The irrelevance of Bell inequalities in Physics : Comments on the DRHM paper

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Abstract

It was shown in [1], cited in the sequel as DRHM, that upon a correct use of the respective statistical data, the celebrated Bell inequalities cannot be violated by quantum systems. This paper presents in more detail the surprisingly elementary, even if rather subtle related basic argument in DRHM, and does so together with a few comments which, hopefully, may further facilitate its wider understanding.

1. Introduction

The paper [1], cited in the sequel as DRHM, presents an argument which shows that the violation of the celebrated Bell inequalities in quantum mechanics is due only to a rather elementary, even if somewhat subtle error made in the way the statistical data are handled. In this regard, it is particularly useful and timely for a proper pursuit, and thus understanding, of quantum mechanics to become aware of the following :

- There is a growing literature, as also pointed out in DRHM, which presents dissent with the celebrated result of Bell, [2].

- In quantum mechanics, quite unlike in other branches of physics, there are well established rather important facts which happen not to be widely enough known, and consequently, opinions denying them still flourish in many quarters. One such example is the urban legend that there cannot be hidden variables, since von Neumann proved that back in the early 1930s. And yet in 1935, Grete Hermann discovered an error in it which invalidates that proof, [3]. Another example is the 1995 rather well known and often cited book of Asher Peres, [5], which clearly indicates on the back cover that "It makes no use of the uncertainty principle or other ill-defined notions." And till today, a much limited awareness of that fact can be experienced in the quantum community, not to mention the lack of any more serious study which would look into the relationship between the uncertainty principle and quantum mechanics. On the contrary, that principle is still widely seen as being foundational, and countless consequences of it are deduced inside and outside of the realms of the quanta. And if another example may be needed, one can mention the 1935 statement of von Neumann, [6], that "I would like to make a confession which may seem immoral: I do not believe in Hilbert space anymore." Nevertheless, the Quantum Mechanics 101 texts still introduce and build the whole theory on Hilbert spaces.

In view of such a state of affairs, it may be appropriate to approach the result in DRHM in a way less usual nowadays when, due to information overload, one simply and instantly tends to disregard many things, and do so being further impelled by what one likes to consider ones very good and highly reliable "physical intuition".

Last, and not least, the result in DRHM is in fact of extreme importance. Indeed, the alleged violation of the Bell inequalities by quantum systems attained from the start its celebrity status, and consequently shook up considerably the realms of quantum foundations after four decades of sole supremacy of the Copenhagen Interpretation, due to what was seen as their obvious implications regarding such fundamental quantum related issues as locality, realism, and so on. In this regard there were some who even saw the Bell inequalities as a first in the history of science and philosophy, when basic philosophical principles could effectively be subjected to empirical verification.

But now, in view of DRHM, all such thinking has to fall away since the Bell inequalities turn out to fail to be violated by quantum systems. It is in this sense, therefore, that the Bell inequalities can be considered quite irrelevant.

2. Experimental Data Collection and Processing

We are, in physics, on various occasions interested in obtaining, and then processing data from measurements. The first aspect one has to take into account is how the data is collected, and the second one is post-measurement analysis, that is, how this data is processed. These two stages may interact and thus condition one another. Also, often they seem so trivially obvious, and thus quite automatically satisfied in our practice that - as it turns out in the case of the Bell inequalities - not sufficient care is exhibited in each and every situation when dealing with data.

In the situation relevant for the Bell inequalities, as dealt with in DRHM and detailed here, the data from measurements and their subsequent processing have several specific features which will, step by step, be made apparent in the sequel.

First, we start with sets of individual data S each of which will always only take one of two values. For convenience, these two values can be chosen as 1 and -1. Thus for these individual data we have $S \in \{-1, 1\}$.
These data S are organized in n-tuples $(S_1, \ldots, S_n)$, where $n \geq 1$ can in principle be any integer, although regarding the Bell inequalities it will be sufficient to consider only $n = 1, 2, 3, 4$. Any individual measurement we perform leads to such an n-tuple. The number of such measurements can be given by any integer number $M \geq 1$, thus leading to the following organization of data

$$Y^{(n)} = \{ \, (\, S_{1,\alpha}, \ldots, S_{n,\alpha} \,) \mid \alpha = 1, \ldots, M \, \} \tag{1}$$

where we recall that $S_{i,\alpha} = \pm 1$.

