Spontaneous Symmetry Breaking in Nonlinear Dynamic System

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Abstract

Spontaneous Symmetry Breaking(SSB), in contrast to explicit symmetry breaking, is a spontaneous process by which a system governed by a symmetrical dynamic ends up in an asymmetrical state. So the symmetry of the equations is not reflected by the individual solutions, but it is reflected by the symmetrically coexistence of asymmetrical solutions. SSB provides a way of understanding the complexity of nature without renouncing fundamental symmetries which makes us believe or prefer symmetric to asymmetric fundamental laws. Many illustrations of SSB are discussed from QFT, everyday life to nonlinear dynamic system.

Contents

1 Introduction

The concept of symmetry dominates modern fundamental physics, both in quantum theory and in relativity. While symmetry plays a crucial role in modern physics, its dual concept "symmetry breaking" is also very important. Spontaneous Symmetry Breaking(SSB), in contrast to explicit symmetry breaking, is a spontaneous process by which a system governed by a symmetrical dynamic ends up in an asymmetrical state. It thus describes systems where the equations of motion or the Lagrangian obey certain symmetries, but the lowest energy solutions do not exhibit that symmetry. So the symmetry of the equations is not reflected by the individual solutions, but it is reflected by the symmetrically coexistence of asymmetrical solutions. An actual measurement reflects only one solution, representing a breakdown in the symmetry of the underlying theory. "Hidden" is perhaps a better term than "broken" because the symmetry is always there in these equations.

2 SSB in quantum field theory

Without spontaneous symmetry breaking, the local gauge principle requires the existence of a number of bosons as force carrier. However, some particles (the W and Z bosons) would then be predicted to be massless. While in reality, they are observed to have mass. To overcome this, spontaneous symmetry breaking is augmented by the Higgs mechanism to give these particles mass. It also suggests the presence of a new particle, the Higgs boson, reported as possibly identifiable with a boson detected in 2012. Spontaneous symmetry breaking occurs whenever a given field in a given Lagrangian has a nonzero vacuum expectation value. The Lagrangian appears symmetric under a symmetry group, but after randomly select a vacuum state the system no longer behaves symmetrically.

2.1 Real scalar field example

Consider the scalar Lagrangian given by

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2}_{\text{“kinetic term”}} + \underbrace{\frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{\text{“potential term”}} \tag{1}$$

where ϕ is the scalar field, μ is a sort of “mass” parameter, and λ is the coupling. Observe that there is a symmetry of $\phi \to -\phi$ (a discrete symmetry). We can think of the potential as being

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{2}$$

which has extremes when its derivative is zero. There are two, given by

$$\phi_0 = \pm v = \pm\mu\sqrt{\frac{6}{\lambda}} \tag{3}$$

where the constant v is the “*vacuum expectation value*”.

The vacuum that is, the lowest-energy state is described by a randomly chosen point of these two new extremes. We can then write

$$\phi(x) = v + \sigma(x) \tag{4}$$

and then rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4 \tag{5}$$

where we dropped the constant terms. We see that the symmetry $\phi \to -\phi$ is no longer identifiable.

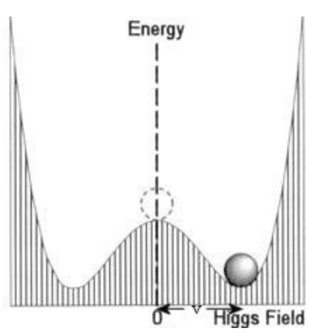


Figure 1: Illustration of SSB in real scalar field case, figure from http://www.zamandayolculuk.com/cetinbal/FJ/IHiggs.jpg

2.2 Complex scalar field example

Consider a complex scalar boson ϕ and $\phi^\dagger$. The Lagrangian will be

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \phi^\dagger \partial_\mu \phi - \frac{1}{2}m^2 \phi^\dagger \phi \tag{6}$$

Naturally we can write this as

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \phi^\dagger \partial_\mu \phi - V(\phi^\dagger, \phi) \tag{7}$$

where

$$V(\phi^\dagger, \phi) = \frac{1}{2}m^2 \phi^\dagger \phi \tag{8}$$

This Lagrangian has the $U(1)$ symmetry which means it's invariant under the transform: $\phi \mapsto e^{i\theta}\phi$.

If we graph $V(\phi^\dagger, \phi)$, plotting V vs. $|\phi|$, we can see a "bowl" with $V_{minimum}$ at $|\phi|^2 = 0$. The vacuum of any theory ends up being at the lowest potential point, and therefore the vacuum of this theory is at $\phi = 0$, as we would expect.

