# Products of Generalised Functions

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**Abstract**

An elementary algebra of products of generalised functions is constructed. A way of multiplying the defined generalised functions with polynomials is also given. The theory is given for single-variable functions but it can be easily generalised to the multi-variable case.

**Key Words:** generalised functions, product of distributions.

## 1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and in partial differential equation (e.g. shock wave solutions in hydrodynamics) see [1]. An important issue, related to products of distributions, is the fact that the product, in the general case, is not well defined in $D'$. This issue is known as the Schwartz impossibility result (see [1] §1.3). In the Schwartz classical theory, only the product between a smooth function and a distribution is well defined. Historically, products of distributions are addressed by means of algebras of generalised functions developed initially by J. F. Colombeau (see [1] and [2]).

The full Colombeau algebra is defined as the quotient algebra (see [1]):

$$\mathcal{G} = \mathcal{M}/\mathcal{I} \tag{1}$$

where $\mathcal{M}$ is a space of moderate functions and $\mathcal{I}$ is an ideal of null functions of $\mathcal{M}$. The elements of $\mathcal{G}$ are sequences obtained from the convolution of the functions to be embedded in $\mathcal{G}$ and the sequence $\rho_\epsilon = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$ with $\rho$ a suitable function called the mollifier. The meaning of moderate and null functions is made precise in the theory.

Aim of this paper is to define and study a new class of generalised functions which extend and include the $\delta^{(p)} \in D'$ (with $p \in \mathbb{N} \cup \{0\}$) and the step discontinuous functions. We will call this generalised function, $\eta$ functions and we will use them to propose a new way to represent products of generalised functions.

## 2 Preliminary Definitions

Before we go to the main point of the paper, we need a few simple definitions. In particular we need to define the sets of functions $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{A}_N$.

**Definition:** *Schwartz space $\mathcal{S}$ : Rapidly Decreasing Functions - We say that a function $f(x)$ is Rapidly Decreasing if:*

$$\begin{cases} f(x) \in C^\infty \\ \lim\limits_{x\to\pm\infty} x^k f^{(p)}(x) = 0 \quad \forall p, k \in \mathbb{N} \cup \{0\} \end{cases} \tag{2}$$

*We define $\mathcal{S}$ to be the set of all Rapidly Decreasing Functions f.*

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**Note:** $\mathcal{S}$ is closed with respect to addition, point-wise multiplication, differentiation and is compliant with the Leibniz rule. It is therefore a differential algebra.

**Note:** $f(x) \in \mathcal{S} \Rightarrow xf(X) \in \mathcal{S}$.

**Definition:** *Set $\mathcal{O}$ : Function of order p - This definition goes in three parts:*

*(1) We say that a function $g^{(1)}$ is of order 1 if:*

$$\begin{cases} g^{(1)} \in \mathcal{S} \\ 0 < \left| \int_{-\infty}^{+\infty} g^{(1)}(x)dx \right| < \infty \end{cases} \tag{3}$$

*(2) For $p > 1$, we say that a function $g^{(p)}$ if of order p if it is the derivative of a function $g^{(p-1)}$ of order p-1.*

*(3) For $p < 1$, we say that a function $g^{(p)}$ if of order p if its derivative is a function $g^{(p+1)}$ of order p+1.*

*We define $\mathcal{O}^p$ to be the set of all functions of order p. We define $\mathcal{O} = \bigcup_{p \in \mathbb{Z}} \mathcal{O}^p$.*

**Definition:** *(Family of functions) - We say that a set $F \subset \mathcal{O}^p$ form a family if for any two functions $g^{(p_1)}, g^{(p_2)} \in F$ exists a function $g^p \in F$ such that differencing a sufficient number of times $g^{(p_1)}$ and $g^{p_2}$ we get $g^p$.*

**Definition:** *(Amplitude of a function) - Elements of the set $\mathcal{O}$ come with an amplitude. Given a functions $g^{(p)} \in \mathcal{O}^p$, the amplitude "A" of $g^{(p)}$ is the integral:*

$$A = \int_{\mathbb{R}} g'(x)dx \tag{4}$$

*where $g'$ belong to the same family of $g^{(p)}$.*

Clearly for $p < 1$, two function of same order and of the same family differ by a polynomial of degree $|p|$. Among all those functions we want to define a special one:

**Definition:** *(Centred Function) - We say that a function $g^{(p)}$ is centred with respect of the y axis if:*

$$\lim_{x \to -\infty} g^{(p)}(x) = 0 \tag{5}$$

*For $p > 0$, all elements of $\mathcal{O}^p$ are centred by definition.*

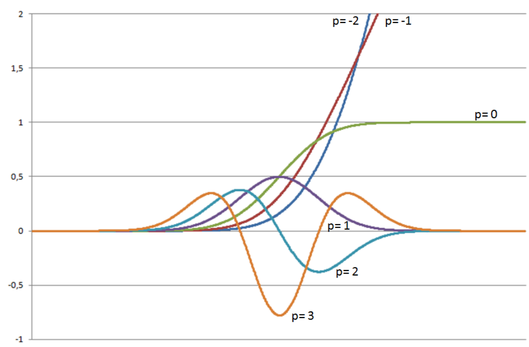


Figure 1: Example of Order p Amplitude 1 Centred Functions

**Note:** $\mathcal{O}$ is closed with respect to addition, point-wise multiplication, differentiation and it is compliant with the Leibniz rule. It is therefore a differential algebra. Closure with respect to multiplication may not be easy to see at first glance and it is due to the fact that any product in $\mathcal{O}$, if differentiated enough times, will give a function that belong to $\mathcal{S}$ and that $\mathcal{S} \subset \mathcal{O}$.

**Note:** $g^{(p)}(x) \in \mathcal{O} \Rightarrow xg^{(p)}(x) \in \mathcal{O}$. This can be seen integrating by parts and showing that the integral is in $\mathcal{O}$.