It is sometime convenient to see each such $Y^{(n)}$ in (1) as the corresponding $M \times n$ matrix

$$Y^{(n)} = \begin{pmatrix} S_{1,1} & \cdots & S_{n,1} \\ S_{1,2} & \cdots & S_{n,2} \\ \vdots & & \\ S_{1,M} & \cdots & S_{n,M} \end{pmatrix} \tag{2}$$

When different runs of measurements of type (1), (2) of the same experiment are performed, subsequent runs can respectively be denoted by

$$\widehat{Y}^{(n)} = \{ (\widehat{S}_{1,\alpha}, \ldots, \widehat{S}_{n,\alpha}) \mid \alpha = 1, \ldots, M \}$$

$$\widetilde{Y}^{(n)} = \{ (\widetilde{S}_{1,\alpha}, \ldots, \widetilde{S}_{n,\alpha}) \mid \alpha = 1, \ldots, M \}$$

and so on.

Now, as a step in the *post-measurement* analysis, one may be interested in subsets of a data set $Y^{(n)} = \{ (S_{1,\alpha}, \ldots, S_{n,\alpha}) \mid \alpha = 1, \ldots, M \}$, subsets corresponding only to $m < n$ data $S_{i,\alpha}$ in each n-tuple in $(S_{1,\alpha}, \ldots, S_{n,\alpha})$. Thus from the respective n-tuples one removes $n-m$ data $S_{i,\alpha}$.
One way to do that is to choose $m < n$ columns $1 \leq i_1 < \ldots < i_m \leq n$ from the matrix (2), and thus obtain the $M \times m$ matrix

$$\Gamma^{(n)}_{i_1, i_2, \ldots, i_m} = \begin{pmatrix} S_{i_1,1} & \cdots & S_{i_m,1} \\ S_{i_1,2} & \cdots & S_{i_m,2} \\ \vdots & & \\ S_{i_1,M} & \cdots & S_{i_m,M} \end{pmatrix} \tag{3}$$

instead of $Y^{(n)}$ in the matrix (2).

For instance, let us consider $n = 3$ and a corresponding $Y^{(3)} = \{ (S_{1,\alpha}, S_{2,\alpha}, S_{3,\alpha}) \mid \alpha = 1, \ldots, M \}$, according to (1), (2).

Let us now take $m = 1$, that is, we shall only select one single $S_{i,\alpha}$ which corresponds to one certain given column $i \in \{1, 2, 3\}$, from each triple $(S_{1,\alpha}, S_{2,\alpha}, S_{3,\alpha})$ in $Y^{(3)}$. According to (3), we denote then by $\Gamma_i^{(3)}$ the set of reduced data thus obtained, which is given by only one of the three outcomes in each triple $(S_{1,\alpha}, S_{2,\alpha}, S_{3,\alpha})$, namely, $S_{i,\alpha}$, where $\alpha = 1, \ldots, M$. It follows that the other two data $S_{j,\alpha}$ in each triple, with $j \in \{1, 2, 3\}$, $j \neq i$, are removed in this specific example. In this way

$$\Gamma_i^{(3)} = \{ S_{i,\alpha} \mid \alpha = 1, \ldots, M \}. \tag{4}$$

Similarly, if $i, j \in \{1, 2, 3\}$, with $i < j$, then according to (3) we have $\Gamma_{ij}^{(3)}$ given by the set of pairs of data chosen out of the triples in $Y^{(3)}$, as follows

$$\Gamma_{ij}^{(3)} = \{ (S_{i,\alpha}, S_{j,\alpha}) \mid \alpha = 1, \ldots, M \}. \tag{5}$$

As it happens, however, related to the Bell inequalities, there is no need to go beyond (4) and (5) in the above kind of pre-processing of data.

As for the general situation in (1) - (3), in the particular case when $m = n$, then clearly $i_1 = 1, \ldots, i_n = n$, and we simply have $\Gamma_{i_1, \ldots, i_n}^{(n)} = Y^{(n)}$.

When on the other hand $m < n$ in the general situation in (1) - (3), then every given particular $\Gamma_{i_1, \ldots, i_m}^{(n)}$ can, of course, be seen as being itself some specific $Y^{(m)}$.

3. A Crucially Important Fact in Processing Data

Let us illustrate the source of one of the possible *errors* made in the processing of measurement data, namely, the particular error which has so far led to the *wrong* conclusion that quantum systems violate the Bell inequalities. In general terms, this error in processing measurement data is in the typically *inadmissible* relationship which turns out to be assumed between the set

$$\Gamma^{(n)}_{i_1,\ldots,i_m}, \quad 1 \leq i_1 < \ldots < i_m \leq n \tag{6}$$

with a given $m < n$, a set selected out from a given and well defined $Y^{(n)}$, and on the other hand, a *presumed* to exist set

$$Y^{(m)},\ \widehat{Y}^{(m)},\ \widetilde{Y}^{(m)}, \ldots \tag{7}$$

Of relevance for the Bell inequalities, it is crucially important to note the particular case of the above inadmissible relationship in (6), (7), namely, when $n = 3$ and $m = 2$, and one consequently deals with a set of $\Gamma^{(3)}_{ij}, \ \ 1 \leq i < j \leq 3$, a set obtained according to (3) from a well defined, given $Y^{(3)}$, and on the other hand with certain presumed to exist $Y^{(m)}, \widehat{Y}^{(m)}, \widetilde{Y}^{(m)}$.

