A Particular Solution Formula For Inhomogeneous Second Order Linear Ordinary Differential Equations

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ABSTRACT

A formula for Particular solutions to any Linear Second Order Inhomogeneous Ordinary Diffrential equations is presented. For second order ODEs these make the methods of undetermined coefficients and variation of parameters obsolete.

Finding a particular solutions to a linear inhomogeneous ordinary differential equation has always been a process of determining homogeneous solutions, and then adding any particular solution of the inhomogeneous equation. The well-known methods of undetermined coefficients and variation of parameters have long been the standard in determining this particular solution. The former has sometimes been considered 'ad-hoc', and both can be intricate. A relatively simple formula is presented here which allows the particular solution to be written and evaluated immediately.

The following theorem, which is a fundamental basis to proving the theorem for the particluar solution formula, is long well-known (see [1] or [3]).

Theorem I.1: If s is an arbitrary differentiable function, V is an integrable function, and y is a twice-differentiable function, then:

$$\exists y \ni y = e^{\int s dx}$$
$$\Leftrightarrow y'' + Py' + (-s' - s^2 - sV)y = 0.$$

The path to the following theorems establishing the particular solution formula came through the following analysis.

Consider first a linear first order ODE:

$$u' + Pu = Q = e^{-\int Pdx}\left(ue^{\int Pdx}\right)' \Rightarrow u = e^{-\int Pdx}\int Qe^{\int Pdx}dx$$

So, if: $u = Ay' + By + C$

$$\Rightarrow Ay' + By + C = e^{-\int Pdx}\int Qe^{\int Pdx}dx$$

$$\Rightarrow y' + \frac{B}{A}y = \frac{1}{A}e^{-\int Pdx}\int Qe^{\int Pdx}dx - \frac{C}{A}$$

$$\Rightarrow y = e^{-\int \frac{B}{A}dx}\int\left(\frac{1}{A}e^{-\int Pdx}\int Qe^{\int Pdx}dx - \frac{C}{A}\right)e^{\int \frac{B}{A}dx}dx$$

$$(Ay' + By + C)' + P(Ay' + By + C) = Q$$

$$\Rightarrow Ay'' + A'y' + By' + B'y + C' + PAy' + PBy + PC = Q$$

$\therefore$

$$y = e^{-\int \frac{B}{A}dx}\int\left(\frac{1}{A}e^{-\int Pdx}\int Qe^{\int Pdx}dx - \frac{C}{A}\right)e^{\int \frac{B}{A}dx}dx$$

$$\Rightarrow y'' + \left(\frac{A'}{A} + \frac{B}{A} + P\right)y' + \left[\frac{B'}{A} + P\frac{B}{A}\right]y = -\left[\frac{C'}{A} + \frac{PC}{A} - \frac{Q}{A}\right] \qquad (I.1)$$

So: $s = \frac{B}{A}$:

$$y = e^{-\int sdx}\int\left(\frac{1}{A}e^{-\int Pdx}\int Qe^{\int Pdx}dx - \frac{C}{A}\right)e^{\int sdx}dx$$

$$\Rightarrow y'' + \left[s + \left(\frac{A'}{A} + P\right)\right]y' + \left[s' + s\left(\frac{A'}{A} + P\right)\right]y = \frac{Q}{A} - \frac{C'}{A} - \frac{PC}{A}$$

$V = s + \frac{A'}{A} + P$:

$$y = e^{-\int sdx}\int\left(e^{\int (s-V)dx}\int \frac{1}{A}Qe^{-\int (s-V)dx}dx - \frac{C}{A}\right)e^{\int sdx}dx$$

$$\Rightarrow y'' + Vy' + [s' - s^2 + sV]y = \frac{Q}{A} - \frac{C'}{A} - \frac{C}{A}\left(V - s - \frac{A'}{A}\right)$$

$C = 0$, $W = \frac{Q}{A}$:

$$y = e^{-\int sdx}\int\left(\int We^{-\int (s-V)dx}dx\right)e^{\int (2s-V)dx}dx$$

$$\Rightarrow y'' + Vy' + [s' - s^2 + sV]y = W. \qquad (I.2)$$

Verification:

$$y = e^{-\int sdx}\int\left(\int We^{-\int (s-V)dx}dx\right)e^{\int (2s-V)dx}dx$$

$$\Rightarrow \left(ye^{\int sdx}\right)' = \left(\int We^{-\int (s-V)dx}dx\right)e^{\int (2s-V)dx}$$

$$\Rightarrow \left[e^{-\int (s-V)dx}(y' + sy)\right]' = We^{-\int (s-V)dx}$$

$$\Rightarrow e^{-\int (s-V)dx}\left[(y' + sy)' - (s - V)(y' + sy)\right] = We^{-\int (s-V)dx}$$

$$\Rightarrow y'' + sy' + s'y - sy' - s^2y + Vy' + sVy = W$$
$$\Rightarrow y'' + Vy' + (s' - s^2 + sV)y = W$$

$s = -t$:

$$\boxed{\begin{aligned} & y = e^{\int tdx}\int(\int We^{\int(t+V)dx}dx)e^{-\int(2t+V)dx}dx \\ & \Rightarrow y'' + Vy' + [-t' - t^2 - tV]y = W. \end{aligned}} \qquad (I.3)$$

These results may be stated as follows.

