Theoretical study of quantization of magnetic flux in a superconducting ring

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We refined the concepts of electric current and fluxoid, and London's equation that specify quantum phenomena of moving electrons and magnetic flux in a closed circuit similar to a superconducting ring, so as not to violate the uncertainty principle.

On this basic the relation between the electron motion and magnetic flux in a superconductor has been theoretically investigated by means of Faraday's law and/or canonical momentum relation.

The fact that minimum unit of the quantized magnetic flux is hc/2e does not mean the concurrent motion of the two electrons in a Cooper pair as is known so far. However, it is shown to be related with independent motion of the each electron in a superconducting state.

Keywords : flux quantum, fluxoid, london's equation

I. Review

In 1933, Meissner and Ochsenfeld discovered the phenomenon that the superconductor excludes the magnetic field, and then in 1935, H. London and F. London theorized Meissner Effect that occures in the superconductor.

v_d the drift velocity of the electron and E, the electric field are related as shown below based on the Newton's second law in the superconducting state

$$m\frac{dv_d}{dt} = eE, \tag{1}$$

where m and e are the electron's mass and charge.

If we substitute $J = e\rho v_d$ where ρ is the number of electrons per unit volume, for v_d the rearranged formula is

$$\frac{dJ}{dt} = \frac{\rho e^2}{m}E. \tag{2}$$

If we apply formula (2), Ampere's Law $\left(\nabla \times B = \frac{4\pi}{c}J\right)$ and Faraday's Law $\left(\nabla \times E = -\frac{1}{c}\frac{dB}{dt}\right)$ we have

$$\nabla \times \left(\nabla \times \frac{dB}{dt}\right) = -\frac{4\pi\rho e^2}{mc^2}\frac{dB}{dt} \tag{3}$$

We know that $\nabla \cdot B = 0$, so

$$\nabla^2 \frac{dB}{dt} = \frac{4\pi\rho e^2}{mc^2} \frac{dB}{dt}. \tag{4}$$

Therefore, answer for furmula (4) is

$$\frac{dB}{dt} = \frac{dB_0}{dt} e^{-\frac{x}{\lambda}} \qquad B = B_0 e^{-\frac{x}{\lambda}} \tag{5}$$

where x is the inside distance of the superconductor, λ is London penentration depth, and B is the magnetic field.

Formula (5) explains that the superconductor excludes the mangetic field, where its strength exponentially diminishes according to the depth of its permiability into the magnetic field on the surface of the superconductor.

Using formula (2), $\nabla \times A = B$, and Faraday's Law, we have $J = -\frac{\rho e^2}{mc} A$, which is known as London Equation proposed by the London brothers.

In 1950, F. London presented the concept of fluxoid by presenting

$$\frac{c}{e} \oint P \cdot ds_1 = \frac{c}{e} \oint mv_d \cdot ds_1 + \oint A \cdot ds_1 \tag{6}$$

where the left side is floxoid and the first term on the right is the electric current and the second term is the magnetic flux.

Bohr's quantum condition can be applied to the left side of formula (6), thus

$$\oint P \cdot ds_1 = nh, \tag{7}$$

where $n = 1, 2, 3, \ldots$ and h is Planck's constant.

From this, the estimated value of the minimum unit of fluxoid is given by

$$\therefore \ \ \phi = \frac{hc}{e}. \tag{8}$$

Formula (6) can be induced in relation to formula (2). The electrons in the superconductor creates the electric current when accelerated by the force generated by the electric field, E, thus, under the assumption that the electric field E is produced by Faraday's Law, we have

$$m\frac{dv_d}{dt} = eE = -\frac{e}{c}\frac{dA}{dt}. \tag{9}$$

If we rearrange this equation, we have

$$\frac{d}{dt}\left[mv_d + \frac{e}{c}A\right] = 0 \tag{10}$$

$$\left[mv_d + \frac{e}{c}A\right] = P = \text{constant}. \tag{11}$$

Formula (6) is induced by applying line integral on both sides of the formula (11).

Also, formula (6) shows that the external magnetic flux becomes quantized going through the ring hole of superconductor by applying the line integral on its inside path assuming that there is no electric current in the deep inside of the relatively thick superconductor.

$$nh = 0 + \frac{e}{c}\oint A \cdot ds_1, \tag{12}$$

$\phi = \dfrac{hc}{e}$ is induced from the formula (12).

