The Critical Analysis of the Pythagorean Theorem and of the Problem of Irrational Numbers

Temur Z. Kalanov

Home of Physical Problems, Pisatelskaya 6a, 100200 Tashkent, Uzbekistan.
tzk_uz@yahoo.com, t.z.kalanov@mail.ru, t.z.kalanov@rambler.ru

Abstract. The critical analysis of the Pythagorean theorem and of the problem of irrational numbers is proposed. Methodological basis for the analysis is the unity of formal logic and of rational dialectics. It is shown that: 1) the Pythagorean theorem represents a conventional (conditional) theoretical proposition because, in some cases, the theorem contradicts the formal-logical laws and leads to the appearance of irrational numbers; 2) the standard theoretical proposition on the existence of incommensurable segments is a mathematical fiction, a consequence of violation of the two formal-logical laws: the law of identity of geometrical forms and the law of lack of contradiction of geometrical forms; 3) the concept of irrational numbers is the result of violation of the dialectical unity of the qualitative aspect (i.e. form) and quantitative aspect (i.e. content: length, area) of geometric objects. Irrational numbers represent a calculation process and, therefore, do not exist on the number scale. There are only rational numbers.

Key words: Pythagorean theorem, geometry, trigonometry, theory of number, mathematics, physics, philosophy of science.

Introduction

As is known, the Pythagorean theorem – a great discovery by scientists of Ancient East and of Ancient Greece – is one of the starting points and the basis of modern mathematics and of theoretical physics. It gave the key to the solution for many important science problems. At the same time, it spawned a number of unexpected problems, such as problems of incommensurability segments and irrational numbers [1-3]. The Pythagorean theorem is probably the unique proven theorem which spawned the problem of the proof: to date, there is a large number (approximately 400 [4, 5]) of proofs of the Pythagorean theorem in the scientific literature. In my opinion, this fact shows that the essence of the theorem is not completely clear until now. Conceptually, all proofs can be divided into a small number of classes [6]. The most famous of them are as follows: proofs of the area method, the axiomatic and exotic proofs (for example, using differential equations). However, the formal-logical proof which would be able to make fully clear the nature of the theorem is absent.

In my view, the Pythagorean theorem cannot be considered as absolute truth if there is no formal-logical proof of this theorem. The purpose of this paper is to propose the critical analysis of the Pythagorean theorem, the problems of incommensurability of segments and irrational numbers within the framework of the correct methodological basis – unity of formal logic and of rational dialectics [7].

1. The standard derivation of the Pythagorean formula

The Pythagorean theorem read as follows: square of hypotenuse $\overline{AB}$ of right-angled triangle ACB is equal to the sum of the squares of the legs $\overline{AC}$ and $\overline{CB}$ (Figure 1):

$$\left(\overline{AB}\right)^2 = \left(\overline{AC}\right)^2 + \left(\overline{CB}\right)^2 .$$

As is well known, the idea of the proof of the theorem is formulated as follows: if one constructs the square whose sides are equal to the hypotenuse of the given right-angled triangle, then the calculation of the hypotenuse of the right-angled triangle can be reduced to the calculation of the side of the square. The proof of the theorem consists of two parts: first, the construction and integration of geometrical figures (i.e. composition of forms) using a ruler and a compasses; and, second, calculations of the areas of figures and lengths of segments.

1) The construction and integration of geometrical figures (i.e. composition of forms) using a ruler and a compasses is carried out in the following way (Figure 1):

a) one constructs a square $ABEG$ whose sides represent the hypotenuses of the four identical the right-angled triangle ACB, BDE, EFG, GHA;

b) one constructs the square $HCDF$ whose sides are

$$\overline{HC} = \overline{HA} + \overline{AC}, \ \overline{CD} = \overline{CB} + \overline{BD}, \ \overline{DF} = \overline{DE} + \overline{EF}, \ \overline{FH} = \overline{FG} + \overline{GH}.$$

In this case, the square $ABEG$ is inscribed in the square $HCDF$. The square $HCDF$ represents the result of integration (composition) of the square $ABEG$ and four identical right-angled triangles ACB, BDE, EFG, GHA. However, the mathematical expression for the hypotenuse cannot be formulated within the framework of this composition.