Now, let's change the potential. Consider

$$V(\phi^\dagger, \phi) = \frac{1}{2}\lambda m^2 (\phi^\dagger \phi - \Phi^2)^2 \tag{9}$$

where λ and Φ are real constants. Notice that the Lagrangian will still have the global $U(1)$ symmetry from before. But, now if we graph V vs. $|\phi|$, we get

where now the vacuum $V_{minimum}$ is represented by the circle at $|\phi| = \Phi$. In other words, there are an infinite number of vacuums in this theory. And because the circle drawn in the figure above represents a rotation through field space, this degenerate vacuum is parameterized by $e^{i\alpha}$, the global $U(1)$. There will be a vacuum for every value of α, located at $|\phi| = \Phi$.

In order to make sense of this theory, we must *choose* a vacuum by hand. Because the theory is completely invariant under the choice of the $U(1)$ $e^{i\alpha}$, we can choose any α and *define* that as our true vacuum. So, we choose α to make our vacuum at $\phi = \Phi$.

Now we need to rewrite this theory in terms of our new vacuum. We therefore expand around the constant vacuum value Φ to have the new field

$$\phi \equiv \Phi + \alpha + i\beta \tag{10}$$

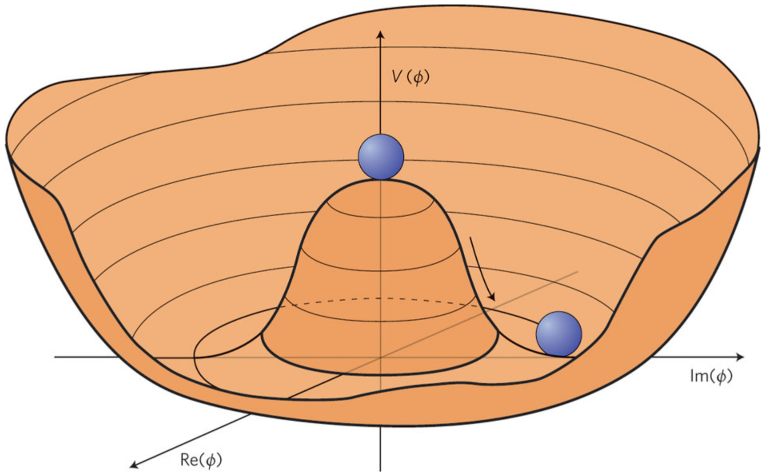


Figure 2: Mexican hat potential function V versus Φ, figure stolen from Eyes on a prize particle Nature Physics 7, 2C3 (2011)

where α and β are new real scalar fields (so $\phi^\dagger = \Phi + \alpha - i\beta$). We can now write out the Lagrangian as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial^\mu[\alpha - i\beta]\partial_\mu[\alpha + i\beta] - \frac{1}{2}\lambda m^2[(\Phi + \alpha - i\beta)(\Phi + \alpha + i\beta) - \Phi^2]^2 \\
&= \left[-\frac{1}{2}\partial^\mu\alpha\partial_\mu\alpha - \frac{1}{2}4\lambda m^2\Phi^2\alpha^2 - \frac{1}{2}\partial^\mu\beta\partial_\mu\beta\right] - \frac{1}{2}\lambda m^2\left[4\Phi\alpha^3 + 4\Phi\alpha\beta^2 + \alpha^4 + \alpha^2\beta^2 + \beta^4\right]
\end{aligned}$$

This is now a theory of a *massive* real scalar field α (with mass $= \sqrt{4\lambda m^2\Phi^2}$), a *massless* real scalar field β, and five different types of interactions (one allowing three α's to interact, the second allowing one α and two β's, the third allowing four α's, the fourth allowing two α's and two β's, and the last allowing four β's.) In other words, there are five different types of vertices allowed in the Feynman diagrams for this theory. Thus, the $U(1)$ symmetry is no longer manifest.

3 Other simple illustrations

3.1 Pencil

It's an amazing idea that the symmetry of the equations could not reflected by the individual solutions, but it is reflected by the symmetrically coexistence of asymmetrical solutions.

First let's consider a pencil, standing up right in the center is possible but unstable. The pencil tend to fall into one stable state randomly. After it fall down, the original symmetry was broken. This is exactly the illustration of the "mexican hat" potential of complex field.