**Notation:** *We may use the notation* $g$ *instead of* $g^{(0)}$, $g'$ *instead of* $g^{(1)}$, $g''$ *instead of* $g^{(2)}$ *and so on.*

**Definition:** *Sets* $\mathcal{A}_N$ *:* $\xi$ *Functions - We say that a function is a* $\xi$ *Function if:*

$$\begin{cases} \xi \in \mathcal{O}^0 \\ \\ \int_{-\infty}^{+\infty} \xi'(x) x^k dx = \begin{cases} 1 & for\ k = 0 \\ 0 & for\ 0 < k \leq N \ \ with\ k, N \in \mathbb{N} \end{cases} \end{cases} \tag{6}$$

*We define* $\mathcal{A}_N$ *to be the set of all* $\xi$ *functions for a given* $N$*.*

**Note:** $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \supset \mathcal{A}_\infty = \emptyset$, because functions with all vanishing momenta do not exist.

# 3 $\eta$ Functions

Aim of this paper is to define and study a new class of generalised functions which extend and include the $\delta^{(p)} \in D'$ (with $p \in \mathbb{N} \cup \{0\}$) and the step discontinuous functions. We will call this generalised function, $\eta$ functions.

**Definition 1.** *Let* $\phi^{(p)} \in \mathcal{O}^p$ *be a function of order* $p$*. We define an* $\eta$ *function of indices* $q$ *and* $p$ *to be the following sequence of functions in the parameter* $n \in \mathbb{N}$*.*

$$\eta^{q,p}_{\phi^{(p)}} = n^q \phi(nx) \tag{7}$$

*where:*

- $q \in \mathbb{Z}$ *is called the growing index.*
- $p \in \mathbb{Z}$ *is called the order and it is the order of the* $\phi^{(p)}$ *function.*
- $\phi^{(p)}$ *is called the base function.*

Note that in the above notation, p is redundant because p is the order of $\phi^{(p)}$ and it is given once we give $\phi^{(p)}$. For the above reason it may be omitted in the notation.

We call the $\eta$ "functions", although they are sequences of functions. The reason is that in some cases they have a limit in $D'$ for n going to infinite and therefore these object can be mapped to those functions. Let us see some examples.

**Examples:** Let $g \in \mathcal{O}^0$ a centred functions of order 0 and amplitude 1. We have:

$$\lim_{n\to\infty} \eta^{0,0}_g = \lim_{n\to\infty} g(nx) = u(x) \tag{8}$$

where $u(x)$ is the Heaviside function.

$$\lim_{n\to\infty} \eta^{1,1}_{g'} = \lim_{n\to\infty} ng'(nx) = \delta(x) \tag{9}$$

since the above limit is the derivative of the previous one.

$$\lim_{n\to\infty} \eta^{p,p}_{g^{(p)}} = \lim_{n\to\infty} n^p g^{(p)}(nx) = \delta^{(p-1)}(x) \tag{10}$$

since the above limit is the previous one differentiated p-1 times. We can therefore map those $\eta$ functions to elements of $D'$.

Let us see how this mapping goes:

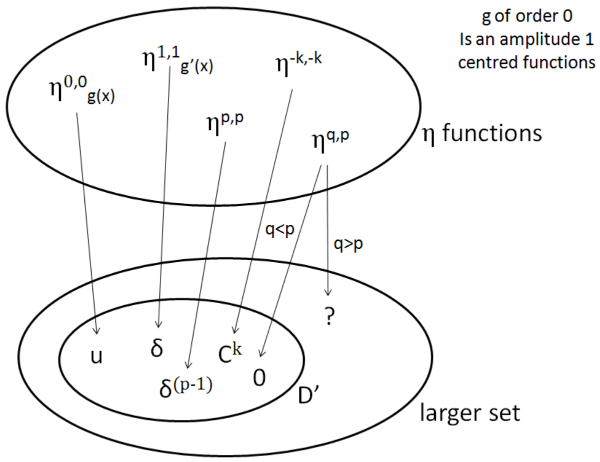


Figure 2: Mapping of $\eta$ functions to D'

From the above picture we see that $\eta^{q,p}$ function are mapped to $D'$ when $q = p$, are mapped to the zero constant function in $D'$ when $q < p$ and have to be mapped to a larger set including $D'$ when $q > p$. We will see later on that one of those functions (that cannot be mapped to $D'$) is $\delta^2$.

**Notation:** *($R^{q,p}$ and $\approx$ Symbol) - According to the Colombeau Algebras theory, by definition two generalised functions $T_1$ and $T_2$ can be associated in $D'$ when $\langle T_1 - T_2, \phi\rangle = 0$ where $\langle\cdot,\phi\rangle$ is the Schwartz functional and $\phi$ is a compact support test function.*

*In our theory this is equivalent to say that $T_1 - T_2 = R(\eta^{q,p})$ with $q - p < 0$. In our notation $R(\eta^{q,p})$ represents a generalised function which is the sum of many $\eta$ functions all of which having (growing index - order) not greater of $q - p$. In this case we say that the functions can be associated in $D'$, meaning they represent the same function in $D'$, and we write $T_1 \approx T_2$.*

# 4 Components of an $\eta$ Function

**Proposition:** *Given any function $\phi^{(p)} \in \mathcal{O}^p$ and an integer $N \in \mathbb{N}$, it exists a unique function $\xi \in \mathcal{A}_N$ and an unique function $r_N(x) \in \mathcal{O}^{N+1}$, such that we have:*

$$\phi^{(p)}(x) = \sum_{k=p}^{N} a_k \xi^{(p)}(x) + r_N(x) \tag{11}$$

*Proof:* For $p > 0$, before we prove that $\xi$ exists and it is unique, we need to show how to evaluate the coefficients $a_k$ once $\xi$ is given.