In some more detail, suppose we are given an experiment and data are collected so as to form the set of triplets $Y^{(3)}$. Suppose further that, according to (3), we extract from $Y^{(3)}$ the three sets of pairs

$$\Gamma^{(3)}_{12},\ \Gamma^{(3)}_{13},\ \Gamma^{(3)}_{23} \tag{8}$$

Now, this step in (8) may at first, and without sufficient care, appear to be equivalent to having collected three sets of pairs of data

$$Y^{(2)},\ \widehat{Y}^{(2)},\ \widetilde{Y}^{(2)} \tag{9}$$

The rather elementary and simple *error* in such an assumption, however, is in the fact that, in general, the data in (8) came from one single $Y^{(3)}$, that is, from M *triplets* of data

$$S_{1,\alpha}, S_{2,\alpha}, S_{3,\alpha}, \quad 1 \leq \alpha \leq M$$

while on the other hand, the data in (9) may in general come from no less than M *sextuples* of data, namely

$$S_{1,\alpha}, S_{2,\alpha}, \widehat{S}_{1,\alpha}, \widehat{S}_{2,\alpha}, \widetilde{S}_{1,\alpha}, \widetilde{S}_{2,\alpha}, \quad 1 \leq \alpha \leq M$$

thus the latter data may in general contain *more* information than the former ones.

For the sake of clarity related to the above rather elementary and simple error in identifying (8) with (9), an error which for long has been missed, let us for a moment elaborate on it in some further detail, even if a more complicated notation is needed for that purpose.

Suppose given a certain system Ω. We perform M times on Ω the following procedure. Each time we make three measurements, and the successive results we denote by

$$S^3_{1,\alpha}, S^3_{2,\alpha}, S^3_{3,\alpha}$$

where $\alpha = 1, \ldots, M$. In this way we obtain the data, see (1), in $Y^{(3)} = \{ \, (S^3_{1,\alpha}, S^3_{2,\alpha}, S^3_{3,\alpha}) \mid \alpha = 1, \ldots, M \, \}$.

Now independently of Ω, and according to (3), for each pair (i, j), with $1 \leq i < j \leq 3$, we perform on the $Y^{(3)}$ obtained above the following M selection procedures. From each triplet $(S^3_{1,\alpha}, S^3_{2,\alpha}, S^3_{3,\alpha})$ we select the pair

$$(S^3_{i,\alpha}, S^3_{j,\alpha})$$

Thus we obtain the three sets of pairs, see (8)

$$\Gamma^{(3)}_{ij} = \{ \, (S^3_{i,\alpha}, S^3_{j,\alpha}) \mid \alpha = 1, \ldots, M \, \}, \quad 1 \leq i < j \leq 3 \qquad (10)$$

This means that we have, in general, three different sets of pairs. However, in view of the way they have been obtained, they *cannot* in general be completely independent, since the *six* columns in their three $M \times 2$ matrices which form them, see (3), come from only the *three* columns of $Y^{(3)}$, see (2).

On the other hand, as far as (9) is concerned, the situation is as follows. First, $Y^{(2)}$ is obtained from the system Ω, by performing

M times two measurements, and the successive results we denote by $S^2_{1,\alpha}, S^2_{2,\alpha}$, where $\alpha = 1, \ldots, M$. In this way, we obtain

$$Y^{(2)} = \{ (S^2_{1,\alpha}, S^2_{2,\alpha}) \mid \alpha = 1, \ldots, M \} \tag{11}$$

So far, we only have one set of pairs, and thus it is necessary to perform two further runs, similar to that which gave us $Y^{(2)}$. These two additional runs we denote by

$$\widehat{Y}^{(2)} = \{ (\widehat{S}^2_{1,\alpha}, \widehat{S}^2_{2,\alpha}) \mid \alpha = 1, \ldots, M \} \tag{12}$$

$$\widetilde{Y}^{(2)} = \{ (\widetilde{S}^2_{1,\alpha}, \widetilde{S}^2_{2,\alpha}) \mid \alpha = 1, \ldots, M \} \tag{13}$$

Clearly, (11) - (13) may in general contain more information about the system Ω, than (10).