However, its measured value from the superconducting ring was

$$\phi = \frac{hc}{2e} = 2.07 \times 10^{-15} T - m^2. \tag{13}$$

Currently, we believe that the supercurrent is carried by pairs of electrons.

II. The problem of the fluxoid concept

1) From the fluxoid formula, $\frac{c}{e}\oint P \cdot ds_1 = \frac{c}{e}\oint mv_d \cdot ds_1 + \oint A \cdot ds_1$, P on the left side of the equation represents the motion of electrons and the first term on the right side, v_d can be interpreted as the drifting velocity of the electron. This interpretation is supported by Bohr's Quantum Condition that is applied to P, which comes from the application of the line integral on the momentum with its motion path. v_d the electric curent of the superconductor may be interpreted as the drift velocity. However, considering the drift velocity is 1mm per second and the velocity of the electron is usually several thousand km per second in the Fermi level range, we can conclude that formula (6) can not characterize the physical principle of the superconductor due to the lack of correspondence between two numbers of both sides of the formula.

2) The formula, $nh = 0 + \frac{e}{c}\oint A \cdot ds_1$ can not provide authentic explanation even though it attempts to indicate that the magnetic flux becomes quantized under the assumed condition of no electric current in the deep inside of the superconductor.

To be specific, by combining $J = -\dfrac{\rho e^2}{mc}A$, Ampere's Law $\nabla \times B = \dfrac{4\pi}{c}J$, and $B = \nabla \times A$, we have $\nabla \times (\nabla \times A) = -\dfrac{4\pi\rho e^2}{mc^2}A$.

Rearranging this equation and using $\nabla \cdot A = 0$ yields $\nabla^2 A = \dfrac{4\pi\rho e^2}{mc^2}A$, then $A = A_0 e^{-\frac{x}{\lambda}}$.

That is, from $J \to 0$ and $A \to 0$ in the deep inside of the superconductor, the accurate definition of formula (13) is given by

$$nh = 0 + 0 \tag{14}$$

This formula shows that the existing evidence that the exterior magnetic flux passing the ring hole of the superconductor becomes quantized is incorrect.

Since there is a flow of the exterior magnetic field in the hole of superconductor even with $J \to 0$ and $A \to 0$ in the deep inside of the superconductor, one can assert that the second term on the right side becomes $\oint A \cdot ds_1 = \phi$ rather than $\oint A \cdot ds_1 = 0$.

However, this proof may show an alternative theory that the first term on the right side of the equation can not be $\oint mv_d \cdot ds_1 = 0$ because the magnetic field in the ring of the superconductor reveals not only the exterior magnetic field but also the magnetic field created by J, the superficial electric current. Therefore, this shows that the evidence of the magnetic flux quantum using the formula (13) calcuating the exterior magnetic field that passes the ring at the line integral path inside of the superconductor is a fallacy.

3) Currently accepted fluxoid concept does not match the experimental result obtained from Little-park. To explain the experimental result, $n = 0$ should be possible on the left side of the formula (6). We can not define $n = 0$ as long as we use the Bohr's quantum condition to define the fluxoid. If $n = 0$, $P = 0$, which contradicts the fact the any particle confined in a certain space can not take 0 value according to the Uncertainty Principle.

Ⅲ. A new theory

The problem shown in the present fluxoid concept essentially comes from the lack of understanding the Canonical Momentum Equation. Since the particle confined within limited space must take other than 0 value, we need to reestablish the concept of the electric current of the superconductor within the definition of the Uncertainty Principle.

In normal circumstances, the electric current is converted to the drift velocity of electron within a conductor. The velocity sum becomes 0 without the electric current. This means there is no the net velocity or momentum. The electrons exist in multiple and gain its own velocity in case of no electric current in the conductor. However, If we total the velocity of all electrons, the net momentum is 0 as given by

$$\sum_k mv_k = 0, \tag{15}$$

where the electric current in the solid is delt under this premise like fomula (15).

The electrons move as shown in formula (16) when E, the constant electric field is maintained for the duration of t, the time.

$$mv_f - mv_i = eEt, \tag{16}$$

where mv_f is the final momentum and mv_i is the initial momentum, which do not take 0 value according to the Uncertainty Principle.