Let us suppose a $\xi$ exists. If we multiply both sides of Eq. (11) by $x^{k-1}$ we have:

$$\int_{\mathbb{R}} x^{k-1}\phi^{(p)}(x)dx = \sum_{k=p}^{N} a_k \int_{\mathbb{R}} x^{k-1}\xi^{(k)}(x)dx + \int_{\mathbb{R}} x^{k-1} r_N(x)dx \tag{12}$$

We integrate by parts the terms on the right hand side of the equation in order to decrease the degree of the monomials in $x$. For terms up to $a_k$ we integrate till we get $\xi'$ in the integral. For others terms we integrate $k-1$ times such that the monomial disappear. We have:

$$\begin{aligned}\int_{\mathbb{R}} x^{k-1}\phi^{(p)}(x)dx &= \ldots + a_{k-1}\frac{(k-2)!}{(-1)^{k-2}}\overbrace{\int_{\mathbb{R}} x\xi'(x)dx}^{=0} + a_k\frac{(k-1)!}{(-1)^{k-1}}\overbrace{\int_{\mathbb{R}} \xi'(x)dx}^{=1} \\ &+ a_{k+1}\frac{(k-1)!}{(-1)^{k-1}}\overbrace{\int_{\mathbb{R}} \xi''(x)dx}^{=0} + \ldots + \overbrace{\int_{\mathbb{R}} x^{k-1}r_N(x)dx}^{=0}\end{aligned} \tag{13}$$

The terms before the one related to $a_k$ are zero because the momenta of $\xi'$ greater then 1 are zero by definition of a $\xi$ function. The terms after $a_k$ are zero because the integrands are of order greater then 1 and $r_N$ has order greater then $N$. The only term left is the one related to the coefficient $a_k$. We have:

$$a_k = \frac{(-1)^{k-1}}{(k-1)!}\int_{-\infty}^{+\infty} x^{k-1}\phi^{(p)}(x)dx \tag{14}$$

We want now to prove that such $\xi$ function exist and it is unique. Given Eq. (11), we define a new function $\phi_0^{(p)} = \phi^{(p)}/a_p$ which has the first component equal to 1 and all other components equal to $b_k = a_k/a_p$.

$$\phi_0^{(p)} = \xi^{(p)} + b_{p+1}\xi_{p+1}^{(p+1)} + b_{p+2}\xi_{p+2}^{(p+2)} + \ldots + r_{1N} \tag{15}$$

Given the above function, we define a new function $\phi_1^{(p)} = \phi_0^{(p)} - b_{p+1}(\phi_0^{(p)})'$ which has the first component equal to 1, the second component equal to 0 (removed) and all other components equal to $c_k = b_k - b_{p+1}b_{k-1}$.

$$\phi_1^{(p)} = \xi^{(p)} + 0 + c_{p+2}\xi_{p+2}^{(p+2)} + \ldots + r_{NN} \tag{16}$$

by iterating this process we can remove one component at a time till we get to the function:

$$\phi_N^{(p)} = \xi^{(p)} \tag{17}$$

Which, once $N$ is given, it is unique and it is the $\xi^{(p)}$ we where looking for. Note that given $N$, if $\xi$ is unique, then also $r_N$ is.

For $p \leq 0$, we can differentiate $\phi^p$ a number $|p|+1$ of times, perform the reasoning as above, and then integrate back to get the original function. □

The meaning of the above proposition is that for any integer $N$, any function of order $p$ is made of components of order $\geq p$ which are given by functions of the same family and which amplitude is give by the coefficients $a_k$.

Let us see the effect on the structure of an $\eta$ function of the above theorem. Given a function $\eta^{q,p}_{\phi^{(p)}}$, we write its definition:

$$\eta_{\phi}^{q,p} = n^q\phi^{(p)}(nx) \tag{18}$$

In the above definition we substitute (11) we have:

$$\eta^{q,p}_{\phi^{(p)}} = n^q\sum_{k=p}^{N} a_k\xi^{(p)}(nx) + n^q r_N(nx) = \sum_{k=p}^{N} a_k\eta^{q,k}_{\xi^{(k)}} + R(\eta^{q,N+1}_{\xi^{(N+1)}}) \tag{19}$$

where the $\eta^{q,k}_{\xi^{(k)}}$ can be considered kind of single order generalised functions meaning that, apart from a function of order $k$, they contain only functions of order $\geq N$ and $R(\eta^{q,N+1})$ represent a negligible generalised function having (growing index - order) not greater then $q-(N+1)$.

Given (14), the coefficient $a_k$ do not depend on $N$ and therefore we can take the limit for $N \to \infty$. We have:

$$\eta_{\phi}^{q,p} = \sum_{k=p}^{\infty} a_k\eta^{q,k}_{\xi^{(k)}} \tag{20}$$

and therefore any $\eta$ function defined by (7) can be expressed as the sum of single order components $\eta^{q,k}_{\xi^{(k)}}$ of constant growing index $q$ and increasing order $k$.

Let us see how this mapping goes:

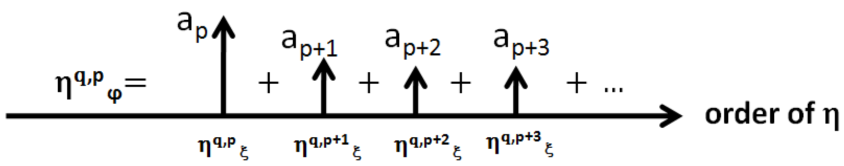


Figure 3: components of an $\eta$ Function

If an $\eta$ function is composed of single order components, the behaviour of the function will be dominated by the leading component which is the component with the higher difference $q - p$.

## 5 An Algebra of $\eta$ Functions

Now that we have defined the $\eta$ functions, we want to make some algebraic structures out of them. First of all we need to define a + operation among $\eta$ functions. This is easy because $\eta$ functions are sequences of functions and their sum can be defined simply as the sequence we get adding term by term the two initial sequences. Multiplication and differentiation can also be defined in the same way.