Theorem II.1: If s is an arbitrary differentiable function, V is an integrable function, and W is a twice integrable function, and y is a twice-differentiable function, such that:

$$y'' + Vy' + (-s' - s^2 - sV)y = 0.$$

then, a particular solution, y_p , to:

$$y_p'' + Vy_p' + (-s' - s^2 - sV)y_p = W,$$

is:

$$y_p = e^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx.$$

proof:

Let:

$$y = e^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx$$

then:

$$y' = se^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx + e^{-\int(s+V)dx}\int We^{\int(s+V)dx}dx$$

$$\begin{aligned} \Rightarrow y'' = {} & s'e^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx + s^2e^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx + \\ & + se^{\int sdx}(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx} + \\ & - (s+V)e^{-\int(s+V)dx}\int We^{\int(s+V)dx}dx + e^{-\int(s+V)dx}We^{\int(s+V)dx} \\ & = (s' + s^2)e^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx + \\ & + s(\int We^{\int(s+V)dx}dx)e^{-\int(s+V)dx} + \\ & - (s+V)e^{-\int(s+V)dx}\int We^{\int(s+V)dx}dx + W \\ & = (s' + s^2)e^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx + \\ & - Ve^{-\int(s+V)dx}\int We^{\int(s+V)dx}dx + W \end{aligned}$$

so:

$$y'' + Vy' = (s' + s^2)e^{\int sdx}\int(\int We^{\int(s+V)dx}dx)e^{-\int(2s+V)dx}dx +$$

$$
\begin{aligned}
&- Ve^{-\int (s+V)dx} \int We^{\int (s+V)dx} dx + W + \\
&+ Vse^{\int sdx} \int (\int We^{\int (s+V)dx} dx) e^{-\int (2s+V)dx} dx + \\
&+ Ve^{-\int (s+V)dx} \int We^{\int (s+V)dx} dx \\
&= (s' + s^2 + Vs) e^{\int sdx} \int (\int We^{\int (s+V)dx} dx) e^{-\int (2s+V)dx} dx + W
\end{aligned}
$$

$$\Rightarrow y'' + Vy' + (-s' - s^2 - sV)y = W$$

□

Corollary II.2: If V is an integrable function, and W is a twice integrable function, and y_h is a twice-differentiable function, such that:

$$y_h'' + Vy_h' + Qy_h = 0.$$

then, a particular solution, y_p , to:

$$y_p'' + Vy_p' + Qy_p = W,$$

is:

$$y_p = y_h \int \left(\frac{1}{y_h^2} \int Wy_h e^{\int Vdx} dx \right) e^{-\int Vdx} dx.$$

proof:

Again, as is well known from [3] : $y = e^{\int sdx} \Rightarrow y'' + Vy' + [-s' - s^2 - sV]y = 0.$

So, if y_h is a solution of: $y'' + Vy' + Qy = 0,$

then $\exists s = \frac{y_h'}{y_h}$ such that $Q = -s' - s^2 - sV.$

So, if y_h is a solution of: $y'' + Vy' + Qy = 0;$

then a particular solution y_p of $y'' + Vy' + Qy = W$ is given by:

$$y_p = y_h \int \left(\frac{1}{y_h^2} \int Wy_h e^{\int Vdx} dx \right) e^{-\int Vdx} dx.$$

Since:

$$e^{\int sdx} = e^{\int \frac{y_h'}{y_h} dx} = e^{\int \frac{dy_h}{y_h}} = e^{\ln y_h} = y_h$$

and:

$$e^{-2\int sdx} = e^{-2\int \frac{y_h'}{y_h} dx} = e^{-2\int \frac{dy_h}{y_h}} = e^{-2\ln y_h} = e^{\ln y_h^{-2}} = y_h^{-2} = \frac{1}{y_h^2}.$$

□

Corollary II.2 manifests the particllar solution formula for second order linear inhomogeneous ordinary diffrential equations. It shows that beyond equations with constant coefficients, that particular solutions can be quite complicated and demanding. Indeed, the method of undetermined coefficients is useless in more demanding problems, and, although using the Wronskian method can be applied, my Corollary II.2 formula requires only one of the homogeneous solutione be determined - and is far more compactly and elegantly written.

References

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[4] Zill, Dennis G.; *A First Course in Differential Equations with Applications, 4th Ed.*; PWS-KENT Publishing Company; Boston, MA; 1989.