Numerical solution of nonlinear sine-Gordon equation with local RBF-based finite difference collocation method

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Abstract

This paper presents the local radial basis function based on finite difference (LRBF-FD) for the sine-Gordon equation. Advantages of the proposed method are that this method is mesh free unlike finite difference (FD) and finite element (FE) methods, and its coefficient matrix is sparse and well-conditioned as compared with the global RBF collocation method (GRBF). Numerical results show that the LRBF-FD method has good accuracy as compared with GRBF.

Keywords: Local RBF-based finite difference (LRBF-FD), Global RBF collocation, sine-Gordon Equation.

1 Introduction

We consider the generalized one-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f(x,t,u), \qquad a < x < b, \quad t \geq 0, \tag{1.1}$$

with the initial conditions

$$u(x,0) = h(x), \quad u_t(x,0) = g(x), \qquad a \leq x \leq b, \tag{1.2}$$

and with boundary conditions

$$u(a,t) = p(t), \quad u(b,t) = q(t), \qquad t \geq 0. \tag{1.3}$$

Sine-Gordon equation arise from many branches of modern physics, for example, the propagation of fluxions in Josephson junctions [1, 2], the motion of a rigid pendula attached to a stretched wire [3], and dislocations in metals [4].

In the last decade, collocation methods on Radial Basis Functions (RBFs) to obtaining the solution ordinary differential equations (ODEs) and partial differential equations (PDEs) have been popular because of the advantage of these methods.the methods do not require any mesh, unlike the finite difference method (FDM) and finite element method (FEM). This property makes these methods to be applied to irregular areas and higher dimensions.

In this paper, we developed the local RBF based finite difference (LRBF-FD) for the solution of Eq. (1.1) and the results are compared with the solutions obtained by global RBF collocation (GRBF). In the local RBF method, the function derivatives are approximated at each node in terms of the function value on a set of points in local stencils. This makes flexibility in choosing these stencils [5].

The layout of the paper is as follows. In Section2, we will explain RBF properties and its application to interpolation. In Section 3, the global RBF collocation (GRBF) for the Equation (1.1) are discussed. The

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formulation of local RBF-based on finite difference (LRBF-FD) for the sine-Gordon equation is derived in Section 4. In Section 5 the results of the numerical experiments and data comparison is shown. Section 5 is devoted to brief conclusion. Finally some references are introduced at the end.

2 RBF interpolation

RBFs are widely used for scattered data interpolation [8]. Given (x_j, u_j) $j = 1, ..., N$ with $x_j \in \mathbb{R}^d, u_j \in \mathbb{R}$, the standard interpolation problem is to find an interpolant as

$$\tilde{u}(x) = \sum_{j=1}^{N} \lambda_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \tag{2.1}$$

where ϕ is a radial basis function and $\|.\|$ denotes the Euclidean distance between points $\mathbf{x}, \mathbf{x}_j$, and $\lambda_j,\ 1 \le j \le N$ are unknown RBF coefficients. The following conditions are imposed on the approximation $\tilde{u}(x)$,

$$\tilde{u}(x_j) = u(x_j), \quad j = 1, \dots, N,$$

leading to a system of equations which can be written in a form as

$$A\lambda = u, \tag{2.2}$$

where

$$A = \begin{pmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_1 - x_2\|) & \cdots & \phi(\|x_1 - x_N\|) \\ \phi(\|x_2 - x_1\|) & \phi(\|x_2 - x_2\|) & \cdots & \phi(\|x_2 - x_N\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_N - x_1\|) & \phi(\|x_N - x_2\|) & \cdots & \phi(\|x_N - x_N\|) \end{pmatrix},$$

$$\text{u} = \begin{bmatrix} u(x_1) & \dots & u(x_N) \end{bmatrix}^T,$$

$$\lambda = \begin{bmatrix} \lambda_1 & \dots & \lambda_N \end{bmatrix}^T.$$

The existence and uniqueness of solution of system (2.2) are guaranteed when ϕ is a positive definite function [8]. Some examples of popular choices of RBFs are given in Table(1). Inverse multi quaric (IMQ), Gaussian (GA) are positive definite functions and multi quaric (MQ), Thin Plate Spline (TPS) are conditionally positive definite functions, that in latter cases one has to add polynomials of a certain degree to the interpolant (2.1).