**Definition:** *Let $\phi_1^{(p_1)}$ and $\phi_2^{(p_2)}$ be two function of order $p_1$ and $p_2$ not necessarily centred with respect to the $y$ axis. We define the sum of two $\eta$ functions as the follows sequence:*

$$\textbf{\textit{Sum:}} \quad \eta^{q_1}_{\phi_1} + \eta^{q_2}_{\phi_2} = n^{q_1}\phi_1^{(p_1)}(nx)n^{q_1} + \phi_2^{(p_2)}(nx) \tag{21}$$

*which is not a proper $\eta$ function if $q_1 \neq q_2$.*

*And the product of $\eta$ functions as follows:*

$$\textbf{\textit{Product:}} \quad \eta^{q_1}_{\phi_1} \cdot \eta^{q_2}_{\phi_2} = n^{q_1}\phi_1(nx)n^{q_1}\phi_1(nx) = n^{q_1+q_2}\phi_1(nx)^{(p_2)}\phi_2^{(p_2)}(nx) = \eta^{q_1+q_2}_{\phi_1\phi_2} \tag{22}$$

*Moreover, let $\phi^{(p)}$ be a function of order $p$ not necessarily centred with respect to the $y$ axis. We define the derivative of an $\eta$ function as follows:*

$$\textbf{\textit{Differentiation:}} \quad \frac{d}{dx}\eta^{q}_{\phi} = \frac{d}{dx}n^{q}\phi^{(p)}(nx) = n^{q+1}\phi'(nx) = \eta^{q+1}_{\phi'} \tag{23}$$

**Definition:** *Let $g \in \mathcal{O}^0$ a function of order zero, we define the set $\mathcal{F}(g)$ to be the set composed of:*

- *For any $p \in \mathbb{Z}$, all $\eta$ functions with base functions $g^{(p)}$ of the same family of $g$.*
- *For any $k \in \mathbb{Z}$, all $\eta$ function with base the monomial $x^k$.*
- *For any smooth function $f$, all $\eta$ function with base function $f(ag(x) + b)$ with $a, b \in \mathbb{R}$.*
- *All finite or infinite linear combinations of finite products of the $\eta$ functions defined above.*

We note that $\mathcal{F}(g)$ is a vector and it is possible to show that it is Cauchy complete (and this is the reason why we require also infinite linear combinations) and therefore $\mathcal{F}(g)$ is an Hilbert space. Moreover $\mathcal{F}(g)$ is equipped with multiplication among $\eta$ functions, which is a bilinear map, and a definition of differentiation which is compliant with the Leibniz rule and therefore $\mathcal{F}(g)$ is also a differential algebra.

## 6 Product of Distributions

We can associate distributions and product of distribution to elements of $\mathcal{F}(g)$ in the obvious way. For example:

$$\begin{array}{lcl} u(x) & \rightarrow & \eta_g^{0,0} \\ \delta(x) & \rightarrow & \eta_{g'}^{1,1} \\ f(u(x))\delta(x) & \rightarrow & \eta_{f(g)g'}^{1,1} \\ \delta(x)\delta'(x)+\delta^2(x) & \rightarrow & \eta_{g'g''}^{3,3}+\eta_{(g')^2}^{2,1} \end{array} \tag{24}$$

and so on.

And therefore $\mathcal{F}(g)$ is our differential algebra of product of distributions. Sums of products of distribution T in $\mathcal{F}(g)$ are the sum of $\eta$ functions with different base functions of the type:

$$T = \eta_{\xi_1}^{(p_1)} + \eta_{\xi_2}^{(p_2)} + \cdots \tag{25}$$

Their structure is determined by the leading terms (i.e. the terms with the higher difference $q-p$).

**Definition:** *We say that a generalised function $T \in \mathcal{F}(g)$ is determinate if the order and amplitude of its leading terms do not depend from $g$. We say that a generalised function $T \in \mathcal{F}(g)$ is indeterminate it is not determinate.*

We note explicitly that the theory developed in this paper is completely equivalent to the Colombeau algebras and that the set $\mathcal{F}(g)$ is basically a subset of the set $\mathcal{M}$ of the Colombeau theory, mentioned in the introduction, in a disguised form. We show this with an example. We evaluate the generalised function $A(\rho, x) \in \mathcal{M}$, of the Coombeau theory, associate to the product $\delta^2$. If $B(\rho, x) \in \mathcal{M}$ is the generalised function associated to $\delta$ we have:

$$A(\rho, x) = (B(\rho, x))^2 = \left( \int_{-\infty}^{+\infty} \delta(\tau)\rho_\epsilon(x-\tau)d\tau \right)^2 = \left( \frac{1}{\epsilon}\rho\left(\frac{x}{\epsilon}\right)\right)^2 \tag{26}$$

where $\rho$ is the mollifier. By setting $\epsilon = \frac{1}{n}$ we see that $A(\rho, x) \in \mathcal{M}$ and $\eta_{(g')^2}^{2,1} \in \mathcal{F}(g)$ (see example above) are the same sequence. Moreover, the $\eta$ functions in particular and the elements of $\mathcal{F}(g)$ in general are moderate function by construction.

## 7 Order of the Product of Functions

For the definition of product of distributions given in the previous paragraph to make sense, the order of the product of $N$ centred functions $g^{(p_i)} \in \mathcal{F}(g)$ of the same family, with $i = 1 \ldots N$, should not depend from $g$. This seems to be true (see Appendix A1).

Moreover, the order of the product of the above product of functions $g^{(p_i)} \in \mathcal{F}(g)$ with a monomial $x^k$, with $k \in \mathbb{Z}$, should not depend from $g$. This seems to be true in most relevant cases (see Appendix A1.).

In this paragraph we will give a few results evaluated numerically for the product of two functions.