And as shown in DRHM, and seen also in the sequel, it is the error in equating three pairs (10)) selected from one and the same set of triples, with pairs selected from three different sets of pairs (11) - (13), which leads to the wrong conclusion that quantum systems violate the Bell inequalities.

4. Averages

The Bell inequalities, as well as their extensions considered in DRHM, involve *averages* of data defined as follows. Suppose given $Y^{(n)} = \{ (S_{1,\alpha}, \ldots, S_{n,\alpha}) \mid \alpha = 1, \ldots, M \}$, then we consider the corresponding average

$$F^{(n)} = (\sum_{1 \leq \alpha \leq M} (S_{1,\alpha} \ldots S_{n,\alpha}))/M \tag{14}$$

while for $m < n$ and $1 \leq i_1 < \ldots < i_m \leq n$, we consider the average

$$F^{(n)}_{i_1,\ldots,i_m} = (\sum_{1 \leq \alpha \leq M} (S_{i_1,\alpha} \ldots S_{i_m,\alpha}))/M \tag{15}$$

5. The Boole Inequalities

It was shown in [7] by Boole that in every experiment in which one collects a data set of triples $Y^{(3)}$ and then associates with them the data sets of pairs $\Gamma_{12}^{(3)}, \Gamma_{13}^{(3)}$ and $\Gamma_{23}^{(3)}$, the following inequalities will hold

$$| \, F_{ij}^{(3)} \pm F_{ik}^{(3)} \, | \leq 1 \pm F_{jk}^{(3)} \tag{16}$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2,)$.

Of course, (16) comes from the very simple property of every triplet (S_1, S_2, S_3), with $S_i \in \{-1, 1\}$, for $1 \leq i \leq 3$, namely

$$| \, S_i S_j \pm S_i S_k \, | \leq 1 \pm S_j S_k \tag{17}$$

with $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2,)$.

Fundamental Remark

The inequalities (17) are *purely mathematical*. In particular, their proof depends in absolutely *no* way on anything else, except the mathematical properties of the set $\mathbb{Z}$ of positive and negative integers, set seen as a linearly ordered ring, [9].

As for the inequalities (16), they are a direct mathematical consequence of the inequalities (17), and thus again, their proof depends in absolutely *no* way on anything else, except the mathematical properties of the set $\mathbb{R}$ of real numbers, set seen as a linearly ordered field, [9].

It is, therefore, bordering on the amusing tinted with the ridiculous, when any sort of so called "physical" meaning or arguments are enforced upon these inequalities - be it regarding their proof, or their connections with issues such as realism and locality in physics - and are so enforced due to a mixture of lack of understanding of rather elementary and quite obviously simple mathematics, to which is added an irresistible tendency among physicists to use their "infallible physical intuition" in absolutely every realm possible ...

In this regard, it is one of the major merits of DRHM to have pointed out clearly and repeatedly, even if in terms less hard than above, the essential and so far hardly known fact that the EBBI inequalities such as (17) simply *cannot* be violated either by classical, or by quantum physics. And they cannot be violated, precisely due to the fact that they only depend on mathematics, and of course, logic.

As a consequence, since as seen later, the EBBI contain as particular cases the Bell inequalities, these inequalities *cannot* be violated by classical or quantum physics.

Therefore, the Bell inequalities turn out to be *irrelevant* to physics, be it classical or quantum, for that matter.

6. A Mathematical Comment

At first, it may look quite strange, if not rather incomprehensible, that inequalities like those in (16) hold under such general conditions. Added to that one may wonder how can one ever come up with them ?

However, as seen next, there is a certain rather simple heuristic argument which may point in their direction.

Let us, in this regard, start with the general setup in (1) - (3) which can be written in the following equivalent form. Let $K^n = \{-1, 1\}^n$ the set of vertices of the n-dimensional cube $[-1, 1]^n$ in $\mathbb{R}^n$. Then every $Y^{(n)} = \{ (S_{1,\alpha}, \ldots, S_{n,\alpha}) \mid \alpha = 1, \ldots, M \}$ in (1) is in fact but a vector valued mapping of the set $\{1, \ldots, M\}$ into the set of vertices K^n which is of course a subset of $\mathbb{R}^n$, namely

$$\{1, \ldots, M\} \ni \alpha \longmapsto V_\alpha = (S_{1,\alpha}, \ldots, S_{n,\alpha}) \in K^n. \qquad (18)$$

Clearly, K^n has 2^n elements, thus there are $(2^n)^M$ different possible $Y^{(n)}$.

