Saint-Venant's Principle: Rationalized and Rational

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This is the abstract. The problem of statement of Saint-Venant's Principle is concerned. Statement of Boussinesq or Love is ambiguous so that its interpretations are in contradiction with each other. Rationalized Statement of Saint-Venants Principle of elasticity is suggested to rule out the ambiguity of Statements of Boussinesq and Love. Rational Saint-Venant's Principle is suggested to fit and guide applications of the principle to fields of continuum physics and cover the analogical case as well as the non-analogical case discovered and discussed in this paper . " Constraint-free " problems are suggested and " Constraint-free " Rational Saint-Venant's Principle or Rational Saint-Venant's Principle with Relaxed Boundary Condition is developed to generalize the principle and promote its applications to fields of continuum physics . Applications of Analogical Rational Saint-Venant's Principle and " Constraint-free " Rational Saint-Venant's Principle are exemplified, emphasizing " properness " of the boundary-value problems. Three kinds of properly posed boundary-value problems, i.e., the boundary-value problem with the undetermined boundary function, the boundary-value problem with the implicit boundary condition and the boundary-value problem with the explicit boundary condition, are suggested for both " constrained " and " constraint-free " problems.

1 Introduction

In 1855 Saint-Venant published his famous " principle " [1,2]. Boussinesq (1885) and Love (1927) announce statements of Saint-Venant's Principe respectively [3, 4]. Trusdell (1959) asserts, from the perspective of Rational Mechanics, that if Saint-Venant's Principle of equipollent loads is true, it " must be a mathematical consequence of the general equations " of linear elasticity [5]. It is obvious that Saint-Venant's Principle has become an academic attraction for contributors of Rational Mechanics [6-9] .

Authors understand the principle, actually, in different ways. We analyze the two different interpretations of the statements of Boussinesq and Love in this paper and find out that the interpretations are in contradiction with each other, and the statements are ambiguous. Therefore, statement of Saint-Venant's Principle needs to be rationalized. Then Rationalized Statement of Saint-Venants Principle , whose interpretation is itself other than something else, is suggested to rule out the elements of irrationality of Statements of Boussinesq and Love.

Authors try to use Saint-Venant's Principle by means of analogy in areas of continuum physics [6-9]. Rational Saint-Venant's Principle is suggested here to fit and guide applications of the principle to fields of continuum physics and cover the analogical case as well as the non-analogical case discovered and discussed in this paper.

We find that the constraint posed frequently by means of analogy for discussion of Saint-Venant's Principle is unnecessary for Saint-Venant's decay and could be ruled out , and so " constraint-free " problems are suggested and " Constraint-free " Rational Saint-Venant's Principle or Rational Saint-Venant's Principle with Relaxed Boundary Condition is developed in this paper to generalize the principle and promote its applications to fields of continuum physics.

Applications of Analogical Rational Saint-Venant's Principle and " Constraint-free " Rational Saint-Venant's Principle are exemplified, emphasizing " properness " of the boundary-value problems. Three ways are suggested to propose three kinds of properly posed boundary-value problems: the boundary-value problem with the undetermined boundary function, the boundary-value problem with the implicit boundary condition and the boundary-value problem with the explicit boundary condition, for both " constrained " and " constraint-free " problems.

2 Boussinesq's and Love's Statements of Saint-Venant's Principle

Boussinesq's Statement : " An equilibrated system of external forces applied to an elastic body, all of the points of application lying within a given sphere, produces deformation of negligible magnitude at distances from the sphere which are sufficiently large compared to its radius. " [3]

Love's Statement : " According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part. " [4]

3 Interpretations of the Statements

There are two different interpretations, literal and connotative, of Boussinesq's or Love's Statement:

1. Effect Localization : The equilibrium system of forces "produces" localized effects or deformations in the elastic body (literal interpretation);

2. Decay of Deformation Field: The deformation field, which is resulted from the solution of the elastic boundary-value problem for which the equilibrium system of forces formulates its near-boundary conditions (boundary conditions of the small near-end in the vicinity of the origin of coordinates for example), decays monotonically with the distances from the loaded boundary and tends to zero as the distances tend to infinite (connotative interpretation).

