Free Fermions on causal sets.

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Abstract

We construct a Dirac theory on causal sets; a key element in the construction being that the causet must be regarded as emergent in an appropriate sense too. We further notice that mixed norm spaces appear in the construction allowing for negative norm particles and "ghosts". This work extends the theory initiated in [1, 2]

1 Introduction

Retrieving continuum concepts from a pure discrete setting or visa versa is always tricky as one often has the tendency to forget that they must be natural from the discrete perspective too and in case of Fermi fields on causal sets; this problem has defied anybody up till now. In contrast to the basic spacetime geometric objects one requires for the scalar field; the concepts of vierbein and Clifford bundle are mandatory for the continuum description. Given the complete absence of such concepts for causal set theory, the best one may hope for is the existence of an object which has no counterpart in the continuum and replaces the aformentioned mathematical gadgets. The Dirac operator is replaced by what we call a hidden structure which allows one to construct the relevant Green's function akin to the methods in the scalar case. As is standard for such enterprise, this section starts from fairly conventional ideas and then moves gradually in the discrete direction in which some equalities are highlighted and others are merely approximated. I am aware I could have presented section two more formally but have chosen not to do so in order to allow the reader to see himself what is essential and what not. Section three contains the general construction of Dirac theory and reveals the possibility of negative norm particles and ghosts which have to enter the prescription. The last section finally gives an explicit computation of massless Dirac theory on the diamond for a particular hidden structure and comments on possible different physical theories for distinct hidden structures.

2 The relevant Fermionic Green's kernels.

We start this section by retrieving some results and terminology from Lorentzian geometry. The one parameter family of Dirac operators $D_m = i\gamma^a e^{\mu}_a \nabla_\mu - m$ in signature $(+---)$ with global Lorentz covariance and retarded Green's kernels

$R_m(x,y)$ obeys

$$D_m R_m(x,y) = \frac{\delta^n(x,y)}{\sqrt{-g(y)}}\,1.$$

Denoting by $G_{R,m}(x,y)$ the retarded solution to the Klein Gordon equation, one calculates that, with

$$S_m(x,z) = -\int R_m(x,y)R_{-m}(y,z)\sqrt{-g(y)}d^n y$$

the following equality holds

$$(\Box^2 + m^2)S_m(x,z) = \frac{\delta^n(x,z)}{\sqrt{-g(z)}}\,1$$

and since S_m is retarded, $S_m(x,z) = G_{R,m}(x,z)\,1$. We define now two different automorphisms $\star$ and $\overline{a}$ by $c^\star = c, \overline{c} = \overline{c}$ and $-(\gamma^a)^\star = \gamma^a = -\overline{\gamma^a}$. One notices then that $-R^\star_{-m}(x,y) = R_m(x,y)$ and $\overline{R}_m(x,y) = R_m(x,y)$; discretizing according to the causal set scheme then gives that

$$\sum_y R_m(x,y)R^\star_m(y,z) = \rho\,1 \otimes G_{R,m}(x,z)$$

implying $[R_m, R^\star_m] = 0$ and the reality condition that $\overline{R}_m = R_m$. Denoting by $\tilde{}$ the composition of $\overline{a}$ with the reversion, taking into account that $-D_{-m}G_{R,m}(x,y) = R_m(x,y)$, one arrives at $\widetilde{R}_m(x,y) = R_m(x,y)$. Causality is then implemented at the discrete level by demanding that R_m has support on the support of the incidence matrix union the diagonal and global Lorentz covariance $U = u\delta_{x,y}$, $\widetilde{u}u = 1$ implying that $u^\star = u$, translates as

$$R'_m = U^{-1}R_m U.$$

Actually any transformation with the appropriate symmetries in the commutant of $G_{R,m}$ leads to a recalibration of R_m. We could try to show that such transformations lead to unitarily equivalent theories. We have that

$$R_m(x,y) = -mG_{R,m}(x,y) - i\gamma^a e^\mu_a \partial_\mu G_{R,m}(x,y)$$

and $e^\mu_a\partial_\mu G_{R,m}(x,y) = [P_a(I)]\,(x,y)$ is a polynomial expression in terms of the incidence matrix I where $I(x,y) = 0$ unless x precedes y in which case it equals one. Hence

$$\sum_y (-mG_{R,m}(x,y) - i\gamma^a\,[P_a(I)]\,(x,y))(-mG_{R,m}(y,z) + i\gamma^b\,[P_b(I)]\,(y,z)) = \rho G_{R,m}(x,z).$$

One notices that not enough information is present in the causal set itself to find a unique solution in this way; the problem being that too many expressions can fit these equations and that the dimension is put in by hand.