**Product of Functions:** Let $g^{(p_1)}, g^{(p_2)} \in \mathcal{O}^p$, two centred functions of amplitude 1 and belonging to the same family, we have that the order of the product $g^{(p_1)}g^{(p_2)}$ has order/amplitude given by the following table:

| Ord./Amp. | $p_2=-2$ | $p_2=-1$ | $p_2=0$ | $p_2=1$ | $p_2=2$ | $p_2=3$ |
|---|---|---|---|---|---|---|
| $p_1=-2$ | -4/6 | -3/3 | -2/1 | 1/? | 1/? | 1/? |
| $p_1=-1$ | | -2/2 | -1/1 | 1/? | 1/? | 1/? |
| $p_1=0$ | | | 0/1 | $1/\frac{1}{2}$ | 1/? | 2/? |
| $p_1=1$ | | | | 1/? | 2/? | 1/? |
| $p_1=2$ | | | | | 1/? | 2/? |
| $p_1=3$ | | | | | | 1/? |

Table 1 : Order of the Product of Functions (? = Indeterminate)

**Product of a Function for a Monomial:** Let $g^{(p)} \in \mathcal{O}^p$ and $k \in \mathbb{Z}$, we have that the order of the product of a centred function $g^{(p)}$ of amplitude 1 with $x$ has order/amplitude given by the following table:

| Ord./Amp. | $p=-2$ | $p=-1$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ |
|---|---|---|---|---|---|---|
| $xg^{(p)}$ | -3/3 | -2/2 | -1/1 | 1 or 2/? | 1/-1 | 2/-2 |

Table 2 : Order/amplitude of the function times x (? = Indeterminate)

**Product of Step and Delta Functions:** Let $g \in \mathcal{O}^0$ and $f$ a smooth function, we have that the order of the product $f(g)$ is 0 and the order of $f(g)g'$ is 1.

**More Results:** In Appendix A.1 a few more initial results on the order of products of functions of a given order is presented with a sketch of a more formal proofs.

**Example:** We have seen that $\delta(x) \in D'$ can be mapped to $\eta^{1,1}_{g'} \in \mathcal{F}(g)$. The square of a function of order 1 has order 1. We have:

$$\delta(x) = \eta^{1,1}_{g'} = a_1\eta^{1,1}_{\xi'} + a_2\eta^{1,2}_{\xi''} + \dots \tag{27}$$

The leading term is $\eta^{1,1}_{\xi'}$ which has $q-p=0$. Using (14), and keeping in mind that g has amplitude 1, we have:

$$a_1 = \int_{\mathbb{R}} g'(x)dx = 1 \tag{28}$$

and therefore $\delta(x)$ is determinate.

**Example:** now we can say that $\delta^2(x)$ can be mapped to $\eta^{2,1}_{(g')^2} \in \mathcal{F}(g)$. We have:

$$\delta^2(x) = \eta^{2,1}_{(g')^2} = a_1\eta^{2,1}_{\xi'} + a_2\eta^{2,2}_{\xi''} + \dots \tag{29}$$

The leading term is $\eta^{2,1}_{\xi'}$ which has $q-p=1$. Using (14), we have this time:

$$a_1 = \int_{\mathbb{R}} (g'(x))^2 dx = \text{depends on g} \tag{30}$$

and therefore $\delta^2(x)$ is indeterminate.

## 8 Product of Step and Delta Functions

We want to show now why we wanted to include terms as $f(g)$ in $\mathcal{F}(g)$ with f smooth (see also [4]).

**Proposition:** *Let f be a locally integrable function and h(x) be a step discontinuous function defined as:*

$$h(x) = \begin{cases} a & x < 0 \\ b & x > 0 \end{cases} \tag{31}$$

*we have:*

$$f(h(x))\delta(x) \approx \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\delta(x) \tag{32}$$

*where $\approx$ has to be intended as the association relation in the Colombeau algebras.*

We have that $h(x) = a + (b-a)u(x)$, where $u(x)$ is the Heaviside unitary step discontinuous function. Ler $g \in \mathcal{O}^0$ be a centred function of amplitude 1. The function $f(g)g'$ is of order 1 regardless the function $g$. We have:

$$f(h(x))\delta(x) = \eta^{1,1}_{f(a+(b-a)g)g'} = a_1\eta^{1,1}_{\xi'} + a_2\eta^{1,2}_{\xi''} + \dots \tag{33}$$

The leading term is $\eta^{1,1}_{\xi'}$. Using (14) we have:

$$\begin{aligned} a_1 &= \int_{\mathbb{R}} f(a+(b-a)g(x))g'(x)dx \\ &= \frac{1}{b-a}\int_{\mathbb{R}} f(\overbrace{a+(b-a)g(x)}^{h(x)})d(\overbrace{a+(b-a)g(x)}^{h(x)}) \\ &= \frac{1}{b-a}[F(a+(b-a)g(x))]^{+\infty}_{-\infty} = \frac{1}{b-a}[F(x)]^b_a \qquad (34) \\ &= \frac{1}{b-a}(F(b)-F(a)) = \frac{1}{b-a}\left(\int_a^b f(x)dx\right) \end{aligned} \tag{35}$$

where $F(x)$ is a primitive of $f(x)$. Since $\eta^{1,1}_{\xi'} = \delta(x)$ this prove the proposition. □

Generalised functions f(a+bu(x)), with $a, b \in \mathbb{R}$ and $u(x)$ the Heaviside function, and their product with delta functions, are interesting products we want to include in our theory.

## 9 Products with Polynomials

Now we want to extend our product of generalised functions to products involving polynomial. Once again if we use definition (7). For a monomial $x^k$ with $k \in \mathbb{Z}$, we have:

$$x^k \cdot \eta^q_\phi = x^k n^q \phi(nk) = n^{q-k}(nx)^k\phi(nk) = \eta^{q-k}_{x^k\phi} \tag{36}$$

Now for $p > 1$, given any $\eta^{q,p}_{g^{(p)}} \in \mathcal{F}(g)$ we know that the order of $xg^{(p)}$ is p-1. Taking also into account (36), we have:

$$\eta^{q,p}_{g^{(p)}} = a_p\,\eta^{q,p}_{\xi_1^{(p)}} + a_{p+1}\,\eta^{q,p+1}_{\xi_1^{(p+1)}} + a_{p+2}\,\eta^{q,p+2}_{\xi_1^{(p+2)}} + \dots \tag{37}$$