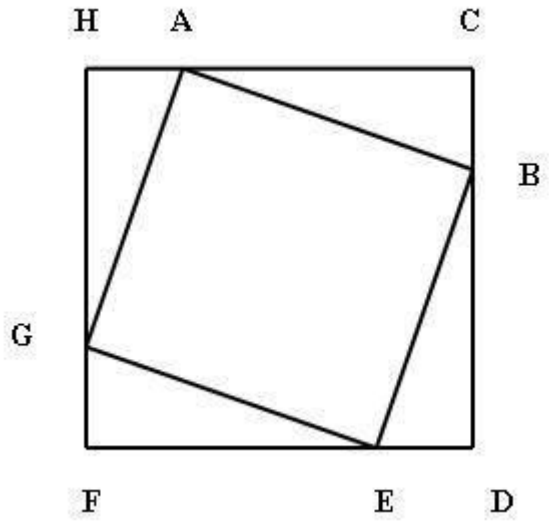

Figure 1.

The square $HCDF$ as the union (composition) of the square $ABEG$ and of the four identical right-angled triangles ACB, BDE, EFG, GHA. Hypotenuses of the triangles coincide with sides of the square $ABEG$.

2) Calculations of the areas of the geometrical figures and lengths of the segments are carried out with the help of decomposition of the square $HCDF$ into component forms in the following way (Figure 2):

a) one designates sides of the triangle (lengths of the segments): $\overline{AC} = a$, $\overline{CB} = b$, $\overline{AB} = c$;

b) one assumes that the segments a, b, c have a single measure, but the measure is not defined;

c) the square $HCDF$ is treated as the composition of the following geometric figures: the square $HKOM$ with sides a; the square $DLON$ with sides b; the rectangle $KCNO$ (made up of two identical right triangles); the rectangle $MOLF$ (made up of two identical right-angled triangles); the square $DLON$;

d) the area of the square $HCDF$ is

$$(a + b)^2 = a^2 + b^2 + 2ab;$$

e) the sum of the areas of the two rectangles $KCNO$ and $MOLF$ (i.e. the sum of the areas of four identical right-angled triangles) is $2ab$;

f) the sum of the areas of the squares $HKOM$ and $DLON$ is

$$(a + b)^2 - 2ab = a^2 + b^2;$$

g) the sum of the areas $a^2 + b^2$ of the squares $HKOM$ and $DLON$ is the area of a geometric figure. One assumes (without ground) that the area of this figure is always equal to the area c^2 of the square $ABEG$. (In my opinion, this assumption is not free of objections). In this case, one obtains the following relation:

$$a^2 + b^2 = c^2.$$

This relation represents a mathematical statement of the Pythagorean theorem;

h) one assumes (without ground) that the quantity $\sqrt{a^2 + b^2}$ is always equal to the length of side of the square $ABEG$ and, therefore, equal to the length of the hypotenuse c:

$$\sqrt{a^2 + b^2} = c.$$

(In my opinion, this assumption is not free of objections).

Calculations by this formula show that the hypotenuse c (i.e. the side of the square) takes rational or irrational values: for example, if $a = 3$ and $b = 4$, then $c = \sqrt{3^2 + 4^2} = \sqrt{5^2} = 5$; and if $a = b = 1$, then $c = \sqrt{2}$. This fact is the basis for the following standard statement: the lengths of the hypotenuse and the leg (i.e. the lengths of the diagonal and the sides of the square) are incommensurable quantities.