Given now $1 \leq m < n$, there are C^n_m different possible choices $1 \leq i_1 < \ldots < i_m \leq n$. And to each of them there corresponds a $\Gamma^{(n)}_{i_1, \ldots, i_m}$, each of them being a vector valued mapping of the set $\{1, \ldots, M\}$ into the set of vertices K^m which this time is a subset of $\mathbb{R}^m$, namely

$$\{1, \ldots, M\} \ni \alpha \longmapsto W_{\alpha, i_1, \ldots, i_m} = (\, S_{i_1, \alpha}, \ldots, S_{i_m, \alpha} \,) \in K^m \qquad (19)$$

It follows that there are $C^n_m (2^m)^M$ such different possible $\Gamma^{(n)}_{i_1, \ldots, i_m}$.

Now, the particularity of the vector valued mappings (18), (19) is obvious when compared, for instance, with arbitrary mappings of $\{1, \ldots, M\}$ into $\mathbb{R}^n$, respectively, into $\mathbb{R}^m$.
Therefore, one can expect that these mappings may satisfy certain properties, and do so individually, or in certain of their combinations, or possibly, transformations.

Two questions, therefore, arise here : what relations they may satisfy, and in which of their combinations, or possibly, transformations they may satisfy them.

The simplest relations would, of course, be equalities. However, in view of the fact that the mappings (18), (19) are arbitrary between the respective domains and ranges, it is not likely that the specific properties those mappings may be captured by equalities.
And then, inequalities are the natural immediately more general possible relations to consider.

As for the second question above, again, in view of the fact that the mappings (18), (19) are arbitrary between the respective domains and ranges, it is not likely that those mappings in themselves may be involved in inequalities. Rather, one may expect that certain transformations of those mappings could possibly do so. And then, one of the simplest and most natural such transformations are various averages. In particular, the mappings (18) lead to the averages (14). And then, as particular cases, the mappings (19) lead to the averages (15).

Thus the Boole inequalities may impress rather by the fact that they give expression to the particularity of the mappings (18) and (19) in the simplest nontrivial possible way : through inequalities of averages.

7. EBBI : the Extended Boole Bell Inequalities

The inequalities in section 5 are not sufficient for the proper study of the Bell inequalities, and for that purpose they have to be generalized to inequalities involving non-negative functions of binary variables, as is done next. This is how we are led in this section to what are called the Extended Boole Bell Inequalities, or in short, EBBI.

Once again, and quite regrettably as far as many in the physics community are concerned, it cannot be overemphasized that the inequalities in section 5, as much as those in the present section, are purely *mathematical*, and as such, they have absolutely *no* need for any kind of so called "physical" considerations in their proofs.

Therefore, let us repeat once more that it is one of the major merits of DRHM to have pointed out so clearly the essential and so far hardly known fact that the inequalities in section 5, as well as those in this section, simply *cannot* be violated either by classical, or by quantum physics. And they cannot be violated, precisely due to the fact that they only depend on mathematics, and of course, logic.

In the next subsections 7.1.-7.4., we recall, according to DRHM, a number of inequalities regarding non-negative real valued functions of two, respectively, three binary variables. These inequalities are characterized by *necessary* and *sufficient* conditions. In view of that it becomes perfectly clear when those inequalities hold, and on the contrary, when they do not hold. Consequently, it becomes equally clear when certain inequalities are true, or for that matter false, for three pairs of data extracted from three data, see the Comments in subsection 7.4.

In this way, the customary *error* in claiming that the Bell inequalities are violated in the quantum context is pointed out.

7.1. Inequalities with a Function of Two Binary Variables

First we start with functions of *two* binary variables, and with real values, namely

$$f^{(2)} : K^2 \ni (S_1, S_2) \longmapsto f^{(2)}(S_1, S_2) \in \mathbb{R} \tag{20}$$

and note that each such function can be represented as

$$f^{(2)}(S_1, S_2) = (\, E_0^{(2)} + S_1 E_1^{(2)} + S_2 E_2^{(2)} + S_1 S_2 E^{(2)} \,)/4 \tag{21}$$

for $(S_1, S_2) \in K^2$, where

$$E_0^{(2)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} f^{(2)}(S_1, S_2) \tag{22}$$

$$E_1^{(2)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} S_1 \, f^{(2)}(S_1, S_2) \tag{23}$$

$$E_2^{(2)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} S_2 \, f^{(2)}(S_1, S_2) \tag{24}$$

$$E^{(2)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} S_1 \, S_2 \, f^{(2)}(S_1, S_2) \tag{25}$$

Theorem 7.1.