Examples cited here of the interpretations are:

Though he declares that Love's statement is not very clear, and suggests his statement of the principle, Mises does not go so far as to leave the way of establishing " produced " strains and stresses, which are negligible [10].

Sternberg supplies a general proof of the amended Saint-Venant's Principle suggested by Mises [11].

Zanaboni understands Saint-Venant's Principle in the way of Effect Localization, trying to formulate the principle in terms of decay of " work and energy " [12-15].

Timoshenko and Goodier write that " The expectation that such a system, applied to a small part of the surface of the body, would give rise to localized stress and strain only, was enunciated by Saint-Venant in 1855 and came to be known as Saint-Venant's principle. " However, they take the way of Decay of Deformation Field, by finding the eigen-solutions of the boundary-value problem of the rectangular region, so as to discuss the so called " End Effects " of the equilibrium system of forces [16].

The terminology " decay of Saint-Venant end effects " is mentioned in the research papers where deformation fields are dealt with [17, 18].

4 Ambiguity of the Interpretations

The interpretations are ambiguous and our arguments are :

A. If the Effect Localization is true, the equilibrium system of forces has no effect on the deformation at the infinity, and then the deformation at the infinite would be " produced " by the boundary forces on the sub-surface at the infinity, which is also an equilibrium system though all the surface-force components are equal to zero. However, at most, only three , instead of six, stress components for three dimensional problems or two, instead of three, stress components for two dimensional problems are determined to be zero by the stress boundary conditions at the infinity. Therefore, the deformation field at the infinity is uncertain, which is in contradiction with the interpretation of Decay of Deformation Field that implies that the deformation field at the infinity is definitely determined.

B. It is logical from A that the effects of the equilibrium system of forces should not be localized if the deformation field could be definitely and stably built up. Logically, the effects of the equilibrium system of forces extend to the the infinite, covering the entire deformation field. In fact, the stress boundary conditions formulated on the small part of the surface of the body by the equilibrium system of forces must be satisfied by the solutions of deformation, which should be interpreted as the effects of the equilibrium system of forces on the deformation field as a whole, without exception of the region at the infinity.

C. At most, only three , instead of six, stress components for three dimensional problems or two, instead of three, stress components for two dimensional problems are determined by the near-boundary conditions formulated by the equilibrium system of forces, and the effects of the equilibrium system of forces are uncertain and vague even in the closest vicinity of the loading. Then serious and difficult questions are unavoidable :

Does the zero-force system at the infinity extend its effects to the vicinity of the near-boundary and make the stress components otherwise uncertain there fixed ?

What effects does the equilibrium system of forces exert over the deformation field, and what else effects does the zero-force system apply on it ?

Is decay of deformations, if any, is manifestation of the effects of the equilibrium system of forces alone ?

It is impossible to make any judgement whether the effects of the equilibrium system of forces decay with distances or not unless those questions are fully answered. There exists even such a possibility that the effects at the infinity are as active as in the vicinity of the near-boundary though the deformations may decay, considering the argument B .

5 Rationalized Statement of Saint-Venant's Principle of Elasticity

We see, from the last section, that the interpretations of the statement of Boussinesq or Love are ambiguous . Furthermore, it is logical to infer that the elements of irrationality of the statement of Boussinesq or Love itself are behind the ambiguity. Therefore, it is necessary to rationalize statement of Saint-Venant's Principle and state the principle in an unambiguous way so that its interpretation is the statement itself other than something else. We suggest the following rationalized statement:

" If a properly posed boundary-value problem of elasticity is defined in a body of infinite dimension and on its boundary and the boundary condition of a small sub-surface of the body is formulated by an equilibrium system of forces, otherwise the body would be free, the solutions of deformation decay monotonically with the distances from the loaded sub-surface and tend to zero as the distances tend to infinite. ”

It is extremely important that the rationalized statement features “ properness ” of the boundary-value problem of elasticity and distinguishes itself from the Decay of Deformation Field interpretation of Boussinesq’s or Love’s Statement by emphasizing “ a properly posed boundary-value problem ”. Essentially, the issue of Saint-Venant’s Principle is of “ properness ” of boundary-value problems posed for elasticity. No ill-posed problem connotes solutions of Saint-Venant’s decay.