In numerical results, we will focus on MQ RBF because MQ RBF is computationally attractive and has become more popular in practical applications. Other RBFs can be implemented in the same way.

3 The global RBF collocation method(GRBF)

In the global RBF collocation method for a time-depended problem, the solution $u(x, t)$ is approximated by a linear combination of RBFs as

$$u(x, t) = \sum_{j=1}^{N} \lambda_j \phi(\|\mathbf{x} - \mathbf{x}_j\|) + \alpha, \tag{3.1}$$

where $\phi(\|\mathbf{x} - \mathbf{x}_j\|) : \mathbb{R}^d \to \mathbb{R}$ is a MQ=$\sqrt{r^2 + c^2}$ with center $\mathbf{x}_j \in R^d,\ \lambda_j,\ \ j = 1, ..., N$ are undetermined RBF coefficients which evolve with time and α is a constant that because of the MQ=$\sqrt{r^2 + c^2}$ is conditionally positive definite function of degree one, we have added this constant to interpolant. The centers of the RBFs are chosen

Table 1: Some popular RBFs

Name	$\phi(r,c)$	*Conditions*
Gausian	$Exp(-\frac{r^2}{c^2})$	$c > 0$
inverse multiquadric	$(c^2+r^2)^{\beta/2}$	$c > 0,\ \beta < 0$
multiquadric	$(c^2+r^2)^{\beta/2}$	$c > 0,\ \beta > 0,\ \beta \notin 2\mathbb{N}$
Thin plate spline	$r^\beta logr$	$\beta > 0,\ \beta \in 2\mathbb{N}$

from the domain $\Omega = \{x_i | i = 2, ..., N-1\}$ and on the boundary $\partial\Omega = \{x_1, x_N\}$. For simplicity, we consider the case that the centers coincide with the set of collocation points.

We consider a sine-Gordon equation of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \beta \frac{\partial u(x,t)}{\partial t} = \mathfrak{L}u(x,t) - f(x,t,u), \qquad a \le x \le b, \quad t > 0, \tag{3.2}$$

where $\mathfrak{L}$ is of the form

$$\mathfrak{L} \equiv \frac{\partial^2}{\partial x^2}$$

Now, by discretize (3.2) according to the following θ-weighted scheme

$$\frac{u^{n+1}(x) - 2u^n(x) + u^{n-1}(x)}{(dt)^2} + \beta \frac{u^{n+1}(x) - u^{n-1}(x)}{2(dt)} = \theta(\mathfrak{L}u^{n+1}(x)) + (1-\theta)(\mathfrak{L}u^n(x)) + f(x, t^n, u^n(x)),$$

where $0 \le \theta \le 1$, and dt is the time step size. Thus we have

$$(2+\beta(dt))u^{n+1}(x) - 2(dt)^2\theta(\mathfrak{L}u^{n+1}(x)) = 2(dt)^2(1-\theta)(\mathfrak{L}u^n(x)) + 2(dt)^2 f(x, t^n, u^n(x)) + 4u^n(x) + (\beta(dt)-2)u^{n-1}(x) \tag{3.3}$$

Substituting equation (3.1) in (3.3) and in the boundary conditions lead to a system of equations which can be solved at $t = t^{n+1}$.

