A Biased Classical Theory of Color – Hannay Angle and MacAdam Ellipse

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Abstract. There is still much unfinished work, and understanding, in order to set the classical theory of light and colors on a physically firm basis. The present work advances a point of view touching this issue. The main starting thesis is that the Planck's physical theory of light carries a special meaning, which allows us to select the general algebra to be used in treating the classical geometry of colors. This algebra is the one related to the SL(2,R) Lie group. In this framework, the classical MacAdam ellipse representing the uncertainty in deciding a color within trichromatic theory of colors, is closely related to the classical Hannay angle. This relation is explained. Further on, a special representation of trichromacy can result in a special representation of QCD itself. According to this representation, the QCD can be considered as the quantum counterpart of the classical theory of colors, much in the same manner in which the quantum mechanics is considered the quantum counterpart of the classical mechanics.
Key Words: theory of colors, MacAdam ellipse, quantum chromodynamics, Yang-Mills fields, Hannay angle, Riemannian geometry, trichromacy, dichromacy, blackbody radiation

Introduction

From a purely statistical theoretical point of view, the Planck moment in the physics of light reveals two distinct sides. The first one is the heuristic side, closely related to the Gaussian aspect of the statistics of light fluctuations. According to Max Born, this was the source of inspiration in establishing the famous connection between the fluctuations of the spectral density and the equilibrium temperature of the blackbody radiation, leading to the idea of quantum. The second side of the Planck moment is the proper quantum side, whereby the distributions of probability characterizing the blackbody radiation are of the quadratic variance function type. The contemporary theory of light colors, and of colors in general, seems mainly related to the first side of this moment of physics.

Conceived this way, the physical theory of light is indeed the source of inspiration for the classical theory of colors, considered as qualities of light itself, or generated by light in the structured matter. Using the suggestion properly, it leads to a noncommutative dynamics related to – or rather generated by – the classical Hannay angle, connected this time to color. The color is thereby imagined rather as a flux, in an abstract three-dimensional space. The theory is thus legitimately noncommutative from an algebraic point of view, involving innate fields of Young-Mills type which represent the basic colors. These can be both innate to the light, or expression of its interaction with matter, but the present work does not go into such details, even though they are essential. They will be reported in the future.

Rather, the present work touches a few points from the classical theory of colors, following however a unique leading idea, namely that between the classical concept of color, involving even the known subjective characterization depending on the physiology of eye, and the quantum number representing the color in quantum chromodynamics, there should be no discontinuity. It seems indeed that, from theoretical point of view, if we can talk today of a dynamics of color, this can be done via a classical mechanistic theory obtained by analogy with QCD. There does not appear to be a point of continuity between the classical theory of colors per se and the QCD, so that this last one appears as only a conventional name for a dynamics of some internal degrees of freedom. However, it occured to us that the concept of asymptotic freedom, which impacted so much the theory of strong interactions via QCD, has a deeper meaning for the positive knowledge. Namely the position of QCD with respect to the classical theory of colors, is exactly the same as the position quantum mechanics with respect to classical mechanics. This means, in particular, that the missing link in the classical (or even quantum!) theory of light, in order to make it a genuine Yang-Mills theory, is the color. Thus, for once, one might explain why the noncommutativity is the essential ingredient for the asymptotic freedom in the case of strong interactions.

Statistical Geometry of a Light Plane

The Planck's original Gaussian is uncorrelated (**Mazilu, 2010**). When considered, however, in the general, correlated form, the probability density of this Gaussian would be:

$$p_{XY}(x,y) = \frac{\sqrt{ac-b^2}}{2\pi} \exp\left\{-\frac{1}{2}\left(ax^2+2bxy+cy^2\right)\right\} \tag{1}$$

where X and Y are the two characteristic fluctuation processes, originally constituting the thermal light at low and high temperature respectively. The classical theory of color has an interesting twist on this statistics.

Indeed, in the classical theory of color, we don't specify these two random processes by temperature regimes, because in general we cannot associate a physical temperature with the color. The problem of associating a temperature to the color was not solved yet (**MacAdam, 1977**), and we don't think will be ever solved. For once, the thermodynamically defined absolute temperature is not physically supported for light as classically defined. This issue led to Planck theory in the first place. On the other hand, from a statistical point of view, the temperature goes into a parameter characterizing the distribution of colors in a more elaborate way than it does in the Planck statistics. Thus, let's just say, for the sake of argument, that in the case of light measurements in general we have to do with two stochastic processes X and Y, participating in the composition of a color. If ever in need of a statistical evaluation of the parameters a, b, c of the density from equation (1) above, we have at our disposal the maximum information entropy principle, for instance, giving their values by

$$a = \frac{\mathrm{var}(y)}{D}, \quad c = \frac{\mathrm{var}(x)}{D}, \quad b = -\frac{\mathrm{cov}(xy)}{D}, \quad D \equiv \mathrm{var}(x)\,\mathrm{var}(y) - [\mathrm{cov}(xy)]^2 \tag{2}$$

Here 'var' and 'cov' denote the variance and the covariance of the experimental data on X and Y.