6 Rational Saint-Venant’s Principle

Authors try to promote application of Saint-Venant’s Principle by means of analogy widely in areas of continuum physics [6-9]. One of the typical examples is the Model Problem where the “ self-equilibration ” condition is posed on, say, the entry distribution of heat flux [6, 7]. “ End Effects ” is mentioned , for example, for the problem of Stokes flow [19]. It is necessary and significant to develop a rational Saint-Venant’s Principle, which fits and guides applications of the principle to fields of continuum physics and covers the analogical case of application as well as the non-analogical case discovered and discussed in this paper (See Section 7, Section 8 and Section 9). We suggest the rational statement as:

“ For a properly posed boundary-value problem of continuum physics, which is defined in an infinite domain and on its boundary and whose only non-zero function of boundary condition is defined on a small sub-boundary, the solutions of the problem decay monotonically with the distances from the sub-boundary and tend to zero as the distances tend to infinite. ”

We will exemplify the Rational Saint-Venant’s Principle in the following sections.

7 Rational Saint-Venent’s Principle concerning Laplace Equation in Cartesian Coordinates

7.1 Application 1: Analogical Rational Saint-Venent’s Principle concerning Laplace Equation in Cartesian Coordinates

7.1.1 Boundary-value Problem

The boundary-value problem in Cartesian Coordinates is:

$$\Delta u = u_{,\alpha\alpha} = 0 \quad on\ D \tag{1}$$
$$D = \{(x_1, x_2)|0 < x_1 < l,\ -c < x_2 < c\ \},$$

$$x_2 = \pm c: \quad u = 0, \tag{2}$$

$$x_1 = l: \quad u \to 0 \quad as\ \ l \to \infty, \tag{3}$$

$$x_1 = 0: \quad u = f(x_2), \tag{4}$$

$$\frac{1}{2c}\int_{-c}^{c} f(x_2)dx_2 = 0 \tag{5}$$

which is the constraint of the mean of the undetermined function $f(x_2)$ and is understood as the “ self-equilibration ” condition of the function as well.

The field function u could have a number of physical interpretations such as temperature field, electrical potential of an electrostatic field, potential of electrical current and velocity potential of fluid flow etc.

7.1.2 Solutions of the Problem

1. It is from Eq.(1), Eq.(2) and Eq.(3) that

$$u(x_1,x_2) = \sum_{n=1}^{N} A_n \sin(\frac{n\pi}{c}x_2)\ \exp(-\frac{n\pi}{c}x_1) + \sum_{k=0}^{K} B_k \cos((k+\frac{1}{2})\frac{\pi}{c}x_2)\ \exp(-(k+\frac{1}{2})\frac{\pi}{c}x_1) \tag{6}$$

where

$$N < \infty, \ \ K < \infty,$$

and B_k are controlled by

$$\sum_{k=0}^{K} (-1)^k \frac{4c}{(2k+1)\pi} B_k = 0 \tag{7}$$

because of the constraint Eq.(5).

From Eq.(6) we have

$$\lim_{x_1 \to \infty} u(x_1,x_2) = 0, \tag{8}$$

which is of Saint-Venant’s decay.

2. The implicit solution of $f(x_2)$ is from Eq.(4) and Eq.(6) and given by

$$\int_{-c}^{c} f(x_2)\sin(\frac{n\pi}{c}x_2)dx_2 = cA_n \ \ (n = 1,2,...N), \tag{9}$$
$$\int_{-c}^{c} f(x_2)\sin(\frac{n\pi}{c}x_2)dx_2 = 0 \ \ \ (n > N, N < \infty),$$
$$\int_{-c}^{c} f(x_2)\cos((k+\frac{1}{2})\frac{\pi}{c}x_2)dx_2 = cB_k \ \ \ (k = 1,2,...K),$$
$$\int_{-c}^{c} f(x_2)\cos((k+\frac{1}{2})\frac{\pi}{c}x_2)dx_2 = 0 \ \ (k > K, K < \infty).$$

3. The explicit solution of $f(x_2)$ is identified from Eq.(4) and Eq.(6) as

$$f(x_2) = \sum_{n=1}^{N} A_n \sin(\frac{n\pi}{c}x_2) + \sum_{k=0}^{K} B_k \cos((k+\frac{1}{2})\frac{\pi}{c}x_2), \quad (10)$$
$$(N < \infty, K < \infty),$$

where B_k are controlled by Eq.(7) because of the constraint Eq.(5).