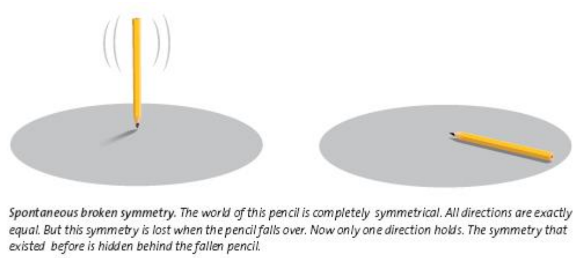


Figure 3: Pencil illustration of SSB, from http://web.hallym.ac.kr/ physics/course/a2u/ep/ssb.htm

3.2 Stick

Consider another everyday life example: a linear vertical stick with a compression force applied on the top and directed along its axis. The physical description is obviously invariant for all rotations around this axis. As long as the applied force is mild enough, the stick does not bend and the equilibrium configuration (the lowest energy configuration) is invariant under this symmetry. When the force reaches a critical value, the symmetric equilibrium configuration becomes unstable and an infinite number of equivalent lowest energy stable states appear, which are no longer rotationally symmetric. The actual breaking of the symmetry may then easily occur by effect of a (however small) external asymmetric cause, and the stick bends until it reaches one of the infinite possible stable asymmetric equilibrium configurations.

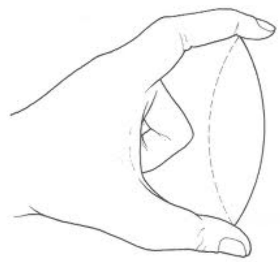
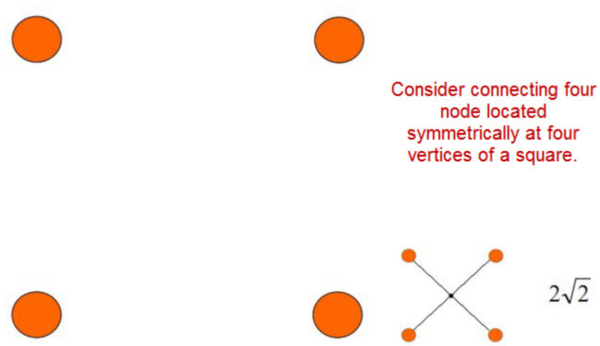


Figure 4: Stick illustration, figure from stick

3.3 Shortest connecting path

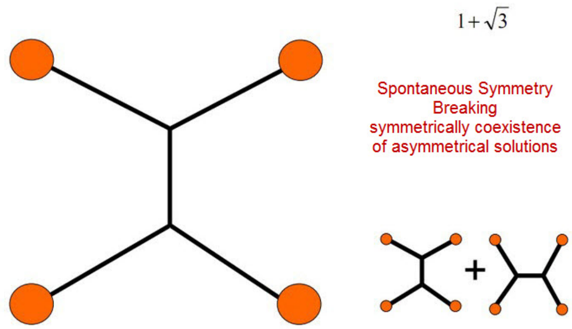
The above examples are essentially the same, all the new lowest energy solutions are asymmetric but are all related through the action of the symmetry transformations $U(1)$.

Here is yet another elegant example. Consider connecting four node located symmetrically at four vertices of a square. A simple solution you can draw by hand at once is like the following:



This solution fully maintain the original symmetry of a foursquare, and in fact it's a very satisfactory solution with total length $2\sqrt{2} \doteq 2.828$.

However, the best solution turns out to be like this one:



The total length is $1 + \sqrt{3} \doteq 2.732$, and the symmetry of the solution is less than the original system.

So the symmetry of the original problem is reflected by the symmetrically coexistence of asymmetrical solutions each of which has less or even no symmetry than the original full symmetry.

4 SSB in Nonlinear Dynamic System

Perhaps, the best illustrations could be given by attractor(s) of nonlinear dynamic system. Here we only consider three-dimensional autonomous systems with two quadratic terms, similar to the Lorenz system.

4.1 Symmetrical system with symmetrical attractor

The symmetry of the algebraic equations determine the symmetry of its geometric dynamics. In fact, symmetry plays an important role in generating chaos, which somehow determines the possible shape of a resulting attractor.

For example, both the Lorenz system and the Chen system have the z-axis rotational symmetry, and they both generate a two-scroll butterfly-shaped symmetrical attractor.

Lorenz system[2] is described by

$$\begin{cases} \dot{x} = \sigma\,(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy\,, \end{cases} \tag{11}$$

which is chaotic when $\sigma = 10,\ r = 28,\ b = \frac{8}{3}$.

Chen system[3] is described by

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = -bz + xy\,, \end{cases} \tag{12}$$

which is chaotic when $a = 35,\ b = 3,\ c = 28$. Both Lorenz and Chen systems have z-axis rotational symmetry so that the system algebraic equations remain the same when (x, y, z) is transformed to $(-x, -y, z)$.