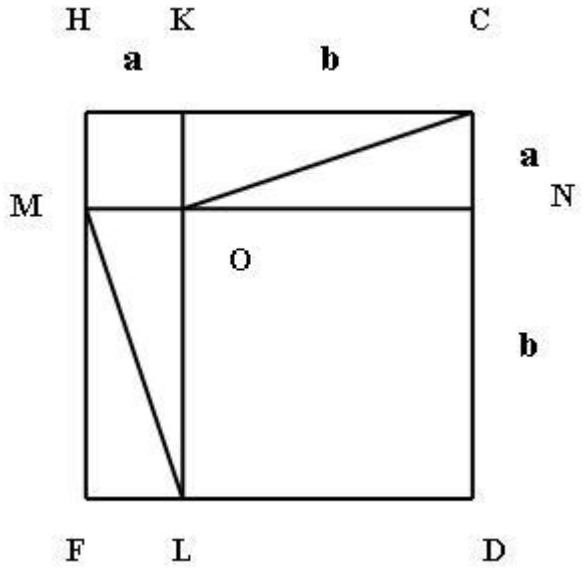

Figure 2.
Decomposition of the square *HCDF* into component figures (forms). The area of the square *HCDF* and the areas of component figures are related by $(a + b)^2 = a^2 + b^2 + 2ab$.

2. Analysis of the Pythagorean theorem

The standard statement that the lengths of the hypotenuse and leg (i.e., the lengths of the diagonal and the side of the square) are incommensurable quantities seems plausible, but it is ungrounded. Formal-logical proof of falseness of the standard statement is based on the following arguments:

a) the result of the squaring operation (i.e., raising to the second power) on any quantity c – the result of the direct mathematical operation – is the area c^2 of the square. The result of the extraction of the square root of the area c^2 of the square – the result of the inverse mathematical operation – is the length c of the side of square;

b) the result of the squaring operation (i.e., raising to the second power) on any quantity $(a + b)$ – the result of the direct mathematical operation – is the area $(a + b)^2$ of the square. The result of the extraction of the square root of the area $(a + b)^2$ of the square – the result of the inverse mathematical operation – is the length $(a + b)$ of the side of square;

c) the square root of the area $a^2 + b^2$ of the geometric figure which have form of a square (Figure 2) represents the length $\sqrt{a^2 + b^2}$ of the segment which is the side of the square. This statement is the first aspect of the essence of the Pythagorean theorem;

d) the square root of the area $a^2 + b^2$ of the geometric figure which does not have form of a square (Figure 2) represents the length $\sqrt{a^2 + b^2}$ of the segment which is the side of the square. This statement is the second aspect of the essence of the Pythagorean theorem;

e) the standard sense of the Pythagorean theorem is the following statement: sum $S_{(a \times a\ square)} + S_{(b \times b\ square)}$ of the areas of the squares constructed on the sides a and b is always an area of some square; the area of this square is always equal to the area $S_{(c \times c\ square)}$ of the square built on the hypotenuse c:

$$S_{(a \times a\ square)} + S_{(b \times b\ square)} = S_{(c \times c\ square)}.$$

Therefore, the standard mathematical relation between the areas of the squares $HKOM$, $DLON$, and $ABEG$ has the following form (Figures 1 and 2):

$$a^2 + b^2 = c^2.$$

The vulnerability of this relation is that this relation does not lead to the statement that the geometrical figure having the area $a^2 + b^2$ is identical with the square having the area c^2;

f) if two geometrical figures have the equal areas $S = S'$ and the same form, then they are identical: these figures coincide with each other under superposition. And vice versa, if the figures coincide with each other under superposition (i.e., are identical), then they have the equal areas and the same form.

g) the form (for example, triangular form or square form) is a qualitative determinacy of geometrical figure. The area and extension (length of side) is a quantitative determinacy of the geometrical figure. From the formal-logical point of view, quantitative determinacy of geometrical object belongs to qualitative determinacy of geometrical object. In accordance with the logical law of identity, the left-hand side and the right-hand side of the mathematical (quantitative) relation must belong to one and the same qualitative determinacy of object. For example, the logical (qualitative) relation between forms,

(square form) = (square form),

expressing the logical law of identity requires that the left-hand and right-hand sides of the mathematical relation should describe one and the same form which is a square;

h) the practice (i.e., the construction of figures) shows that the geometrical form having the area $S_{(a \times a\ square)} + S_{(b \times b\ square)}$ can be transformed into a square form in some cases but can not be transformed to a square form in other cases;

i) if the geometric form having the area $S_{(a \times a\ square)} + S_{(b \times b\ square)}$ can be transformed into the square form whose area is equal to the area $S_{(c \times c\ square)}$ of another square, then the Pythagorean theorem,

$$\sqrt{a^2 + b^2} = c,$$

is true one. In this case, the Pythagorean theorem satisfies the logical law of identity of forms,

(square form) = (square form),

and logical the law of lack of contradiction of forms,

(non-square form) ≠ (square form).