We now persue the viewpoint that the causal set itself is *emergent* from a spinor perspective in the following sense: consider $\mathbb{C}l_{\mathbb{R}}(p,q)$ and the trace state Tr on it; then $K \in \mathbb{C}l_{\mathbb{R}}(p,q) \otimes \mathbb{R}^{n\times n}$, $K = A_{a\ldots c}\gamma^a \ldots \gamma^c$ with $A_{a\ldots c}(x,y) \in \{0,1,-1\}$

having support on the union of the diagonal and the support of I is weakly positively or negatively hidden in I if and only if

$$\mathrm{Tr}(KK^{\star}) = \pm I,$$

while it is positively or negatively strongly hidden if and only if

$$KK^{\star} + K^{\star}K = \pm 2\,1 \otimes I.$$

In contrast to the continuum theory, we take the trace since an exact equality would be too strong; in case of one Clifford generator, weakly hidden structures are also strongly hidden. Take for example the causal set associated to $\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)$ then $\mathbb{C}l_{\mathbb{R}}(1,0)$ nor $\mathbb{C}l_{\mathbb{R}}(0,1)$ do accomodate for a K such that $K^2 = I$ or $KK^{\star} = I$. $\mathbb{C}l_{\mathbb{R}}(1,0)$ does accomodate for a positively hidden structure

$$K = \left(\begin{array}{cc} 0 & 1 \\ 0 & 1-\beta \end{array}\right)$$

since it obeys $KK^{\star} = \left(\begin{array}{cc} 0 & 1+\beta \\ 0 & 0 \end{array}\right)$ with $\beta^2 = 1$. In $\mathbb{C}$, no positively hidden structure exists which is hopeful since it selects the correct signature (i.e. positive instead of imaginary mass). More in general,

$$K = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1+\beta & -\beta \\ 0 & 0 & 0 \end{array}\right)$$

and

$$L = \left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 1+\beta & -\beta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1+\beta \end{array}\right)$$

show that the one dimensional structure persists since

$$KK^{\star} = \left(\begin{array}{ccc} 0 & 1-\beta & \beta \\ 0 & 0 & 1+\beta \\ 0 & 0 & 0 \end{array}\right)$$

and

$$LL^{\star} = \left(\begin{array}{cccc} 0 & 1-\beta & \beta & 0 \\ 0 & 0 & 1+\beta & -\beta \\ 0 & 0 & 0 & 1-\beta \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and similar negatively hidden structures exist. The causet

$$I = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

is one dimensional from the spinor structure as

$$K = \left(\begin{array}{ccc} 0 & 0 & \beta \\ 0 & 0 & \beta \\ 0 & 0 & 1-\beta \end{array}\right)$$

is positively hidden in I and likewise is

$$K = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \beta \\ 0 & 0 & 1+\beta \end{pmatrix}$$

negatively hidden. For the causal diamond, the reader can find a positively[1] and negatively hidden structure associated to $\mathbb{C}l_{\mathbb{R}}(1,0)$ and it remains a question to find the need for higher dimensional algebra's. If K is hidden in I, so is $K^\star = \overline{K}, -K, -K^\star$. For the advanced case, we may use $\widetilde{K}^T$ since

$$I^T(x,y) = \mathrm{Tr}(\widetilde{KK^\star})^T(x,y) = \mathrm{Tr}(\widetilde{K}^T(\widetilde{K}^T)^\star)(x,y)$$

in accordance with the continuum theory. This implies the use of indefinite metric spaces with as scalar product, in the previous examples,

$$\langle v|w\rangle = \begin{pmatrix} \overline{v}_1 & \overline{v}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Denoting the latter matrix by δ and assuming that β is given by