If we sum up the velocity of all electrons using the formula (16), $\sum_k (mv_{fk} - mv_{ik})$ becomes $\sum_k mv_{fk} - \sum_k mv_{ik}$, $\sum_k mv_{ik} = 0$ according to the formula (15), then, we have the final $\sum_k mv_{fk} = \sum_k eE_k t$.

If we sum it up, from formula (15) the sum of the beninning momentum becomes 0, and remain the sum of the velocity element that are in the same direction in the eletric field E within the electron's end momentum elements.

Therefore, we present the drift velocity, v_d that was used by London brothers leads to

$$mv_f - mv_i = mv_d = eEt. \tag{17}$$

Using this formula, we have

$$m\frac{dv_f}{dt} - m\frac{dv_i}{dt} = m\frac{dv_d}{dt} = eE. \tag{18}$$

Since $m\frac{dv_i}{dt}$ in formula (18) is not relevant to the electric field, E, then its value becomes $m\frac{dv_i}{dt} = 0$. Using Faraday's law $E = -\frac{1}{c}\frac{dA}{dt}$, we have

$$m\frac{dv_f}{dt} = m\frac{dv_d}{dt} = eE = -\frac{e}{c}\frac{dA}{dt}. \tag{19}$$

The difference between formula (19) and (1) is that there are more accelerated terms in the velocity of the electron, these term are those of the drift velocity and equivalence.

From the formula (19), we have

$$m\frac{dv_f}{dt} = -\frac{e}{c}\frac{dA}{dt} \tag{20}$$

$$m\frac{dv_d}{dt} = -\frac{e}{c}\frac{dA}{dt}. \tag{21}$$

Examining these two formulas, we can speculate that F. London did not distinguish between (20) and (21) while developing the fluxoid concept.

Formula (20) provides a formula that is similar to the relational expression of the Canonical Momentum

$$mV(\text{initial}) = mv(\text{final}) + \frac{e}{c}A. \tag{22}$$

From formula (21), we can find London Equation and a new concpt of fluxoid

$$\text{constant} = mv_d + \frac{e}{c}A, \tag{23}$$

where the constant includes 0, v_d is the drift velocity of the electron, A is vector potential. Since a constant is calcuated by adding two term, Bohr's quantum condition can not be applied here due to its irrelavance to the momentum of electron. If a constant is 0, it would coincide with London Equation, otherwise it indicates a new fluxoid concept.

In formula (22), mV and mv represent the momentum that indicates the status of each electron, thus, Bohr's quantum condition can be applied and can not take 0 for their velocity based on the Uncertainty Principle.

The following is an attempt to show that the magnetic flux become quantized in the superconductor with a hole like the one in the ring of a superconductor.

For convenience, we can substitute mv_1 and mv_2 for mV and mv into the formula (22), so

$$mv_1 = mv_2 + \frac{e}{c}A. \tag{24}$$

Apply dot product on both sides of formula (24) with v_1, then

$$mv_1 \cdot v_1 = mv_2 \cdot v_1 + \frac{e}{c}A \cdot v_1. \tag{25}$$

By applying dot product on both sides of formula (24) with v_2, then we have

$$mv_1 \cdot v_2 = mv_2 \cdot v_2 + \frac{e}{c}A \cdot v_2. \tag{26}$$

The combination of (25) and (26) becomes

$$mv_1 \cdot v_1 + mv_1 \cdot v_2 = mv_2 \cdot v_1 + \frac{e}{c}A \cdot v_1 + mv_2 \cdot v_2 + \frac{e}{c}A \cdot v_2. \tag{27}$$

This shows that the second term of the left side of formula (27) and the first term of the right are identical. If we rearrange (27),

$$mv_1 \cdot v_1 = mv_2 \cdot v_2 + \frac{e}{c}A \cdot v_1 + \frac{e}{c}A \cdot v_2. \tag{28}$$

Applying (integral) calculus on formula (28), we have

$$\int mv_1 \cdot v_1 dt = \int mv_2 \cdot v_2 dt + \frac{e}{c}\int A \cdot v_1 dt + \frac{e}{c}\int A \cdot v_2 dt. \tag{29}$$

This can be simplified to

$$\int mv_1 \cdot ds_1 - \int mv_2 \cdot ds_2 = \frac{e}{c}\int A \cdot ds_1 + \frac{e}{c}\int A \cdot ds_2. \tag{30}$$

Bohr's quantum condition can be applied to the two term of the left side of formula (30) because the momentum of electrons becomes quantized. Thus,

$$n_1 h = n_2 h + \frac{e}{c}\phi_1 + \frac{e}{c}\phi_2, \tag{31}$$

where n_1 and n_2 are integer with vaues other than 0.