We will prove the Uniqueness of Solution of the problem in the next subsection as an argument for properness of the posed problem.

7.1.3 Uniqueness of Solution

If u_1 and u_2 are two solutions of the problem , suppose

$$\bar{u} = u_1 - u_2. \quad (11)$$

Then we have

$$\Delta\bar{u} = \bar{u}_{,\alpha\alpha} = 0, \quad on\ D \quad (12)$$
$$D = \{(x_1, x_2)|0 < x_1 < l,\ -c < x_2 < c\},$$

$$x_2 = \pm c: \quad \bar{u} = 0, \quad (13)$$

$$x_1 = l: \quad \bar{u} \to 0 \quad as \quad l \to \infty, \quad (14)$$

$$x_1 = 0: \quad \bar{u} = 0, \quad (15)$$

$$\frac{1}{2c}\int_{-c}^{c} \bar{u}dx_2 = 0. \quad (16)$$

It is easy to find the solution

$$\bar{u} = 0, \quad (17)$$

then

$$u_1 = u_2, \quad (18)$$

and Uniqueness of Solution of the problem is proved. This proof should be added to the argument of the properness of the boundary-value problem posed by Equations (1) - (5).

7.1.4 Properly Posed Boundary-value Problems of Analogical Rational Saint-Venent's Principle concerning Laplace Equation in Cartesian Coordinates

Then we propose the three kinds of boundary-value problems for the discussion of Saint-Venant's Principle:

1. The boundary-value problem with the undetermined boundary function $f(x_2)$. The problem, whose solutions are Eq.(6), Eq.(8), Eq.(9) and Eq.(10), is posed by Equations (1) - (5), as is discussed in Sec.7.1.1 and Sec.7.1.2 .

2. The boundary-value problem with the implicit boundary condition of $f(x_2)$. The problem , whose solutions are Eq.(6), Eq.(8) and Eq.(10) , is posed by Equations (1) - (5) and Eq.(9) .

3. The boundary-value problem with the explicit boundary condition of $f(x_2)$. The problem, whose solutions are Eq.(6) and Eq.(8), is posed by Equations (1) - (4) and Eq.(10).

Each boundary-value problem mentioned yields Saint-Venant's decay of u by Eq.(6) and Eq.(8). Therefore, each of them is a boundary-value problem properly posed for the Rational Saint-Venant's Principle. The principle is of " analogical type " because Eq.(5) (or Eq.(7)), the " self-equilibration " condition, is posed for the problem.

7.2 Application 2: Properly Posed Boundary-value Problems of " Constraint-free " Rational Saint-Venent's Principle concerning Laplace Equation in Cartesian Coordinates

In fact, the constraint Eq.(5) is not necessary for yielding Saint-Venant's decay of Eq.(6) and Eq.(8), and could be ruled out when proposing the boundary-value problems. We propose the three kinds of boundary-value problems for discussion of the " Constraint-free " Rational Saint-Venant's Principle or the Rational Saint-Venant's Principle with Relaxed Boundary Condition:

1. The boundary-value problem with the undetermined boundary function $f(x_2)$. The problem, whose solutions are Eq.(6), Eq.(8), Eq.(9) and Eq.(10) excluding the control Eq.(7), is posed by Equations (1) - (4).

2. The boundary-value problem with the implicit boundary condition of $f(x_2)$. The problem , whose solutions are Eq.(6), Eq.(8) and Eq.(10) excluding the control Eq.(7) , is posed by Equations (1) - (4) and Eq.(9).

3. The boundary-value problem with the explicit boundary condition of $f(x_2)$. The problem , whose solutions are Eq.(6) and Eq.(8) excluding the control Eq.(7) , is posed by Equations (1) - (4) and Eq.(10) excluding the control Eq.(7).

Each boundary-value problem mentioned here yields Saint-Venant's decay of u by Eq.(6) and Eq.(8). Therefore, each of them is a constraint-free boundary-value problem properly posed for the " Constraint-free " Rational Saint-Venant's Principle in Cartesian coordinates. The principle is of " non-analogical type " because Eq.(5) (or Eq.(7)), the " self-equilibration " condition, is excluded from the problem.