Given any function $f^{(2)} : K^2 \ni (S_1, S_2) \longmapsto f^{(2)}(S_1, S_2) \in \mathbb{R}$, then

$$f^{(2)} \geq 0 \tag{26}$$

if and only if

$$E_0^{(2)} \geq 0, \qquad |\, E_1^{(2)} \pm E_2^{(2)} \,| \leq E_0^{(2)} \pm E^{(2)} \tag{27}$$

7.2. Inequalities with a Function of Three Binary Variables

Let us now consider functions of *three* binary variables, and with real values, namely

$$f^{(3)} : K^3 \ni (S_1, S_2, S_3) \longmapsto f^{(3)}(S_1, S_2, S_3) \in \mathbb{R} \tag{28}$$

then again, each such function can be represented as

$$
\begin{aligned}
f^{(3)}(S_1,S_2,S_3) = (\, & E_0^{(3)} + S_1E_1^{(3)} + S_2E_2^{(3)} + S_3E_3^{(3)} + \\
& +S_1S_2E_{12}^{(2)} + S_1S_3E_{13}^{(2)} + S_2S_3E_{23}^{(2)} + \\
& +S_1S_2S_3E_{123}^{(3)} \,)/8
\end{aligned} \tag{29}
$$

for $(S_1,S_2,S_3) \in K^3$, where

$$
E_0^{(3)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \sum_{S_3=\pm 1} f^{(3)}(S_1,S_2,S_3) \tag{30}
$$

$$
E_i^{(3)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \sum_{S_3=\pm 1} S_i \, f^{(3)}(S_1,S_2,S_3) \tag{31}
$$

$$
E_{ij}^{(3)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \sum_{S_3=\pm 1} S_i \, S_j \, f^{(3)}(S_1,S_2,S_3) \tag{32}
$$

$$
E^{(3)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \sum_{S_3=\pm 1} S_1 \, S_2 \, S_3 \, f^{(3)}(S_1,S_2,S_3) \tag{33}
$$

with $i = 1,2,3$ and $(i,j) = (1,2),(1,3),(2,3)$.

Theorem 7.2.

Given any function $f^{(3)} : K^3 \ni (S_1,S_2,S_3) \longmapsto f^{(3)}(S_1,S_2,S_3) \in \mathbb{R}$. Then

$$
f^{(3)} \geq 0 \tag{34}
$$

implies the inequalities

$$
| \, E_{ij}^{(3)} \pm E_{jk}^{(3)} \, | \leq E_0^{(3)} \pm E_{jk}^{(3)} \tag{35}
$$

where $(i,j,k) = (1,2,3),(2,3,1),(3,1,2)$.

Conversely, given any four real numbers $E_0^{(3)}, E_{12}^{(3)}, E_{13}^{(3)}, E_{23}^{(3)} \in \mathbb{R}$, such that the inequalities

$$
| \, E_{ij}^{(3)} \, | \leq E_0^{(3)} \tag{36}
$$

and

$$| E_{ij}^{(3)} \pm E_{jk}^{(3)} | \leq E_0^{(3)} \pm E_{jk}^{(3)} \tag{37}$$

hold for $(i,j) = (1,2), (1,3), (2,3)$, then exists a function

$$f^{(3)} : K^3 \ni (S_1, S_2, S_3) \longmapsto f^{(3)}(S_1, S_2, S_3) \in \mathbb{R} \tag{38}$$

for which

$$f^{(3)} \geq 0 \tag{39}$$

and

$$E_0^{(3)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \sum_{S_3=\pm 1} f^{(3)}(S_1, S_2, S_3) \tag{40}$$

as well as

$$E_{ij}^{(3)} = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \sum_{S_3=\pm 1} S_i \, S_j \, f^{(3)}(S_1, S_2, S_3) \tag{41}$$

with $(i,j) = (1,2), (1,3), (2,3)$.

7.3. Inequalities with Three Functions of Two Binary Variables

Theorem 7.3.

Let

$$\begin{aligned} f^{(2)} &: K^2 \ni (S_1, S_2) \longmapsto f^{(2)}(S_1, S_2) \in \mathbb{R} \\ \widehat{f}^{(2)} &: K^2 \ni (S_1, S_2) \longmapsto \widehat{f}^{(2)}(S_1, S_2) \in \mathbb{R} \\ \widetilde{f}^{(2)} &: K^2 \ni (S_1, S_2) \longmapsto \widetilde{f}^{(2)}(S_1, S_2) \in \mathbb{R} \end{aligned} \tag{42}$$

defined by, see (22), (25)

$$\begin{aligned} f^{(2)}(S_1, S_2) &= (\, E_0^{(2)} + S_1 \, S_2 \, E^{(2)} \,)/4 \\ \widehat{f}^{(2)}(S_1, S_2) &= (\, \widehat{E}_0^{(2)} + S_1 \, S_2 \, \widehat{E}^{(2)} \,)/4 \\ \widetilde{f}^{(2)}(S_1, S_2) &= (\, \widetilde{E}_0^{(2)} + S_1 \, S_2 \, \widetilde{E}^{(2)} \,)/4 \end{aligned} \tag{43}$$