4 The RBF collocation method based on finite difference(RBF-FD)

Let $\mathfrak{L}$ is differential operator. The finite difference method approximates derivatives of a function $u(x)$ at point $x = x_i$ by

$$\mathfrak{L}u(x_i) \cong \sum_{j=1}^{n} w_{(i,j)}^{(k)} u(x_j) \tag{4.1}$$

where $w^{(k)}$ are called the FD weights at node points x_i. The unknown coefficients $w_{(i,j)}$ are obtained by using polynomial interpolation or Taylor series [10, 11]. In the RBF-FD approach the weights of the FD formulas are obtained using the RBF interpolation technique. To derive these weights an interpolant (2.1) is considered in a Lagrangian form as

$$s(x) = \sum_{j=1}^{n} \psi_j(x) u(x_j), \tag{4.2}$$

where $\psi_j(x_k) = \delta_{jk}, \quad k = 1, ..., N.$

Using this method, we determine the approximated derivative for each interior node in a local support region. Support region for interior node is defined as

$$\textit{support region } (x_i) = \{x_j \mid 0 \leq \|x_j - x_i\| \leq R\} \qquad i = 2, ..., N-1.$$

To derive RBF-FD formula, for example at the node x_1, we approximate the differential operator using the Lagrangian from of the RBF interpolant i.e.,

$$\mathfrak{L}u(x_1) \approx \mathfrak{L}s(x_1) = \sum_{j=1}^{n} \mathfrak{L}\psi_j(x_1) u(x_j), \tag{4.3}$$

where n is the number of the nodes in support region. From (4.1) and (4.3) we get

$$w_{(1,i)} = \mathfrak{L}\psi_i(x_1).$$

The number of points in support region can be variable, this property makes this method flexible. Once the LRBF-FD method is applied to discretize the spatial derivatives in the governing Equation (1.1) and approximating RBF weights for $\mathfrak{L} \equiv \frac{\partial}{\partial x^2}$ we obtain at any interior node $x_i, \quad i = 2, ..., N-1$,

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(dt)^2} + \beta \frac{u_i^{n+1} - u_i^{n-1}}{2(dt)} = \theta(\sum_{j=1}^{n} w_{(i,j)} u^{n+1}(x_j)) + (1-\theta)(\sum_{j=1}^{n} w_{(i,j)} u^n(x_j)) - f(x_i, t^n, u_i^n),$$

so we get

$$(2 + \beta(dt))u_i^{n+1} - 2(dt)^2\theta(\sum_{j=1}^{n} w_{(i,j)} u^{n+1}(x_j)) =$$

$$2(\mathrm{dt})^2(1-\theta)(\textstyle\sum_{j=1}^{n} w_{(i,j)} u^n(x_j)) - 2(dt)^2 f(x_i, t^n, u_i^n) + 4u_i^n + (\beta(dt) - 2)u_i^{n-1}, \text{ (4.4)}$$

where n is the number of nodes in support region. Equation (4) is written for each interior node. Substitruting the function value u_1, u_N from the boundary conditions (1.3) leads to a system of equations which can be written in matrix form as

$$A\mathbf{u} = B,$$

where $\mathbf{u}$ is the vector of the unknown function values at whole of the interior nodes at $t = t^{n+1}$ and B is the vector of the right hand of the Equation(4) along with the boundary terms.

Note that matrix A is sparse and well-conditioned and can hence be effectively inverted. The sparsity of A is depend of distribution of nodes and size of support region R. The sparsity pattern of A for different size R is shown in Figures(1,2,3).