8 Rational Saint-Venent's Principle concerning Laplace Equation in Cylindrical Coordinates

8.1 Application 3: Analogical Rational Saint-Venent's Principle concerning Laplace Equation in Cylindrical Coordinates

8.1.1 Boundary-value Problem

The axisymmetrical problem is dealt with and the boundary-value problem in Cylindrical Coordinates is:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{\partial^2 u}{\partial z^2} = 0 \quad on\ R \tag{19}$$
$$R = \{(r,\theta,z)|0 < r < a,\ 0 \le \theta < 2\pi, 0 < z < l\ \},$$

$$r = 0: \ \ u < \infty, \tag{20}$$

$$r = a: \ \ u = 0, \tag{21}$$

$$z = l: \ \ u \to 0 \ \ as\ \ l \to \infty, \tag{22}$$

$$z = 0: \ \ u = f(r), \tag{23}$$

$$\frac{2}{a^2}\int_0^a f(r) r dr = 0 \tag{24}$$

which is the constraint of the mean of the undetermined function $f(r)$ and is understood as the " self-equilibration " condition of the function as well.

8.1.2 Solutions of the Problem

1. It is from Eq.(19), Eq.(20), Eq.(21) and Eq.(22) that

$$u = \sum_{n=1}^{N} D_n J_0(c_n \frac{r}{a}) \exp(-\frac{c_n}{a} z), \tag{25}$$

where

$$N < \infty,$$

$J_0(c_n \frac{r}{a})$ is the Bessel function , c_n is the n^{th} positive zero of J_0, D_n are controlled by

$$\sum_{n=1}^{N} \frac{D_n}{c_n} J_1(c_n) = 0 \tag{26}$$

because of the constraint Eq.(24).

From Eq.(25) we have

$$\lim_{z\to\infty} u(r,z) = 0, \tag{27}$$

which is of Saint-Venant's decay.

2. The implicit solution of $f(r)$ is from Eq.(23) and Eq.(25) and given by

$$\int_0^a f(r) J_0(c_n \frac{r}{a}) r dr = \frac{a^2 J_1^2(c_n)}{2} D_n \quad (n = 1,2,...N), \tag{28}$$
$$\int_0^a f(r) J_0(c_n \frac{r}{a}) r dr = 0 \quad (n > N, N < \infty).$$

3. The explicit solution of $f(r)$ is identified from Eq.(23) and Eq.(25) as

$$f(r) = \sum_{n=1}^{N} D_n J_0(c_n \frac{r}{a}), \ \ (N < \infty), \tag{29}$$

where D_n are controlled by Eq.(26).

8.1.3 Properly Posed Boundary-value Problems of Analogical Rational Saint-Venent's Principle concerning Laplace Equation in Cylindrical Coordinates

Then we propose the three kinds of boundary-value problems for the discussion of Saint-Venant's Principle of the axisymmetrical problem in cylindrical coordinates:

1. The boundary-value problem with the undetermined boundary function $f(r)$. The problem , whose solutions are Eq.(25), Eq.(27), Eq.(28), Eq.(29) and Eq.(26), is posed by Equations (19) - (24), as is discussed in Sec. 8.1.1 and Sec.8.1.2.

2. The boundary-value problem with the implicit boundary condition of $f(r)$. The problem , whose solutions are Eq.(25), Eq.(27) , Eq.(29) and Eq.(26), is posed by Equations (19) - (24) and Eq.(28).

3. The boundary-value problem with the explicit boundary condition of $f(r)$. The problem, whose solutions are Eq.(25) and Eq.(27), is posed by Equations (19) - (23) , Eq.(29) and Eq.(26) .

Each boundary-value problem mentioned yields Saint-Venant's decay of u by Eq.(25) and Eq.(27). Therefore, each of them is an axisymmetrical boundary-value problem properly posed for the Rational Saint-Venant's Principle in Cylindrical Coordinates (the proof of Uniqueness of Solution is omitted). The principle is of " analogical type " because Eq.(24) (or Eq.(26)), the " self-equilibration " condition, is posed for the problem.