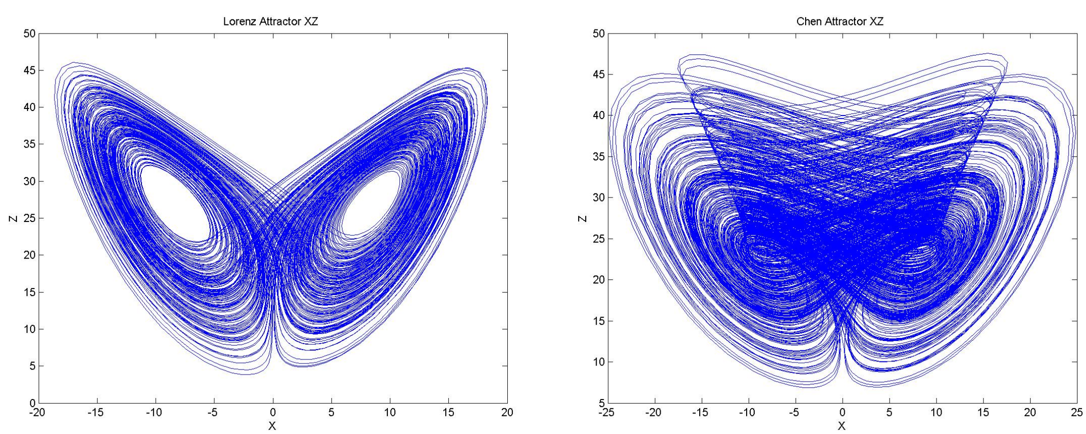
There are totally six possible quadratic nonlinear terms: xy, yz, xz, x^2, y^2 and z^2 in a 3D quadratic equation. Restricted by the z-axis rotational symmetry, the nonlinear terms in the second equation of the 3D system in interest must be either xz or yz, while in the third equation they must be either xy or z^2, or x^2, or y^2.

Based on the above observations, to maintain the z-axis rotational symmetry, the most general form of a 3D autonomous system with only linear and quadratic terms seems to be the following:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y + m_1xz + m_2yz \\ \dot{z} = a_{33}z + m_3xy + m_4x^2 + m_5y^2 + m_6z^2 + c\,, \end{cases} \tag{13}$$

where all coefficients are real constants.

The Lorenz attractor and Chen attractor are shown in Fig. 5.



(i) (ii)

Figure 5: Lorenz attractor and Chen attractor

4.2 Explicit symmetry breaking

The explicit symmetry breaking can easily realized by adding terms that break the symmetry of the equation. For example, we add a constant term in the second equation, then break the z-axis rotational symmetry:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy + m \\ \dot{z} = -bz + xy\,, \end{cases} \tag{14}$$

The resulting attractor is not symmetrical due to the asymmetrical dynamic equation.

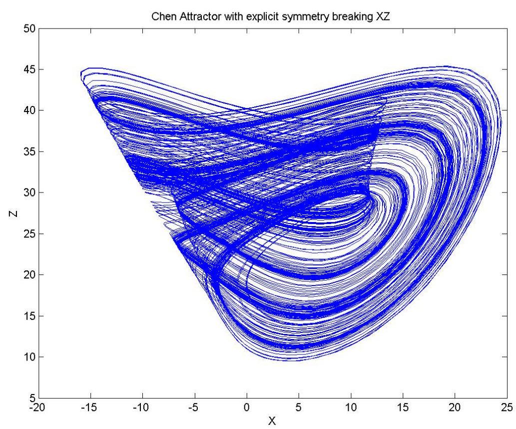


Figure 6: Explicit symmetry breaking of Chen system with m=20

4.3 SSB of nonlinear system

Constrained by the algebraically symmetry of the system equations, the above systems generate symmetrical two-scroll attractors. But this is not always the case, because this statement is based on the postulation that the dynamic system has only one attractor.

A very interesting phenomena is nonlinear systems may have multiple coexisting attractors. One such a example[4] is described by

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - 0.1y - yz \\ \dot{z} = -0.5z + x^2 - k\,, \end{cases} \tag{15}$$

where k is a real parameter. It can generate coexisting chaotic attractors, as shown in Fig. 7.