Moreover we have:

$$\eta^{q-1,p-1}_{xg^{(p)}} = b_{p-1}\,\eta^{q-1,p-1}_{\xi_2^{(q-1)}} + b_p\,\eta^{q-1,p}_{\xi_2^{(p)}} + b_{p+1}\,\eta^{q-1,p+1}_{\xi_2^{(p+1)}} + \dots \tag{38}$$

Using (14) we have:

$$b_k = \frac{(-1)^{(k-1)}}{(k-1)!}\int_{\mathbb{R}} x^{k-1}\left(xg^{(p-1)}\right)dx = -k\frac{(-1)^{(k)}}{k!}\int_{\mathbb{R}} x^k g^{(p-1)}dx = -ka_{k+1} \tag{39}$$

From which we see clearly that the product of a generalised function of order $p$, with $x$, lower the order of the generalised function by 1. To sum up, we have:

$$x\eta^{q,p} = -(p-1)\eta^{q-1,p-1} \text{ for } p > 1 \tag{40}$$

and in particular for $s = p - 1$ (there is a difference of one between order of $\eta$ and of $\delta$ functions):

$$x\delta^{(s)} = -s\delta^{(s-1)} \text{ for } s > 0 \tag{41}$$

which is a well known result in literature (compare with [3]).

For $p = 1$, (i.e. $s = p - 1 = 0$), multiplying a function of order 1 with $x$, we cannot further lower its order. This is because $g^{(1)} \in \mathcal{S}$ and by definition there is no way to get a function of order 0 by multiplying and element of $\mathcal{S}$ with a polynomial.

The product of $g^{(1)} \in \mathcal{S}$ by $x$ gives a function of order 2 if $g$ is symmetric and 1 if it is not. We get therefore a function of order and amplitude indeterminate. With a fancy notation we may write:

$$x\eta^{q,1} = ?\eta^{q-1,?} \tag{42}$$

where the order ? means 1 or 2.

For $p < 1$, multiplying a function of order p by x we lower again its order. It is not difficult to show that:

$$x\eta^{q,p} = -(p-1)\eta^{q-1,p-1} \tag{43}$$

which is similar to the result for $p > 1$.

**Example:** If $vp\frac{1}{x}$ is the Cauchy principal value of $\frac{1}{x}$ then we have:

$$0 \cdot \frac{1}{x} = (\delta(x) \cdot x) \cdot vp\frac{1}{x} = \delta(x) \cdot \left(x \cdot vp\frac{1}{x}\right) = \delta(x) \tag{44}$$

which is absurd.
By using our theory we know that $x\delta(x) = ?\eta^{0,?}_{x\phi} \Rightarrow 0 = x\delta(x) - ?\eta^{0,?}_{x\phi}$. We have:

$$0 \cdot \frac{1}{x} = (x \cdot \delta(x) - ?\eta^{0,?}_{x\phi}) \cdot \frac{1}{x} = \delta(x) - \frac{1}{x}?\eta^{0,?}_{x\phi} = \delta(x) - \delta(x) \tag{45}$$

a results that now makes sense.

# 10 Products with Smooth Functions

Let $f$ be a smooth function, we want to study the product:

$$f(x)\eta^{q,p} \tag{46}$$

By Taylor expanding we have:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots \tag{47}$$

For $\eta^{q,p}$ we have:

$$\eta^{q,p} = b_p \eta^{q,p}_{\xi^{(p)}} + b_{p+1}\eta^{q,p+1}_{\xi^{(p+1)}} + \dots \tag{48}$$

And therefore the product is an infinite sum of product by $\eta$ functions by monomials and we know how to handle that.

Let us see how these product components are mapped to $\eta$ functions:

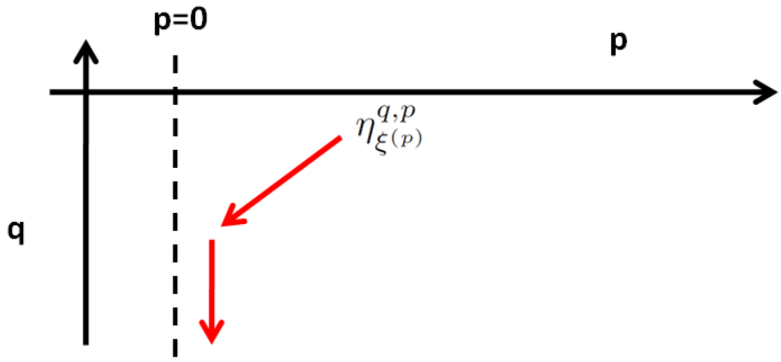


Figure 4: Product with Smooth functions

As a general statement, we may say that the components of $\eta^{q,p}$, when multiplied by the monomials of the smooth function, get decreased in both order and growing index till they hit the wall of the order $p = 1$ where they are still lowered in growing index but cannot be lowered in order any more. The final tructure of the generalised function is therefore defined by this components.

The final generalised function will have many components with $q > p$ and therefore dominated by the other leading terms. This components will not in general show up in further products except for very specific cases.

**Examples:** In the Schwartz theory of distributions we have:

$$f(x)\delta'(x) \quad \rightarrow \quad \langle f\delta', \phi\rangle = \langle \delta', f\phi\rangle \tag{49}$$

where $f$ is a smooth function, $\phi$ is a compact support test function and $\langle \cdot, \phi\rangle$ is the Schwartz functional.

Using the definition of derivative in the Schwartz theory of distributions we have:

$$\langle \delta', f\phi\rangle = -\langle \delta, \frac{d}{dx}(f\phi)\rangle = -f'(0)\phi(0) - f(0)\phi'(0) \tag{50}$$

which means (see [5]):

$$f(x)\delta(x) = -f(0)\delta(x) + f(0)\delta'(x) \tag{51}$$

In our theory we have:

$$f(x)\delta'(x) = (f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots)(\eta^{2,2}_{\xi(2)} + \eta^{2,3}_{\xi(3)} + \ldots) \tag{52}$$

and according to Eq. (41) we have:

$$f(x)\delta'(x) = f'(0)x\eta^{2,3}_{\xi(3)} + f(0)\eta^{2,2}_{\xi(2)} + R(\eta^{2,2}) \approx -f'(0)\delta(x) + f(0)\delta'(x) \tag{53}$$

where $R(\eta^{2,2})$ is a generalised function with terms having (growing index - order) not greater then 0.