Consequently, in this case, the Pythagorean theorem is the correct expression of the law of identity of quantity:

(rational number) = (rational number).

j) if the geometry figure has the area $S_{(a \times a\ square)} + S_{(b \times b\ square)}$ equal to the area $S_{(c \times c\ square)}$ of another square but it cannot be transformed into a square form, then the Pythagorean theorem is not true, i.e.,

$$\sqrt{a^2 + b^2} \neq c.$$

In this case, the Pythagorean theorem is a quantitative expression of the following logical (qualitative) error:

(non-square form) = (square form),
(irrational number) = (rational number).

These arguments are sufficient grounds for the following conclusion which expresses the essence of the Pythagorean theorem:
1) the Pythagorean theorem is true if the union (composition) of squares *HKOM* and *DLON* can be transformed to a square which is identical with the square *ABEG* ;
2) the Pythagorean theorem is a logical error and, therefore, leads to appearance of irrational numbers if the union (composition) of squares *HKOM* and *DLON* cannot be transformed to a square which is identical with the square *ABEG*;
3) the correct formulation of the Pythagorean theorem is as follows: the square c^2 of the hypotenuse of right-angled triangle is equal to the sum $a^2 + b^2$ of the squares of the legs if the sum $a^2 + b^2$ represents the area of a square; if the sum $a^2 + b^2$ does not represent an area of a square, then the relation $a^2 + b^2 = c^2$ is not true and has only the nature of the approximate equality;
4) the standard theoretical proposition that the lengths of the hypotenuse and the leg (in other words, the lengths of the diagonal and the side of the square) are incommensurable quantities should be substituted for the following correct statement: the theoretical proposition of the existence of incommensurable segments contradicts to the practice (i.e. construction); incommensurability of segments is a mathematical fiction, consequence of violations of the logical law of identity of forms.

3. Analysis of the problem of irrational numbers

In view of the fact that the Pythagorean theorem leads to the appearance of irrational numbers, it is necessary to expose the true sense of irrational numbers. As is well known, an infinite non-periodic decimal fraction is called an irrational number [8, 9]. For example:

$$\sqrt{2} = 1,4142136\ldots, \quad \pi = 3,1415926535\ldots, \quad e = 2,7182818284\ldots .$$

In these expressions, the left-hand side is the symbol of mathematical operations, and the right-hand side is the number with dots. Dots designate the continuing process of calculation. From the formal-logical point of view, irrational numbers are expressed with the law of identity of concepts as follows:

(mathematical operation) = (calculation process)

where the left-hand side and right-hand side are identical concepts. The continuing process of calculation cannot be used as a number in practice because the mathematical process is not a number. This statement is expressed with the formal-logical law of lack of contradiction:

(non-number) ≠ *(number).*

If the calculation process is stopped by operator, the result of the process (i.e. the result of the mathematical operation) can be written as follows: $\sqrt{2} \approx 1,4142136$, $\pi \approx 3,1415926535$, $e \approx 2,7182818284$. The result of the mathematical process is a finite (rational) number which is used in practice. This statement expresses the requirement of the formal-logical law of identity:

(number) = (number).

The practical importance of rational numbers is that rational numbers are a result of measurements. Just these and only these numbers exist on number scale OX: if a straight line has the property of continuity, then any point on the line OX corresponds only to a rational number which is called abscissa of this point, and vice versa. Irrational numbers do not exist on number scale OX.