If

$$f^{(2)},\ \widehat{f}^{(2)},\ \widetilde{f}^{(2)} \geq 0 \tag{44}$$

then the inequalities hold

$$\begin{aligned} |\, E^{(2)} \pm \widehat{E}^{(2)} \,| &\leq 3E_0^{(2)} - |\, \widetilde{E}^{(2)} \,| \\ |\, E^{(2)} \pm \widetilde{E}^{(2)} \,| &\leq 3E_0^{(2)} - |\, \widehat{E}^{(2)} \,| \\ |\, \widetilde{E}^{(2)} \pm \widehat{E}^{(2)} \,| &\leq 3E_0^{(2)} - |\, E^{(2)} \,| \end{aligned} \tag{45}$$

Theorem 7.4.

Let

$$\begin{aligned} f^{(2)} &: K^2 \ni (S_1, S_2) \longmapsto f^{(2)}(S_1, S_2) \in \mathbb{R} \\ \widehat{f}^{(2)} &: K^2 \ni (S_1, S_2) \longmapsto \widehat{f}^{(2)}(S_1, S_2) \in \mathbb{R} \\ \widetilde{f}^{(2)} &: K^2 \ni (S_1, S_2) \longmapsto \widetilde{f}^{(2)}(S_1, S_2) \in \mathbb{R} \end{aligned} \tag{46}$$

defined by, see (22) - (25)

$$\begin{aligned} f^{(2)}(S_1, S_2) &= (\, E_0^{(2)} + S_1\, E_1^{(2)} + S_2\, E_2^{(2)} + S_1\, S_2\, E^{(2)} \,)/4 \\ \widehat{f}^{(2)}(S_1, S_2) &= (\, \widehat{E}_0^{(2)} + S_1\, \widehat{E}_1^{(2)} + S_2\, \widehat{E}_2^{(2)} + S_1\, S_2\, \widehat{E}^{(2)} \,)/4 \\ \widetilde{f}^{(2)}(S_1, S_2) &= (\, \widetilde{E}_0^{(2)} + S_1\, \widetilde{E}_1^{(2)} + S_2\, \widetilde{E}_2^{(2)} + S_1\, S_2\, \widetilde{E}^{(2)} \,)/4 \end{aligned} \tag{47}$$

Then

I) There exists a function

$$f^{(3)} : K^3 \ni (S_1, S_2, S_3) \longmapsto f^{(3)}(S_1, S_2, S_3) \in \mathbb{R} \tag{48}$$

such that

$$
\begin{aligned}
f^{(2)}(S_1, S_2) &= \sum_{S_3=\pm 1} f^{(3)}(S_1, S_2, S_3) \\
\widehat{f}^{(2)}(S_1, S_3) &= \sum_{S_2=\pm 1} f^{(3)}(S_1, S_2, S_3) \\
\widetilde{f}^{(2)}(S_2, S_3) &= \sum_{S_1=\pm 1} f^{(3)}(S_1, S_2, S_3)
\end{aligned}
\tag{49}
$$

if and only if

$$
E_0^{(2)} = \widehat{E}_0^{(2)} = \widetilde{E}_0^{(2)},\ E_1^{(2)} = \widehat{E}_1^{(2)},\ E_2^{(2)} = \widetilde{E}_1^{(2)},\ \widehat{E}_2^{(2)} = \widetilde{E}_2^{(2)} \tag{50}
$$

II) If in (47) we have

$$
f^{(2)},\ \widehat{f}^{(2)},\ \widetilde{f}^{(2)} \geq 0 \tag{51}
$$

and in addition, the relations, see (50)

$$
E_0^{(2)} = \widehat{E}_0^{(2)} = \widetilde{E}_0^{(2)},\ E_1^{(2)} = \widehat{E}_1^{(2)},\ E_2^{(2)} = \widetilde{E}_1^{(2)},\ \widehat{E}_2^{(2)} = \widetilde{E}_2^{(2)} \tag{52}
$$

together with the inequalities

$$
\begin{aligned}
|\, E^{(2)} \pm \widehat{E}^{(2)} \,| &\leq E_0^{(2)} \pm \widetilde{E}^{(2)} \\
|\, E^{(2)} \pm \widetilde{E}^{(2)} \,| &\leq E_0^{(2)} \pm \widehat{E}^{(2)} \\
|\, \widetilde{E}^{(2)} \pm \widehat{E}^{(2)} \,| &\leq E_0^{(2)} \pm E^{(2)}
\end{aligned}
\tag{53}
$$