8.2 Application 4: Properly Posed Boundary-value Problems of " Constraint-free " Rational Saint-Venent's Principle concerning Laplace Equation in Cylindrical Coordinates

In fact, the constraint Eq.(24) is not necessary for yielding Saint-Venant's decay of Eq.(25) and Eq.(27), and could be ruled out when proposing the boundary-value problem. We propose the three kinds of boundary-value problems for discussion of the " Constraint-free " Rational Saint-Venant's Principle :

1. The boundary-value problem with the undetermined boundary function $f(r)$. The problem, whose solutions are Eq.(25), Eq.(27), Eq.(28)and Eq.(29) excluding the control Eq.(26), is posed by Equations (19) - (23).

2. The boundary-value problem with the implicit boundary condition of $f(r)$. The problem , whose solutions are Eq.(25), Eq.(27) and Eq.(29) excluding the control Eq.(26), is posed by Equations (19) - (23) and Eq.(28).

3. The boundary-value problem with the explicit boundary condition of $f(r)$. The problem , whose solutions are Eq.(25) and Eq.(27) excluding the control Eq.(26) , is posed by Equations (19) - (23) and Eq.(29) excluding the control Eq.(26).

Each boundary-value problem mentioned yields Saint-Venant's decay of u by Eq.(25) and Eq.(27). Therefore, each of them is a constraint-free axisymmetrical boundary-value problem properly posed for the " Constraint-free " Rational Saint-Venant's Principle in Cylindrical Coordinates. The principle is of " non-analogical type " because Eq.(24) (or Eq.(26)), the " self-equilibration " condition, is excluded from the problem.

9 Rational Saint-Venent's Principle concerning Laplace Equation in Spherical Coordinates

9.1 Application 5: Analogical Rational Saint-Venent's Principle concerning Laplace Equation in Spherical Coordinates

9.1.1 Boundary-value Problem

The axisymmetrical problem is dealt with and the boundary-value problem in Spherical Coordinates is:

$$\Delta u = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial u}{\partial R}) + \frac{1}{R^2\sin\psi}\frac{\partial}{\partial\psi}(\sin\psi\frac{\partial u}{\partial\psi}) = 0 \quad (30)$$

$$on\ D$$

$$D = \{(R,\psi,\theta)|a < R < b, 0 \leq \psi \leq \pi, 0 \leq \theta < 2\pi\},$$

$$R = b: \quad u \to 0 \quad as \quad b \to \infty, \quad (31)$$

$$R = a: \quad u = f(\psi), \quad (32)$$

$$\frac{1}{2}\int_0^\pi f(\psi)\sin\psi d\psi = 0 \quad (33)$$

which is the constraint of the mean of the undetermined function $f(\psi)$ and is understood as the " self-equilibration " condition of the function as well.

9.1.2 Solutions of the Problem

1. It is from Eq.(30) and Eq.(31) that

$$u = \sum_{n=0}^{N} C_n R^{-(n+1)} P_n(\cos\psi), \quad (34)$$

where

$$N < \infty,$$

$P_n(\cos\psi)$ is the Legendre function , C_0 is controlled by

$$C_0 = 0 \quad (35)$$

because of the constraint Eq.(33).

From Eq.(34) we have

$$\lim_{R\to\infty} u(R,\psi) = 0, \quad (36)$$

which is of Saint-Venant's decay.

2. The implicit solution of $f(\psi)$ is from Eq.(32) and Eq.(34) and given by

$$\int_0^\pi f(\psi)P_n(\cos\psi)\sin\psi d\psi = \frac{2}{2n+1}a^{-(n+1)}C_n \quad (37)$$
$$(n = 0,1,2,...N),$$
$$\int_0^\pi f(\psi)P_n(\cos\psi)\sin\psi d\psi = 0 \quad (n > N, N < \infty).$$

3. The explicit solution of $f(\psi)$ is identified from Eq.(32) and Eq.(34) as

$$f(\psi)=\sum_{n=0}^{N}C_n a^{-(n+1)}P_n(\cos\psi),\ \ (N<\infty), \tag{38}$$

where C_0 is controlled by Eq.(35).