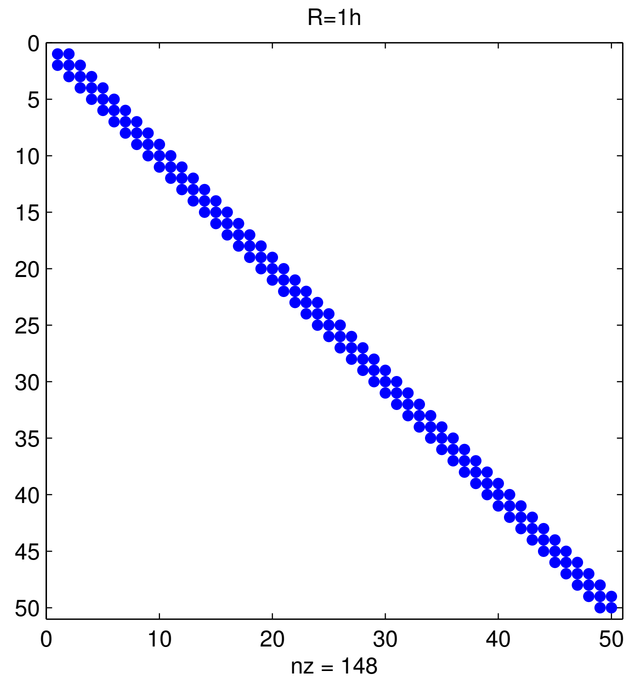

Figure 1: sparsity pattern of matrix A for equidistant points for $R = 1h$

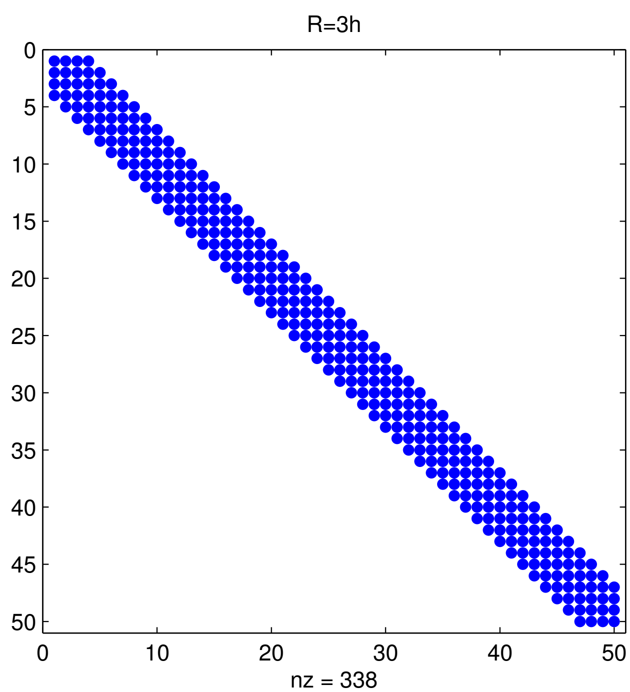

Figure 2: sparsity pattern of matrix A for equidistant points for $R = 3h$

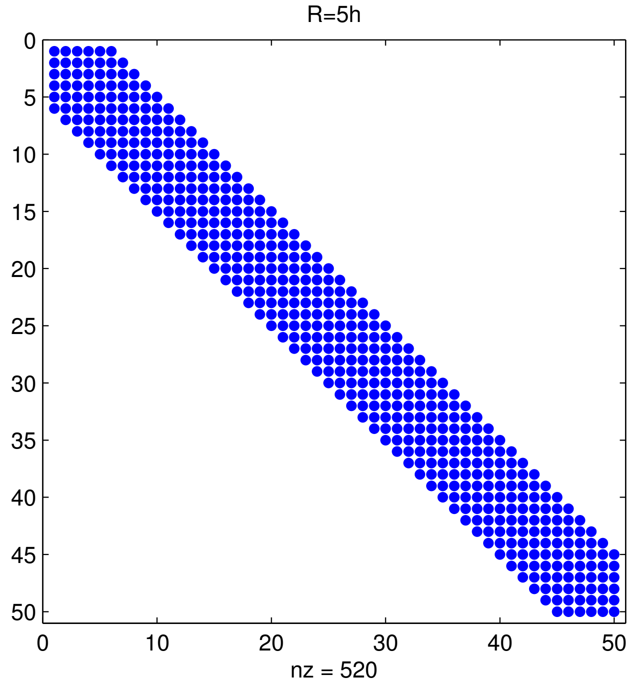

Figure 3: sparsity pattern of matrix A for equidistant points for $R = 5h$

5 Numerical results

We present some numerical results to illustrate the applicability and justify the accuracy and efficiency of our presented method. Some RBFs contain the parameter c that is shape parameter which plays an important role in the application of RBF methods. There is experimental evidence that shows the accuracy of RBF methods significantly depend on the amount of this shape parameter, but choice of the optimal shape parameter for any equations are the open problem yet. In order to analyze the error of the method, we consider root mean square (RMS) error and L_∞ error as

$$L_\infty = \|u(x,t) - u_N(x,t)\|_\infty = \max_{1\leq i\leq N} |u(x_i,t) - u_N(x_i,t)|,$$

$$RMS = \left[\frac{\sum_{i=1}^{N}(u(x_i,t) - u_N(x_i,t))^2}{N}\right]^{\frac{1}{2}}.$$