An important example of irrational numbers is the number $\pi = 3,1415926535...$ also. This number is the corollary of a radian measurement of angles and arcs. In other words, this number appears as a result of comparison of the arc length and of the length of the radius of circle. Indeed, arc lengths of two circles and the lengths of their radii are in the ratio of

$$\frac{l_1}{l_2} = \frac{R_1}{R_2} \quad \text{i.e.} \quad \frac{l_1}{R_1} = \frac{l_2}{R_2}$$

for one and the same central angle. If the angle contains A°, then its radian measure α is equal to

$$\alpha = \frac{A^\circ \pi}{180^\circ}.$$

In accordance with it, the arc length and the length of circle are $l = \alpha R$ and $L = 2\pi R$, respectively. In this example, the irrational number $\pi = 3,1415926535...$ appears as a result of dialectical error: dissolution of the dialectical unity of qualitative determinacy (i.e. form) and of quantitative determinacy (i.e. content: length) of the line segment. The arc and the radius of the circle represent different forms of line: curvilinear form and rectilinear form, respectively. As the construction (i.e., practice) shows, two different forms are not coincided with each other and cannot be reduced to one another. From the point of view of formal logic and classical geometry, this implyies that different forms (i.e., qualities, concepts) can be defined genetically [7] (i.e., can be defined by means of the construction), but they cannot be compared with each other:

(*arc)* ≠ *(radius).*

In accordance with the formal logic, mathematical (quantitative) relations must satisfy the law of identity of forms (concepts):

(arc) = (arc) and *(radius) = (radius).*

It means that there is no correct mathematical relation between arc length and length of radius in classical geometry. Therefore, length of radius R (for example, measured in centimeters) cannot be used as a measure of arc length l (measured in centimeters).

As is well known, the segments of qualitatively different lines have one common aspect: quantitative determinacy (i.e. length). Therefore, in accordance with the formal logic, the lengths of segments of qualitatively different lines can be compared with each other. However, this quantitative (mathematical) comparing segments of different forms is possible only by abstraction (segregation) of the quantitative aspect from of the qualitative aspects (i.e. by destruction of the dialectical unity of the qualitative and quantitative aspects). In this case, the length is considered to be an essential sign of the line, and the form is considered to be an inessential sign of the line:

(length of arc) = (length of radius),
i.e. *(length of line) = (length of line).*

This logical relation leads to the mathematical expression $L = 2\pi R$ which contains the number π. Thus, the number π is a corollary of violation of the dialectical unity of qualitative and quantitative aspects of objects in classical geometry.

4. Discussion

1. The problem of the criterion of truth is the central point of science. There are two criteria of truth of theory: the dialectical criterion, i.e. "criterion of external justification of theory" (Einstein); and the logical criterion, i.e. "criterion of internal perfection of theory" (Einstein). As is well known, practice represents "criterion of external justification of theory": if a theory contradicts to the practice, then the theory is erroneous one. In geometry, it means that the calculations done within the framework of standard mathematical formalism should not contradict to constructions. Formal logic represents "criterion of internal perfection of theory": if a theory contradicts to the formal-logical laws, then the theory is erroneous one. Therefore, the problem of truth of a geometrical theory should be solved within the framework of correct methodological basis – the unity of formal logic and of rational dialectics. The unity of formal logic and of rational dialectics represents theoretical generalization of practice.

2. Indissoluble connection, reciprocal complementarity of constructions (practice) and calculations (theory) is a characteristic property (essence) of geometry [7]. It means that the mathematical theorems and proofs must consider not only the numerical values of quantities (i.e., quantitative determinacy), but also the form (i.e., qualitative determinacy) of geometric objects. The qualitative determinacy of objects is expressed in concepts (for example, in the concepts "straight line", "angle", "arc", "triangle", "square", "form"). Consequently, in geometry, correct formation of a theory and comparison of theory with practice is not possible without the use of formal logic and of rational dialectics. There are two ways to define the concept in formal logic: definition by the nearest genus and specific difference; and genetic definition. The first way is as follows: one brings the defined (superprdinate) concept under the defining (generic) concept (i.e., under another, more general concept); then one indicates the sign which gives opportunity to differ the defined (superprdinate) concept from other concepts. The second way to define the concept is a genetic definition. The genetic definition is the special kind of definition which shows how given object or phenomenon arise. The genetic definition represents a practical method of definition. It represents the criterion of truth and is used in theoretical geometry [7].

