}}]$ to $[0;\, 3\times10^{5}\,\mathrm{m\,s^{-1}}]$; and as before, with the help of Mathematica5, we plot the functions (81) and (82) for $\Lambda_0 = 1\times10^{-10}\,\mathrm{m\,s^{-2}}$.

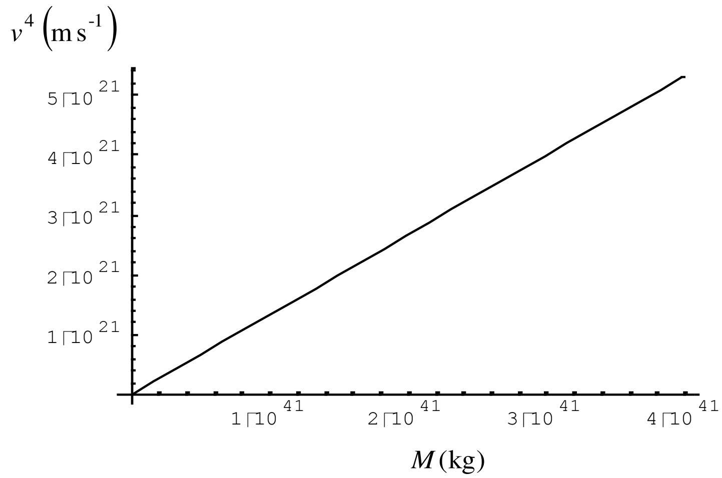

Figure 3. Fourth power of the rotational velocity, v^4, plotted versus the distributed average total mass, M, of a typical galaxy using mass-rotational velocity relation (82) for $\Lambda_0 = 1\times10^{-10}\ \mathrm{m\,s^{-2}}$.

In Fig.3 we have plotted v^4 as a function of M. The graph illustrates perfectly the *correlation* between fourth power of the rotational velocity and the distribution of the average total mass of a typical galaxy. This is in good agreement with the original empirical Tully-Fisher relation and the usual aspect and behavior of the observed curves.

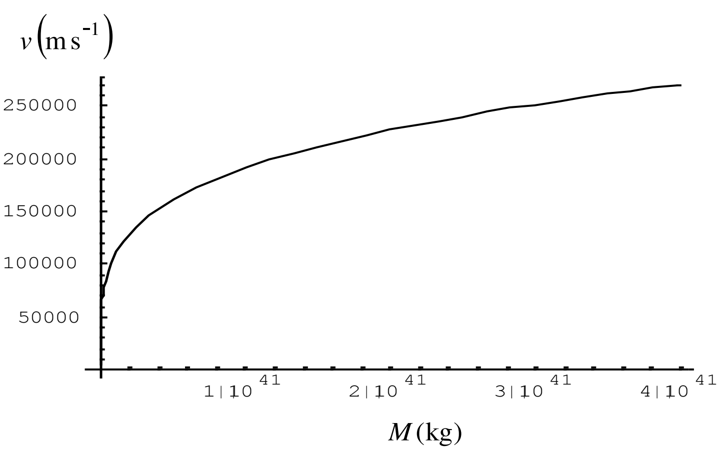

Figure 4. The rotational velocity, v, plotted versus the distributed average total mass M of a typical galaxy using relation (84) for $\Lambda_0 = 1\times10^{-10}\ \mathrm{m\,s^{-2}}$.

In Fig.4, we have plotted the rotational velocity, v, as a function of the distributed average total mass M of a typical galaxy. As we can remark it, the illustrated rotation curve is in good conformity with the standard aspect and behavior of the observed rotation curves. Moreover, this result has a number of interesting implications. First, according to (81) and (82), there is an apparently universal correlation between baryonic mass and rotational velocity *through* the gravitodynamical influence of DM, which is phenomenologically reflected by the presence of the dynamic gravitational accelerations Λ_0 in (81) and (82). Secondly, the mass-rotational velocity relation (81) or (82) provides the physical basis to the empirical Tully-Fisher relation that remained unclear before the CGA advent.

8. Conclusion

The CGA could be regarded as an alternative gravitational model to compare with the others that have already existed for a long time. As we have seen, the CGA enabled us to study and solve some old and new problems related to gravitational phenomena through a novel comprehension and interpretation of the *gravity* itself; the famous Newton's law of gravitation was corrected and reformulated in a new more general form [1,2,3]. In the CGA's context, dark matter 'hypothesis' and MOND paradigm have been finally reconciled with each other; and also the empirical Tully-Fisher relation has found its physical basis.