We consider Equation (1.1) with $a = 0,\ b = 1,\ \beta = 1$ and

$$f(x,t,u) = 2\sin u + \pi^2 \exp(-t)\cos(\pi x) - 2\sin(\exp(-t)(1 - \cos(\pi x))),$$

with the initial conditions

$$\begin{cases} u(x,0) = 1 - \cos(\pi x) & 0 \leq x \leq 1 \\ u_t(x,0) = -1 + \cos(\pi x) & 0 \leq x \leq 1 \end{cases}$$

The exact solution derived in [*]

Table 2: The error of resulting solution of LRBF-FD with $N = 40,\ c = 0.14,\ dt = 0.001$,and $\theta = \frac{1}{2}$

	L_∞			RMS		
t	$R = 1h$	$R = 3h$	$R = 5h$	$R = 1h$	$R = 3h$	$R = 5h$
1	1.77e-3	8.05e-5	6.48e-6	1.16e-3	5.30e-5	3.51e-6
2	1.39e-3	6.98e-5	4.37e-6	9.56e-4	4.78e-5	3.06e-6
3	8.09e-4	4.74e-5	2.92e-6	5.66e-4	3.30e-5	2.01e-6
4	3.74e-4	2.86e-5	1.76e-6	2.68e-4	2.02e-5	1.29e-6
5	1.28e-4	1.60e-5	9.85e-7	9.11e-5	1.15e-5	7.39e-7

Table 3: The error of resulting solution of GRBF with $N = 40,\ c = 0.14,\ dt = 0.001$,and $\theta = \frac{1}{2}$

t	L_∞	RMS
1	8.17e-4	2.14e-4
2	4.12e-4	1.51e-4
3	2.55e-4	1.29e-4
4	1.92e-4	9.96e-5
5	1.34e-4	7.05e-5

$$u(x,t) = \exp(-t)(1 - \cos(\pi x)).$$

We extract the boundary conditions from the exact solutions.

We use the equidistant points in our experiments. We applied our new LRBF-FD method and GRBF method to this example with shape parameter $c = 0.14$ and $dt = 0.001$,$\theta = \frac{1}{2}$. The computed solutions are compared with the exact solution at equidistant points.The RMS errors and L_∞ errors are tabulated in Tables (2-3). It can be seen that numerical results obtained from LRBF-FD method is in good agreement in contrast to the GRBF method. Because of the term $\exp(-t)$, the exact solution tends to zero as t increase. The behavior of convergence is shown in Figures (4,5). For a fixed shape parameter, the error tends to zero as the amount of the points increase.

6 Conclusion

In this paper a local RBF based on FD for the sine-Gordon equation was presented this new method combines advantages FD and RBF together. The proposed method has the attractive merits such as mesh free properties and coefficient matrix is sparse and well-conditioned unlike GRBF. The proposed method is compared with the Global RBF collocation method. The numerical results show that the presented method for the numerical solution of the sine-Gordon is very accurate and has a good accuracy in contrast with GRBF.