## 11 Examples of Products of Generalised Functions

In this section we give a few examples of how the theory developed in this paper can be used.

**Example 1:** Prove the following identity:

$$\delta^2(x) \approx -u(x)\delta'(x) - \frac{1}{2}\delta'(x) \tag{54}$$

Given the product $\delta^2(x)$ we have;

$$\delta^2 = \eta^{2,1}_{(g')^2} = a_1\eta^{2,1}_{\xi_1'} + a_2\eta^{2,2}_{\xi_1''} + a_3\eta^{2,3}_{\xi_1'''} + \ldots \tag{55}$$

In this case the leading term which dominates on all others is $\eta^{2,1}_{\xi_1'}$ which has the higher difference $q-p = 2-1 = 1$. The leading term has definite order but unfortunately its amplitude depend from g. This distribution is therefore undetermined.

We consider now the product $u(x)\delta'(x)$:

$$u\delta' = \eta^{2,1}_{gg''} = b_1\eta^{2,1}_{\xi_2'} + b_2\eta^{2,2}_{\xi_2''} + b_3\eta^{2,3}_{\xi_2'''} + \ldots \tag{56}$$

Also in this case the leading term which dominates on all others is $\eta^{2,1}_{\xi_2'}$ and it has determinate order but indeterminate amplitude. Also this distribution is therefore undetermined.

Using Eq. (14), we evaluate now the amplitude of the leading term of $\delta^2$.

In this case we will use a centred function $g$ of order 0 and amplitude 1. We have:

$$a_1 = \int_{\mathbb{R}} (g')^2 dx = -\int_{\mathbb{R}} gg'' dx = -b_1 \tag{57}$$

From the above we see that the two distribution above, although indeterminate, given $g$ have the same amplitude and $\xi$ function (i.e. $\xi_1 = \xi_2$. Using Eq. (14), we evaluate now the term $a_2$ of the product $\delta^2$.

$$b_2 = \int_{\mathbb{R}} x(g')^2 dx = -\int_{\mathbb{R}} (xg'' + g')\, g dx = -\overbrace{\int_{\mathbb{R}} xgg'' dx}^{=b_2} - \overbrace{\int_{\mathbb{R}} gg' dx}^{=\frac{1}{2}} \tag{58}$$

where the second term is equal to $\frac{1}{2}$ because integrating by parts we have:

$$\int_{\mathbb{R}} gg' dx = \overbrace{[gg]^{+\infty}_{-\infty}}^{=1} - \int_{\mathbb{R}} g'g dx \quad \Rightarrow \quad 2\int_{\mathbb{R}} gg' dx = 1 \tag{59}$$

We have eventually:

$$\delta^2 + u\delta' = (a_1 - a_1)\eta^{1,1}_{\xi'} + (a_2 - a_2 - \frac{1}{2})\delta' + R\left(\eta^{1,3}\right) \tag{60}$$

where $R\left(\eta^{1,3}\right)$ represent a function with leading term not higher of $\eta^{1,3}$. Ignoring this function which is dominated by the other terms in the identity we have:

$$\delta^2(x) \approx -u(x)\delta'(x) - \frac{1}{2}\delta'(x) \tag{61}$$

**Example 2:** Product of 1 times $\delta$ function.

**a)** The product between the constant function 1 and a delta is clearly:

$$1 \cdot \delta(x) = \delta(x) \tag{62}$$

**b)** We consider now the function $sign^2(x)$ which is equal to 1 for $x \neq 0$ and it is not defined in 0. Given the function $f = (2x-1)^2$, we have:

$$sign^2(x)\delta(x) = f(u(x))\delta(x) \tag{63}$$

where $u(x)$ is the Heaviside function. The product $sign^2(x)\delta(x)$ is therefore:

$$sign^2(x)\delta(x) = \eta^{1,1}_{f(g)g'} = a_1\eta^{1,1}_{\xi_1} + a_2\eta^{1,2}_{\xi_1'} + a_3\eta^{1,2}_{\xi_1''} + \ldots \tag{64}$$

where $g(x)$ is a centred zero order function of amplitude 1. The leading term is $a_0$ and using Eq. (14) we have:

$$a_1 = \int_{\mathbb{R}} f(g)g'dx = [F(g(x))]_{-\infty}^{\infty} = [F(x)]_0^1 \tag{65}$$

where $F(x)$ is a primitive of $f(x)$. From the above:

$$a_1 = \int_0^1 f(x)dx = \int_0^1 (2x-1)^2 dx = \left[\frac{4x^3}{3} - 2x^2 + x\right]_0^1 = \frac{1}{3} \tag{66}$$

and therefore:

$$sign^2(x)\delta(x) \approx \frac{1}{3}\delta(x) \tag{67}$$

compare with [2] §1.1 ex. iii and with [4].

**c)** Finally we define the 1(x) function, which is equal to 1 for $x \neq 0$ and it is not defined in 0, to be:

$$1(x) = -1 + \frac{1}{1 - \frac{1}{2}sign^2(x)} \tag{68}$$

Given the function:

$$f(x) = -1 + \frac{1}{1 - \frac{1}{2}x^2} = \frac{1}{2}\frac{x^2}{2 - x^2} \tag{69}$$

We have that $1(x) = \eta_{f(g)}^{0,0}$ where $g(x)$ is an order zero function of amplitude 1 with $g(-\infty) = -1$ and $g(-\infty) = 1$

Following the same reasoning as before we have:

$$1(x)\delta(x) \approx \frac{1}{2}\left[\int_{-1}^1 \frac{x^2}{2-x^2}dx\right]\delta(x) = \left[\frac{\sqrt{2}}{2}\ln\left(\frac{2+\sqrt{2}}{2+\sqrt{2}}\right) - 1\right]\delta(x) \tag{70}$$

From a), b) and c) we see that $1 \approx sign^2(x) \approx 1(x) \in D'$ but their products by a delta give a completly different result. Moreover if $T(x) = sign(x)\delta = \eta^{0,1} \approx 0 \in D'$, $sign(x)T(x) = sign^2(x)\delta(x) \neq 0$.