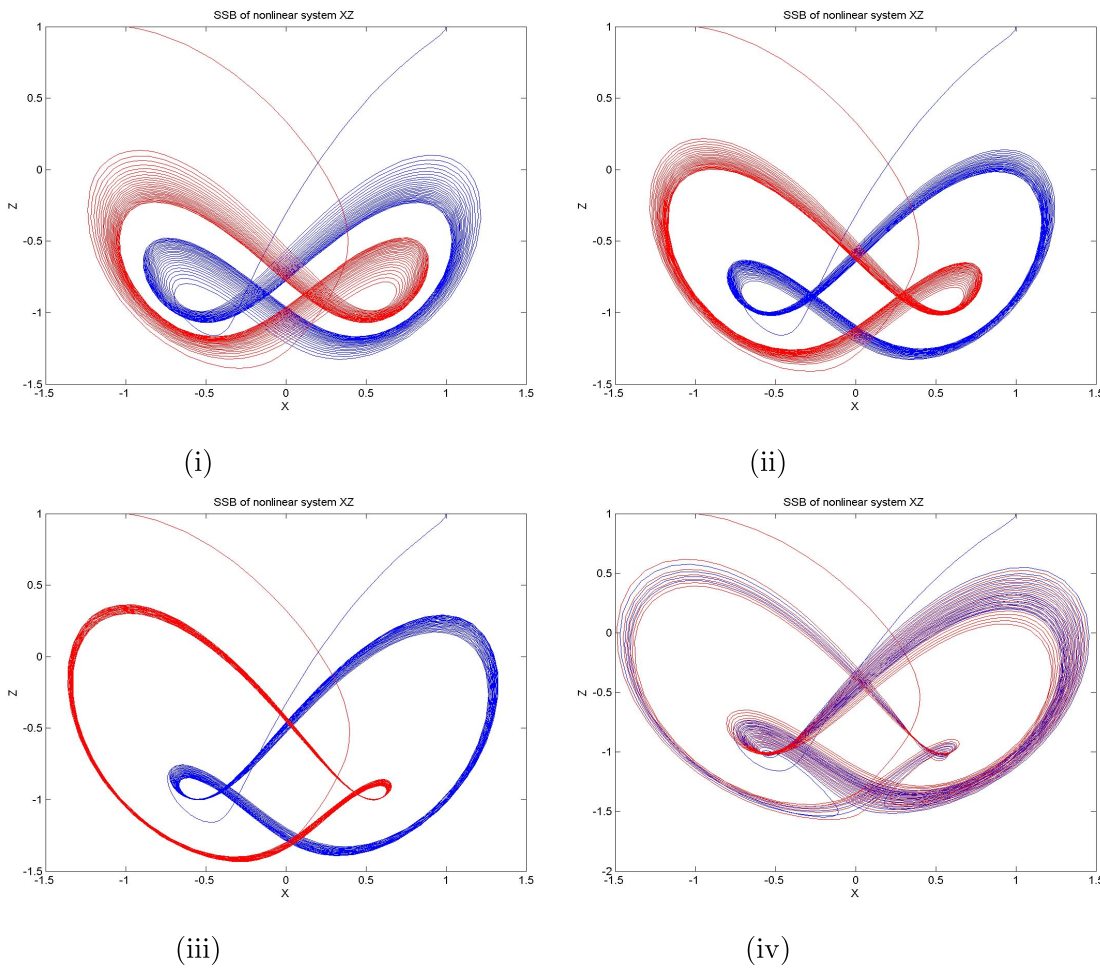


Figure 7: Family-k of chaotic systems with two coexisting attractors: (i) $k = 0.78$, (ii) $k = 0.79$, (iii) $k = 0.8$, (iv) $k = 0.81$, the two trajectories are from two different initial conditions, for the last case the two attractors merge into one and become symmetric.

5 Discussion

Constrained by the algebraically symmetry of the dynamic equations, the attractors generated by the system may or may not preserve the original symmetry. It's still puzzle to me what type of system may break the symmetry while others preserve, or what type of system may have multiple attractor.

What's more, in reality, we may not know the governing equation or fully understand its symmetry. What we can directly observe the physical phenomenon, the trajectory and attractor may be asymmetric, then we may overlook the underlying symmetry. If the observation/attractor can not reflect the underlying symmetry of the fundamental physic law, then the symmetry is hidden".

Thus SSB allows symmetric theories to describe asymmetric reality. SSB provides a way of understanding the complexity of nature without renouncing fundamental symmetries. So we believe or prefer symmetric to asymmetric fundamental laws.

In one word, symmetry is simple and elegant while symmetry breaking makes the world complex and colorful.

References

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[3] Chen, G. & Ueta, T. [1999] Yet another chaotic attractor, *Int. J. Bifur. Chaos* **9**, 1465–1466.

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