9.1.3 Properly Posed Boundary-value Problems of Analogical Rational Saint-Venent's Principle concerning Laplace Equation in Spherical Coordinates

Then we propose the three kinds of boundary-value problems for the discussion of Saint-Venant's Principle of the axisymmetrical problem in spherical coordinates:

1. The boundary-value problem with the undetermined boundary function $f(\psi)$. The problem , whose solutions are Eq.(34), Eq.(36), Eq.(37) and Eq.(38), is posed by Equations (30) - (33), as is discussed in Sec. 9.1.1 and Sec. 9.1.2.

2. The boundary-value problem with the implicit boundary condition of $f(\psi)$. The problem , whose solutions are Eq.(34), Eq.(36) and Eq.(38), is posed by Equations (30) - (33) and Eq.(37) .

3. The boundary-value problem with the explicit boundary condition of $f(\psi)$. The problem, whose solutions are Eq.(34) and Eq.(36), is posed by Equations (30) - (32) and Eq.(38) .

Each boundary-value problem mentioned yields Saint-Venant's decay of u by Eq.(34) and Eq.(36). Therefore, each of them is an axisymmetrical boundary-value problem properly posed for the Rational Saint-Venant's Principle in Spherical Coordinates (the proof of Uniqueness of Solution is omitted). The principle is of " analogical type " because Eq.(33) (or Eq.(35)), the " self-equilibration " condition, is posed for the problem.

9.2 Application 6: Properly Posed Boundary-value Problems of " Constraint-free " Rational Saint-Venent's Principle concerning Laplace Equation in Spherical Coordinates

In fact, the constraint Eq.(33) is not necessary for yielding Saint-Venant's decay of Eq.(34) and Eq.(36), and could be ruled out when proposing the boundary-value problems. We propose the three kinds of boundary-value problems for discussion of the "Constraint-free " Rational Saint-Venant's Principle :

1. The boundary-value problem with the undetermined boundary function $f(\psi)$. The problem , whose solutions are Eq.(34), Eq.(36), Eq.(37) and Eq.(38) excluding the control Eq.(35), is posed by Equations (30) - (32).

2. The boundary-value problem with the implicit boundary condition of $f(\psi)$. The problem , whose solutions are Eq.(34), Eq.(36) and Eq.(38) excluding the control Eq.(35), is posed by Equations (30) - (32) and Eq.(37) .

3. The boundary-value problem with the explicit boundary condition of $f(\psi)$. The problem, whose solutions are Eq.(34) and Eq.(36) excluding the control Eq.(35), is posed by Equations (30) - (32) and Eq.(38) excluding the control Eq.(35).

Each boundary-value problem mentioned yields Saint-Venant's decay of u by Eq.(34) and Eq.(36). Therefore, each of them is a "constraint-free" axisymmetrical boundary-value problem properly posed for the "Constraint-free " Rational Saint-Venant's Principle in Spherical Coordinates. The principle is of " non-analogical type" because Eq.(33) (or Eq.(35)), the " self-equilibration " condition, is excluded from the problem.

10 Conclusion

A. Statement of Boussinesq or Love of Saint-Venants Principle is ambiguous so that its interpretations are in contradiction with each other. Therefore, statement of the principle needs to be rationalized.

B. Rationalized Statement of Saint-Venant's Principle of elasticity , whose interpretation is itself other than something else, is suggested to rule out the ambiguity of statements of Boussinesq and Love.

C. Rational Saint-Venant's Principle is suggested to fit and guide applications of the principle to fields of continuum physics and cover the analogical as well as the non-analogical case of application.

D. " Constraint-free " problems are suggested and " Constraint-free " Rational Saint-Venant's Principle or Rational Saint-Venant's Principle with Relaxed Boundary Condition is developed to generalize the principle and promote its applications to fields of continuum physics.

E. Applications of Analogical Rational Saint-Venant's Principle and " Constraint-free " Rational Saint-Venant's Principle are exemplified , emphasizing " properness " of the boundary-value problems. Three ways are suggested to propose three kinds of properly posed boundary-value problems : the boundary-value problem with the undetermined boundary function, the boundary-value problem with the implicit boundary condition and the boundary-value problem with the explicit boundary condition, for both " constrained " and " constraint-free " problems.

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