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

one observes that $\delta\beta + \beta\delta = 0$ and $\beta^\dagger = \delta\beta\delta = -\beta$ where the adjoint is defined with respect to the indefinite product. That is, the adjoint acts like $\widetilde{\ }$ on $\mathbb{C}l_{\mathbb{C}}(1,0)$ and the total representation space is a $2n$ dimensional with a scalar product of signature (n,n). A spectral theorem for self adjoint matrices exists and one has real eigenvalues associated to k eigenvectors of positive and negative norm respectively and to $n-k$ pairs of null vectors associated to any *complex* eigenvalue. That is, any self adjoint operator A can be written as

$$A = \sum_{r=1}^{k} (\lambda_r |v_r\rangle\langle v_r| + \mu_r |w_r\rangle\langle w_r|) + \sum_{r=1}^{n-k} (c_r |m_r\rangle\langle n_r| + \overline{c}_r |n_r\rangle\langle m_r|)$$

where $\langle v_i|v_j\rangle = \delta_{ij} = \langle w_i|w_j\rangle$, $\langle n_i|m_j\rangle = \delta_{ij}$ and all other scalar products vanish. We construct R_m from K and $K^\star$ so that $\overline{R}_m = R_m$ and the product equality $R_m^\star R_m + R_m R_m^\star = 2\rho\, 1 \otimes G_{R,m}$ holds. In the massless case, where $G_R = 1 + aI$, the unique polynomial up to second order is given by

$$R_0 = \sqrt{\rho}\left[1 + i\sqrt{a}(K - K^\star) + \frac{1}{2}(i\sqrt{a}(K - K^\star))^2\right]$$

[1] The following one is positively hidden

$$\begin{pmatrix} 0 & 1 & -\beta & 0 \\ 0 & 1+\beta & 0 & -\beta \\ 0 & 0 & 1+\beta & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 0 & 1 & \beta & 0 \\ 0 & -1+\beta & 0 & \beta \\ 0 & 0 & 1+\beta & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is negatively hidden.

satisfying

$$R_0^\star R_0 + R_0 R_0^\star = 2\rho(1 + aI + O(4)) \sim 2\rho\, 1 \otimes G_{R,0}$$

up to fourth order[2] for a positive hidden structure K. For a negatively hidden structure

$$R_0 = \sqrt{\rho}\left[1 + i\sqrt{a}(K - K^\star)\right]$$

would give the correct quadrature upto second order in $K, K^\star$. Johnston [2] proposes a mass expansion for the massive propagator

$$R_m = R_0 + \overline{b}R_0^2 + \overline{b}^2 R_0^3 + \ldots = R_0(1 - \overline{b}R_0)^{-1}$$

where $\overline{b}$ is mass assymetric. Summarizing, we construct from a positively hidden structure K a retarded Green's function for the massless case satisfying $R_0 R_0^\star + R_0^\star R_0 = 2\rho(1 + aI + O(4))$ and $\overline{R}_0 = R_0$; from this we compute with a mass series expansion R_m satisfying $\overline{R}_m = R_m$ and to a good approximation the quadrature formula. Furthermore, the radiative Dirac propagator reads

$$\Delta_m = R_m - \widetilde{R}_m^T$$

and we build a theory of fermions in the next section and compute some massless examples in section four in which also different positive hidden structures K are compared.

3 Dirac Theory.

Dirac theory contains the equations

$$\{\psi_{\alpha x}, \psi_{\beta y}\} = 0$$

and

$$\{\psi_{\alpha x}, \overline{\psi}_{\beta y}\} = i\Delta_{\alpha\beta}(x, y)$$

where in our case $\overline{\psi}_{\alpha x} = \psi^H_{\alpha x}\delta \otimes 1$ (H denotes the Hermitian conjugate) so that

$$\{\psi_{\alpha x}, \psi^H_{\beta y}\} = i\Delta_{\alpha\kappa}(x, y)\delta^\kappa_\beta.$$

Using the spectral decomposition of $i\Delta$, this equation is equivalent to

$$\{\psi_{\alpha x}, \psi^H_{\beta y}\} = \sum_{r=1}^{k}\left(\lambda_r v_{r,\alpha x} v^H_{r,\beta y} + \mu_r w_{r,\alpha x} w^H_{r,\beta y}\right) + \sum_{r=1}^{n-k}\left(\kappa_r m_{r,\alpha x} n^H_{r,\beta y} + \overline{\kappa}_r n_{r,\alpha x} m^H_{r,\beta y}\right)$$