3. The universal formalism of sciences is formal logic. Formal logic is a science of laws of correct thinking. The correct scientific thinking must be obeyed the four basic logical laws. The basic form of thought is a concept. Concepts are formed by logical methods such as analysis and synthesis, abstraction and generalization. Abstraction is a mental separation, a mental extraction of the individual, essential signs of the object or phenomenon from the object or

phenomenon and the abandonment of all other (non-essential) signs without consideration. The concept (for example, “straight line”, “angle”, “arc”, “triangle”, “square”, “form”) is a form of thought reflecting and fixing the essential signs of the objects and of phenomena of reality. The essential signs are called the signs which belong to objects of certain genus (for example, triangles) and distinguish them from objects of other genera (for example, quadrilaterals). Thus, the essential signs (for example, form) characterize the appropriate objects and give opportunity to cognize these objects. Inessential signs (for example, size) are called the signs which belong to some objects of given genus but do not characterize them, do not give opportunity to cognize these objects and to distinguish them from objects of other genera. Some of the signs of the given object can be essential signs in one respect (sense), and other signs can be essential signs in another respect (sense).

4. Geometric figure is a system of bound elements. For example, a triangle is a system of three bound segments of straight line. The practical task of construction of a triangle has the exact solution: one can always draw the three intersecting straight lines through three arbitrary points of the plane, which do not lie on a straight line. Segments of these lines always have a common measure (measure is a philosophical category designating the unity of qualitative and quantitative determinacy): the segments can always be measured with any desired order of accuracy (for example, in centimeters, meters, etc.). In other words, the practice shows what the incommensurable segments of straight lines do not exist. However, it does not mean what length of segment of a certain line (for example, length of radius measured in centimeters) is always a measure of other segment of any line (for example, length of appropriate arc measured in centimeters, too). In the view of rational dialectics (systems analysis), explanation is that the properties of the system are not a consequence of the properties of elements of the system. (This knowledge was not available for ancient scientists).

5. The concept of number is one of the basic concepts of mathematics. It originated in ancient time and was gradually widened and generalized in connection with tasks of measurement of lengths, areas, volumes, mass, temperature, speed, strength, etc. The concept of quantity is a generalization of concrete concepts (length, area, mass, etc.). Choosing one of the quantities of given kind as unit, one can express the ratio of any other quantity of the same kind to the unit of measurement by a number. However, the measurement (or calculation) process is not expressed by numbers. A number always represents the result of the measurement (or calculation) process. It means that a number is always finite. Hence, irrational numbers (for example, $\sqrt{2} = 1,4142136\ldots$, $\pi = 3,1415926535\ldots$, $e = 2,7182818284\ldots$) as a result of the measurements (or calculations) do not exist. There are only rational numbers (for example, $\sqrt{2} \approx 1,4142136$, $\pi \approx 3,1415926535$, $e \approx 2,7182818284$) as a result of the measurements or calculations (i.e., as a result of practice).

Conclusion

Thus, the critical analysis leads to the following results:

1. The essence of the Pythagorean theorem is that $a^2 + b^2 = c^2$ represents a conventional (conditional) theoretical proposition in general case: if the sum $a^2 + b^2$ is the area of the square, then the theorem is true one; and if the sum $a^2 + b^2$ is not the area of the square, then the theorem is erroneous one (because, in this special case, it contradicts to the formal-logical laws and leads to appearance of irrational numbers).
2. The standard theoretical statement that the lengths of the hypotenuse c and of the leg a (in other words, the lengths of the diagonal and of the side of the square) are incommensurable quantities should be replaced by the following correct statement: the theoretical proposition of the existence of incommensurable segments contradicts to the practice (i.e., construction); incommensurability of segments is a mathematical fiction, the consequence of violations of the logical law of identity of forms and the law of lack of contradiction.

3. The concept of irrational number arise as a result of violations of the dialectical unity of the qualitative aspect (i.e., form) and quantitative aspect (i.e., content: length, area) of geometric objects. Irrational numbers represent a mathematical process and, therefore, they do not exist on the number scale. There are only rational numbers.

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