hold, then there exists a function

$$
f^{(3)} : K^3 \ni (S_1, S_2, S_3) \longmapsto f^{(3)}(S_1, S_2, S_3) \in \mathbb{R} \tag{54}
$$

such that

$$
f^{(3)} \geq 0 \tag{55}
$$

and, see (49)

$$\begin{aligned} f^{(2)}(S_1,S_2) &= \sum_{S_3=\pm 1} f^{(3)}(S_1,S_2,S_3) \\ \widehat{f}^{(2)}(S_1,S_3) &= \sum_{S_2=\pm 1} f^{(3)}(S_1,S_2,S_3) \\ \widetilde{f}^{(2)}(S_2,S_3) &= \sum_{S_1=\pm 1} f^{(3)}(S_1,S_2,S_3) \end{aligned} \tag{56}$$

III) If for the function

$$f^{(3)} : K^3 \ni (S_1,S_2,S_3) \longmapsto f^{(3)}(S_1,S_2,S_3) \in \mathbb{R} \tag{57}$$

we have

$$f^{(3)} \geq 0 \tag{58}$$

then the corresponding functions in (49) satisfy

$$f^{(2)},\ \widehat{f}^{(2)},\ \widetilde{f}^{(2)} \geq 0 \tag{59}$$

and in addition, we have the inequalities, see (53), (47)

$$\begin{aligned} |\, E^{(2)} \pm \widehat{E}^{(2)} \,| &\leq E_0^{(2)} \pm \widetilde{E}^{(2)} \\ |\, E^{(2)} \pm \widetilde{E}^{(2)} \,| &\leq E_0^{(2)} \pm \widehat{E}^{(2)} \\ |\, \widetilde{E}^{(2)} \pm \widehat{E}^{(2)} \,| &\leq E_0^{(2)} \pm E^{(2)} \end{aligned} \tag{60}$$

7.4. Comments

In view of Theorem 7.4. the following becomes obvious. If and only if three non-negative real valued functions of two binary variables

$$f^{(2)}(S_1,S_2),\ f^{(2)}(S_1,S_3),\ f^{(2)}(S_2,S_3) \geq 0 \tag{61}$$

can be derived - through (49) - from one single non-negative function of three binary variables

$$f^{(3)}(S_1,S_2,S_3) \geq 0 \tag{62}$$

then, and only then, one can replace in the EBBI (35) the superscript (3) with the superscript (2).

In other words, the inequalities

$$| E_{ij}^{(3)} \pm E_{jk}^{(3)} | \leq E_0^{(3)} \pm E_{jk}^{(3)} \tag{63}$$

where $(i,j,k) = (1,2,3), (2,3,1), (3,1,2)$, do *not* imply for $f^{(2)}(S_1, S_2)$ the inequality, see (27)

$$| E_1^{(2)} \pm E_2^{(2)} | \leq E_0^{(2)} \pm E^{(2)} \tag{64}$$

and the similar two inequalities for $f^{(2)}(S_1, S_3)$, $f^{(2)}(S_2, S_3)$, *unless* (61) can be derived from (62) through (49).

7.5. Application to quantum physics

Using the results summarized in subsection 7.4, it is not difficult to see why quantum systems cannot violate (63). The quantum theoretical description of an experiment which involves measurements on n different objects, usually called spin-1/2 particles, yields the probabilities (in Kolmogorov's sense) $P^{(n)}(S_1, \ldots, S_n)$ to observe a particular realization of the two-valued variables $\{S_1, \ldots, S_n\}$ where, by convention, $S_k = \pm 1$ for $1 \leq k \leq n$ [8, 1]. Obviously, these probabilities are non-negative functions and therefore all the results derived above apply.

Taking $n = 3$, we immediately conclude that inequalities (63) must hold *always*. Furthermore, repeating the argument given in Section 7.4, unless $P^{(2)}(S_1, S_2)$, $P^{(2)}(S_1, S_3)$ and $P^{(2)}(S_2, S_3)$ can be derived from $P^{(3)}(S_1, S_2, S_3)$ by summing over S_3, S_2 and S_1, respectively (as is the case when the quantum system is described by a so-called separable state [8, 1]), there simply is no inequality of the type (63) that puts constraints on the values of the $E^{(2)}$'s that are obtained from $P^{(2)}(S_1, S_2)$, $P^{(2)}(S_1, S_3)$ and $P^{(2)}(S_2, S_3)$.

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