where the orthogonality is with respect $v^H\delta \otimes 1w$, $\lambda_r, \mu_r \in \mathbb{R}$ and $\kappa_r \in \mathbb{C}$. Therefore $i\Delta$ splits representation space $\mathcal{K} = \mathbb{C}^2 \otimes \mathbb{C}^n$ in three subspaces $\mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathcal{K}_0$ where $\mathcal{K}_+ = \mathrm{Span}_{\mathbb{C}}\{v_r | r = 1 \ldots k\}$, $\mathcal{K}_- = \mathrm{Span}_{\mathbb{C}}\{w_r | r = 1 \ldots k\}$ and $\mathcal{K}_0 = \mathrm{Span}_{\mathbb{C}}\{m_r, n_r | r = 1 \ldots k\}$. Define then $\mathcal{H}_0$ as the subspace spanned by all eigenvectors or null eigenpairs corresponding to a zero eigenvalue. Then, define for any j such that $\lambda_j \neq 0$, $a_j = \overline{v}_j\psi$, in case $\mu_j \neq 0$ we have that $b^\dagger_j = -\overline{w}_j\psi$

[2] One notices that the right hand side equals the first three terms in the power series expansion of $\exp\left(i\sqrt{a}(K - K^\star)\right)$. Note also, that unlike the continuum, our proposal does not satisfy $-R_0^\star = R_0$ however it does approximately in case the first order term dominates.

and finally in case $\kappa_j \neq 0$ we define $c_j = \overline{n}_j \psi$ and $d_j = \overline{m}_j \psi$. In other words, we pose that

$$\psi = \sum_{j:\lambda_j \neq 0} v_j a_j + \sum_{j:\mu_j \neq 0} w_j b_j^\dagger + \sum_{j:\kappa_j \neq 0} (m_j c_j + n_j d_j)$$

then the above anticommutation relations lead to the following nonzero commutation relations (all other relations vanish)

$$\begin{aligned} \{a_i, a_j^\dagger\} &= \lambda_j \delta_{ij} \\ \{b_i, b_j^\dagger\} &= \mu_j \delta_{ij} \\ \{c_i, d_j^\dagger\} &= \kappa_j \delta_{ij} \end{aligned}$$

as can be computed directly from their definition. For example, an elementary computation yields $\{a_j, b_k\} = -i\overline{v}_j \Delta w_k = 0$ and $\{a_j, a_k^\dagger\} = i\overline{v}_j \Delta v_k = \lambda_j \delta_{jk}$. Now, we are ready to pose the equivalent of the Dirac equations of motion: any $v \in \mathcal{H}_0$ automatically satisfying $\overline{v}\Delta = 0$ gives $\overline{v}\psi = 0$. In short, our theory contains positive and negative norm particles and anti-particles depending whether $\lambda_j, \mu_j > 0$ or smaller than zero. Also, it contains complex ghosts which cannot be ignored from the prescription; the vacuum $|0\rangle$ is then defined as the state annihilated by all a_i, b_j, c_k and d_k. It is rather interesting that negative norm particles enter the prescription of causal set fermions which is no surprise to this author as its use in physics has been advocated in plenty of other places.

4 Examples.

We work out massless theory on a diamond with a positively hidden structure given by

$$K = \begin{pmatrix} 0 & 1 & -\beta & 0 \\ 0 & 1+\beta & 0 & -\beta \\ 0 & 0 & 1+\beta & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $a = \frac{1}{4}$ gives the following retarded propagator

$$R_0 = \frac{1}{8} \begin{pmatrix} 8 & 0 & 1-4i\beta & 0 \\ 0 & 7+4i\beta & 0 & 1-4i\beta \\ 0 & 0 & 7+4i\beta & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}.$$

Hence, the Pauli-Jordan function equals

$$\Delta_0 = \frac{1}{8} \begin{pmatrix} 0 & 0 & 1-4i\beta & 0 \\ 0 & 0 & 0 & 1-4i\beta \\ 1-4i\beta & 0 & 0 & 0 \\ 0 & 1-4i\beta & 0 & 0 \end{pmatrix}$$

and has eigenvalues $\frac{i}{8} + \frac{1}{2}, -\frac{i}{8} - \frac{1}{2}$ plus its conjugates. Each eigenvalue and its conjugate corresponds to two conjugate null pairs and therefore the theory is one of two ghosts. The reader notices that distinct hidden structures can give rise to an inequivalent particle content and we leave such issues for the future.

References

[1] Steven Johnston, Particle propagators on discrete spacetime, Classical and Quantum gravity 25:202001, 2008 and arXiv:0806.3083

[2] Steven Johnston, Quantum fields on causal sets, PhD thesis, Imperial College London, September 20120, arXiv:1010.5514