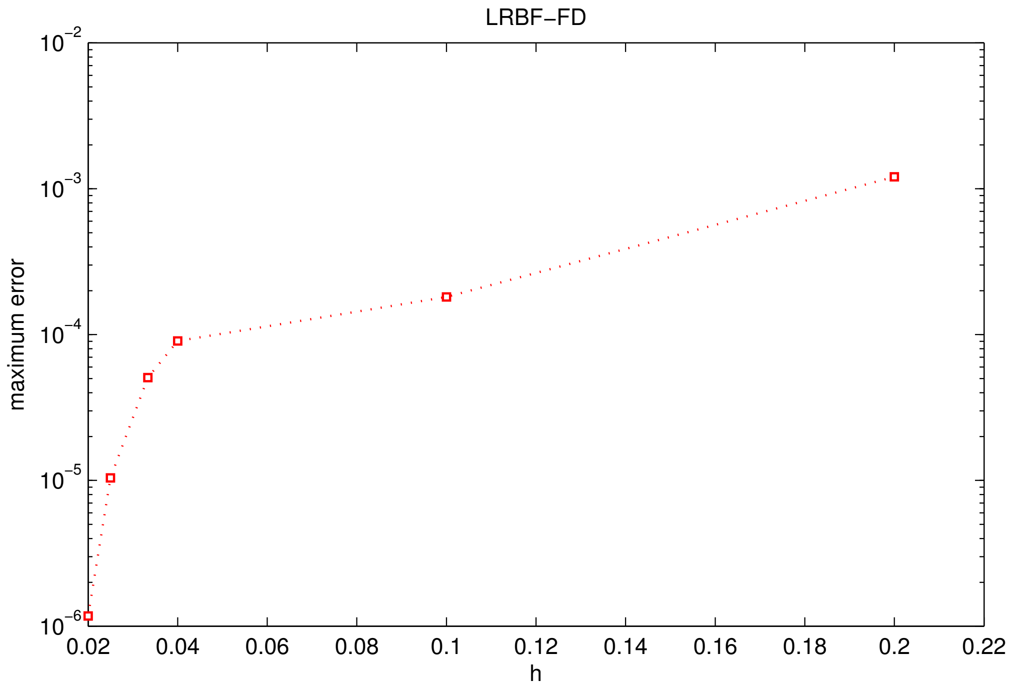

Figure 4: The error L_∞ of the LRBF-FD as function of h with $c = 0.12$ at $t = 5$

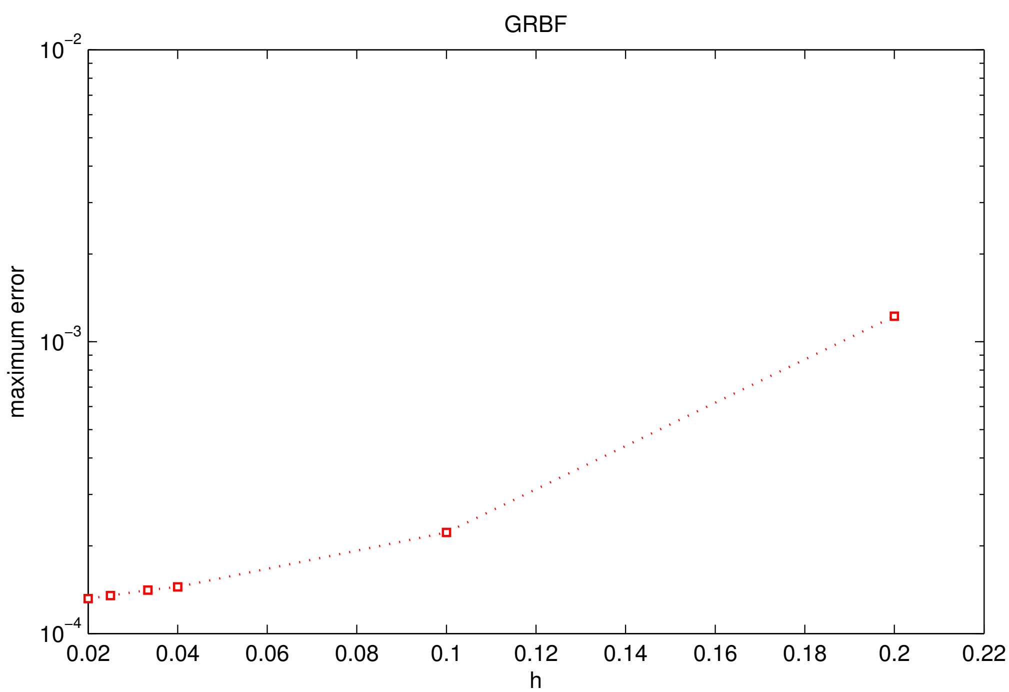

Figure 5: The error L_∞ of the GRBF as function of h with $c = 0.12$ at $t = 5$

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