References

[1] M.E. Hassani, Galilean Electrodynamics **20**, Special Issues 3, 54 (2009)
[2] M.E. Hassani, Galilean Electrodynamics **22**, Special Issues 3, 43 (2011)
[3] To be appeared in Galilean Electrodynamics, viXra:1303.0112 (2014)
[4] M.E. Hassani, Galilean Electrodynamics **24**, Special Issues 3, 42 (2013)
[5] H.A. Lorentz, Proceedings of the Royal Netherlands Academy of Arts and Sciences, 2,559 (1900)
[6] W.De Sitter, MNRAS, 76, 155 (1916)
[7] V.A. Brumberg, P. Bretagnon and G. Francou, in Journée 1991: *systèmes de références spatio-temporels*, edited by N. Capitaine (Observatoire de Paris, Paris, 1991), pp.141-148.
[8] V.A. Brumberg, P. Bretagnon and G. Francou, A&A 275, 651 (1993)
[9] J.M. Weisberg and J.H.Taylor, ApJ, 576, 942 (2002)
[10] R.N. Manchester et al., arXiv: astro-ph/0501665v1 (2005)
[11] J.L. Lopes, Théorie Relativiste de la Gravitation (Masson, Paris, 1993)
[12] J. Weber, General Relativity and Gravitational wave (Interscience, New York, 1961)
[13] R d'Inverno, Introducing Einstein's Relativity (Oxford U. P., New York, 2001)
[14] M. Milgrom, ApJ, **270**, 365 (1983)
[15] M. Milgrom, ApJ, **270**, 371 (1983)
[16] M. Milgrom, ApJ, **270**, 384 (1983)
[17] J. Bekenstein and M. Milgrom, ApJ, **286**, 7 (1984)
[18] M. Milgrom, ApJ, **302**, 617 (1986)
[19] M. Milgrom, Ann. Phys., **229**, 384 (1994)
[20] R. Brada and M. Milgrom, MNRAS, **276**, 453 (1995)
[21] J. Oort, Bull. Astron. Inst. Netherlands, **6**, 249 (1932)
[22] F. Zwicky, Helvetica Physica Acta, **6**, 110 (1933)
[23] S. Smith, ApJ. **83**, 23 (1936)
[24] F. Zwicky, ApJ, **86**, 217 (1937)
[25] A. Finzi, MNRAS, **127**, 21 (1963)
[26] J. W. Moffat, Phy. Letts. **B 355**, 447 (1995)
[27] J. W. Moffat, J. Math. Phys. **36**, 3722 (1995)
[28] S. M. Kent, AJ, **91**, 1301 (1986)
[29] S. Casertano and J. H. van Gorkom, AJ, **101**, 1231 (1991)
[30] A. Bosma, Cel. Mech. Dyn. Astr., **72**, 69 (1998)
[31] J. A. Peacock, arXiv:astro-ph/0002013 (2000)
[32] J. R. Primack, arXiv:astro-ph/0007187 (2000)
[33] D. Glowe et al.; arXiv:astro-ph/0608407 (2006)
[34] J. E. Felten, ApJ, **286**, 3 (1984)
[35] J. Bekenstein, Phys. Rev. D, **70**, 083509 (2004)
[36] B. Famaey and J. Binney, MNRAS, **363**, 603 (2005)

[37] M. Milgrom, *Dark matter in astrophysics and particle physics*, eds. H.V. Klapdor-Kleingrothaus and L. Baudis, Institute of physics Pub., Philadelphia, p.443 (1999)
[38] J. Bekenstein and R. H. Sanders, arXiv: astro-ph/0509519 (2006)
[39] S. M. Faber and J. S. Gallagher, Ann. Rev. Astron. Astrophys., **17**, 135 (1979)
[40] M. Davis *et al.,* ApJ**, 238**, L113 (1980)
[41] S. M. Kent, AJ, **93**, 816 (1987)
[42] L. Hernquist and P. J. Quinn, ApJ, **312**, 17 (1987)
[43] L. S. The and S. D. M. White, AJ, **95**, 1642 (1988)
[44] G. Lake, ApJ, **345**, L17 (1989)
[45] G. Lake and E. D. Skillman, AJ, **98**, 1274 (1989)
[46] (O. E. Gerhard and D. N. Spergel, ApJ, **397**, 38 (1992)
[47] A. Bosma, AJ, 86, 1791 (1981)
[48] V. C. Rubin *et al*., ApJ, **289**, 81 (1985)
[49] J. P. Ostriker and P. J. E. Peebles, ApJ, **186**, 467 (1973)
[50] R. B. Tully and J. R. Fisher, A&A, **54**, 661 (1977)