**Example 3:** Product of Step $\delta$ function.

Given the product:

$$u(x)^n\delta(x) \tag{71}$$

given the function $f = x^2$, the centred function $g \in \mathcal{O}^0$ of order 0, amplitude 1 and using the same resining as example 2 point a) above we have:

$$u^n(x)\delta(x) = f(u(x))\delta(x) = \eta_{f(g)}^{1,1} \approx \left(\int_0^1 x^n dx\right)\delta(x) = \frac{1}{n+1}\delta(x) \tag{72}$$

which the same result given by the Colombeau Algebras (see [1] §3.3).

**Example 4:** Prove the following identity:

$$\delta(x)\delta'(x) \approx -\frac{1}{3}u(x)\delta''(x) + \frac{1}{6}\delta''(x) \tag{73}$$

It is left to the reader to show that using the same reasoning of example 1 above, it is possible to prove the above identity.

# Appendix

## A.1 Order of the Products of Functions

For the theory developed in this paper to make sense, the following statement should be true:

**Hypothesis:** *Let $g^{(p_i)} \in \mathcal{O}$ with $i = 1 \dots N$ be N finite centred functions of the same family. Then the order of their products depends from the orders $p_i$ but do not depends from the function g.*

From numerical evidence this statement seems to be true but I have not a formal proof yet. Moreover the product of several functions in $\mathcal{O}$ of the same family and a monomial $x^k$ with $k \in \mathbb{Z}$ should not depend from $g$ in many relevant cases. This part of the theory should be further developed.

In this Appendix we will prove some statements on the topic which refer to the products of two functions.

**Statement:** *Let $g^{(p)} \in \mathcal{O}$ be a function with $p > 0$, then $(g^{(p)})^2 \in \mathcal{O}$ and it is a function of order 1.*

*Proof:* $(g^{(p)})^2 \in \mathcal{S}$ and therefore $(g^{(p)})^2 \in \mathcal{O}$ with order greater then 0. Its integral is not zero because $(g^{(p)})^2 \geq 0 \ \forall x$ and therefore it is of order 1. □

**Statement:** *Let $g^{(p)} \in \mathcal{O}$ a function with $p > 0$ and $g^{(p+1)}$ its derivative, then $g^{(p)}g^{(p+1)} \in \mathcal{O}$ and it is a function of order 2.*

*Proof:*

$$\frac{d}{dx}(g^{(p)})^2 = 2g^{(p)}g^{(p+1)} \tag{74}$$

and therefore $g^{(p)}g^{(p+1)}$ is of order 2. □

**Statement:** *Let $g^{(p_1)}$, $g^{(p_2)} \in \mathcal{O}$ functions of the same family with $p_2 > p_1 > 0$ and $d = p_1 - p_2$, then $g^{(p_1)}g^{(p_2)} \in \mathcal{O}$ and it is a function of order p with:*

$$p = \begin{cases} 1 & \text{for } d \text{ even} \\ \\ > 1 & \text{for } d \text{ odd} \end{cases} \tag{75}$$

*Proof:* For d even, integration $d/2$ times by parts $g^{(p_1)}g^{(p_2)}$ we get a function of the type $(g^{(r)})^2$ which has finite integral. For d odd, integrating $(d-1)/2$ times by parts $g^{(p_1)}g^{(p_2)}$ we get a function of the type $g^{(r)}g^{(r+1)}$ has integral equal to 0. □

**Statement:** *Let $g^{(p)} \in \mathcal{O}$ be a function with $p > 1$, then $xg^{(p)} \in \mathcal{O}$ and it is a function of order p-1.*

*Proof:* Let us write expansion (11) for $g^{(p)}$

$$g^{(p)} = a_p \xi_1^{(p)} + a_{p+1}\xi_1^{(p+1)} + a_{p+2}\xi_1^{(p+2)} + \dots \tag{76}$$

and suppose that $xg^{(p)}$ is of order p-1. We have:

$$xg^{(p)} = b_{p-1}\xi_2^{(p-1)} + b_p \xi_2^{(p)} + b_{p+1}\xi_2^{(p+1)} + \dots \tag{77}$$

using (14) we have:

$$b_k = \frac{(-1)^{k-1}}{(k-1)!}\int_{\mathbb{R}} x^{k-1}xg^{(p)}(x)dx = -\frac{(-1)^k k}{k!}\int_{\mathbb{R}} x^k g^{(p)}(x)dx = ka_{k+1} \tag{78}$$

and therefore the non-zero components of $xg^{(p)}$, apart from a factor k, are the non-zero components of $g^{(p)}$ shifted by 1. □

**Statement: Let** *$g' \in \mathcal{O}$ be a function of order 1, then $xg^{(p)} \in \mathcal{O}$ and it is a function of indeterminate order (i.e. depending from g).*

*Proof:* If $g'$ is symmetric (i.e. g'(x)=g'(-x), then clearly:

$$\int_{\mathbb{R}} xg'(x)dx = 0 \tag{79}$$

and therefore $xg$ is of order 2. If $g'$ is not symmetric then the above integral in general is different from 0 and the product is of order 1. $\square$

**Statement:** *Let $g^{(p_1)}$, $g^{(p_2)} \in \mathcal{O}$ centred functions of the same family with $p_2, p_1 < 0$, then $g^{(p_1)}g^{(p_2)} \in \mathcal{O}$ and it is a function of order $p = -p_1 - p_2$.*

*Proof:* A function of order $p$ with $p \leq 0$ goes to 0 for $x \to -\infty$ and it goes as a polynomial of degree $|p|$ for $x \to \infty$. Products of such two functions will therefore go to infinity as polynomial of degree $|p_1| + |p_2|$ $\square$

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