Since the magnetic flux in the ring of superconductor is identical regardless of the electron's motion path, we know that $(\phi_1 = \phi_2 = \phi)$, so we have

$$\phi = \frac{hc}{2e}(n_1 - n_2). \tag{32}$$

I attempt to calculate the exterior magnetic flux that passes the ring of a superconductor by using Faraday's Law even though it would be essentially the same proof as presented above. Faraday's Law is

$$\int E \cdot ds = -\frac{1}{c}\frac{d\phi}{dt} \tag{33}$$

We can substitute $eE = \dfrac{dp}{dt}$ into the left side of the formula since there is no electric resistancy in the superconductor, thus,

$$\int \frac{dp}{dt} \cdot ds = -\frac{1}{c}\frac{d\phi}{dt}. \tag{34}$$

If we develop the formula (34),

$$\frac{1}{2}mv_2 \cdot v_2 - \frac{1}{2}mv_1 \cdot v_1 = -\frac{1}{c}\frac{d\phi}{dt}. \tag{35}$$

Applying calculus on both sides of (35)

$$\frac{1}{2}\int mv_2 \cdot v_2 dt - \frac{1}{2}\int mv_1 \cdot v_1 dt = -\int \frac{1}{c}\frac{d\phi}{dt}dt \tag{36}$$

$$\frac{1}{2}\int mv_2 \cdot ds_2 - \frac{1}{2}\int mv_1 \cdot ds_1 = -\int \frac{1}{c}d\phi \tag{37}$$

If magnetic field is delivered under the condition that magnetic field is missing in the ring of the superconductor, we have

$$\frac{1}{2}n_2 h - \frac{1}{2}n_1 h = -\frac{1}{c}(\phi - 0) \tag{38}$$

$$\therefore \ \phi = \frac{hc}{2e}(n_1 - n_2). \tag{39}$$

The exterior magnetic flux becomes quantized in the ring of superconductor as shown above, therefore, using formula (21) the fluxoid is given by

$$\int m\frac{dv_d}{dt}dt = -\frac{e}{c}\int \frac{dA}{dt}dt \tag{40}$$

$$\int d(mv_d) = -\frac{e}{c}\int dA. \tag{41}$$

As shown in the experiment of Little-park, assuming that vortex gets inserted inside of the ring hole of the superconductor, formula (41) evolves by using definite integral

$$[mv_d - 0] = -\frac{e}{c}[A - A_0] \tag{42}$$

$$mv_d + \frac{e}{c}A = \frac{e}{c}A_0. \tag{43}$$

Applying contour integral on both sides of formula (43)

$$\oint mv_d \cdot ds + \frac{e}{c}\oint A \cdot ds = \frac{e}{c}\oint A_0 \cdot ds. \tag{44}$$

Using formula (39) we can rearrange the right side of the formula (44) into

$$\frac{mc}{\rho e^2}\oint J \cdot ds + \oint A \cdot ds = n\phi_0, \tag{45}$$

where $\phi_0 = \dfrac{hc}{2e}$ and $J = e\rho v_d$.

As a new fluxoid concept, formula (45) coincides with the Little-park experiment, where n is integer including 0 value, and is free of problems that may cause a conflict with the Uncertainty Principle.

Ⅳ. Conclusion

The existing fluxoid concept contained a problem that contradicts the Uncertainty Principle by using the Canonical Momentum formula because of no specified distinction between the velocity and the drift velocity of the electron. To resolve this problem, we strictly defined the electric current in the superconducting ring and the fluxoid concept to develop an outcome that two electrons do not accompany in the superconducting state. This result is based on the fact that the minimum value of exterior magnetic flux,ϕ is theoretically $hc/2e$ induced from the Canonical Momentum Equation and Faraday's Law and the premises of Bohr's Quantum Condition must be satisfied before and after the electron's momentum in the superconductor takes its effect on the magnetic field.

This outcome may be a useful reference in research related to the superconducting phenomenon.

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