This characterization of the color measurements – the dichromatic characterization – is closely related to a plane centric affine geometry. That is to say that if one insists in characterizing the measurements of light in a plane, which is obviously the natural way to consider these measurements (**Hoffman, 1966**), the geometry of this plane is the centric affine geometry. The group of this plane geometry is given by the infinitesimal generators

$$X_1 = y\frac{\partial}{\partial x}; \quad X_2 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right); \quad X_3 = -x\frac{\partial}{\partial y} \tag{3}$$

while the group of the space of values a, b, c is given by infinitesimal generators

$$X_1 = -a\frac{\partial}{\partial b} - 2b\frac{\partial}{\partial c}; \quad X_2 = -a\frac{\partial}{\partial a} + c\frac{\partial}{\partial c}; \quad X_3 = 2b\frac{\partial}{\partial a} + c\frac{\partial}{\partial b} \tag{4}$$

These are two realizations of the same $\mathfrak{sl}(2,R)$ algebraical structure. The second one has intransitive action, which allows transitivity only along specific manifolds, given by constant discriminant of the quadratic form from the exponent of equation (1).

The probability density (1) itself can be presented as a joint invariant of the two actions (3) and (4), with the help of Stoka theorem (**Stoka, 1968**). According to this, any joint invariant of the two actions is an arbitrary continuous function of the algebraic formations

$$ax^2 + 2bxy + cy^2, \quad ac - b^2 \tag{5}$$

Obviously (1) is only a special case of this theorem. According to the same theorem, the straight lines through origin x = y = 0 can be presented as joint invariants of two actions (3), while the joint invariants of two actions (4), one in the variables a, b, c, the other in the variables α, β, γ, say, are arbitrary functions of the following three algebraic formations (**Mazilu, 2004**):

$$\alpha\gamma - \beta^2, \ ac - b^2, \ a\gamma + c\alpha - 2b\beta \tag{6}$$

These facts can give good reasons for a few further observations related to the classical theory of colors.

The argument along these lines allows us to put forward the idea of representation of color in connection with MacAdam discovery of the meaning of quadratic forms for which the discriminant (5) is positive (**MacAdam, 1942**). First of all, we have to be a little more specific about the probability densities like that from (1). Thus, for instance, consider that the background color process on a plane of color measurements has the density

$$p_{XY}\left(x, y \middle| \alpha, \beta, \gamma\right) \equiv \frac{\sqrt{\alpha\gamma - \beta^2}}{2\pi} \exp\left\{-\frac{1}{2}\left(\alpha x^2 + 2\beta xy + \gamma y^2\right)\right\} \tag{7}$$

in two variables X and Y, of which we don't know too much for now, other than they are characterized by the statistics α, β and γ. All we know for sure is that X and Y are some kind of projections on unspecified planes, that happen to be experimentally realizable, and that they represent two colors (the so-called property of dichromacy). At this moment, the theory is therefore dichromatic. Now, let us say that the two processes participate somehow to give a third process, and all we know of this *participation* is that it is some kind of addition of them. More specifically, we will suppose that this third process is a kind of weighted sum of the two processes, having the general form

$$Z = \mu X + \nu Y \tag{8}$$

This is, for instance, the case of *initial conditions* in the case of the harmonic oscillator. The participations μ and ν are, in this particular case, given by the two solutions of a second order differential equation. The problem now is to find the probability density of the stochastic process Z. This can be done by following a known statistical routine, and the final result is

$$P_Z(z) = \sqrt{\frac{\alpha\gamma - \beta^2}{2\pi(\alpha\nu^2 - 2\beta\mu\nu + \gamma\mu^2)}} \exp\left\{-\frac{1}{2}\frac{\alpha\gamma - \beta^2}{\alpha\nu^2 - 2\beta\mu\nu + \gamma\mu^2} z^2\right\} \tag{9}$$

This is a Gaussian type probability density, having a zero mean and the variance

$$\sigma_z^2 = \frac{\alpha\nu^2 - 2\beta\mu\nu + \gamma\mu^2}{\alpha\gamma - \beta^2} \tag{10}$$

Such a probability density is particularly attractive in constructing the one related to characterizing the differentials of the three statistics, given their values.

The equation (10) is indication of the nature of an 'intensity variable' so to speak. It satisfies the Stoka theorem, and indicates that the quadratics are fundamental in the statistics related to the trichromacy theory of colors. One can see directly that the trichromacy is due to the fact that there is a bichromatic moment in the theory of color space, related to the experimental procedures. Indeed, from algebraical point of view, the set of binary quadratics like those occuring in the exponent of a bivariate Gaussian, is a linear three-dimensional space. Whence the basic theoretical support for the idea that the color space should be three-dimensional, even though not necessarily Euclidean. This, of course, gives even more reasons for considering the quadratic as fundamental in the theory of light colors.

There should be, therefore, a way to the color of light, giving consistency to the ideas regarding the trichromacy of light colors directly through the general quadratic statistical variable obtained, by dichromacy, in the measurement process:

$$Z(X,Y): z(x,y) \equiv \frac{1}{2}(ax^2 + 2bxy + cy^2) \tag{11}$$

This one then characterizes a specific plane of illumination, *no matter of the orientation of that plane*, because the quadratic is form-invariant by any projection. We have thus to find the probability density of this variable, under condition that the plane of light is characterized by the a priori probability density as given, for instance, in equation (7). That probability density satisfies, of course, the Stoka theorem, and the probability density of Z should also satisfy that theorem, in the precise sense that it must be a function of the algebraical formations from equation (6). This leaves us with a functionally undetermined probability density, even if we impose some natural constraints in order to construct it. Proceeding nevertheless directly, in the usual statistical manner, we find first the characteristic function of the variable (11). As known, this is the expectation of the imaginary exponential of Z, using (7) as probability density:

$$\left\langle e^{i\zeta Z} \right\rangle = \frac{1}{2\pi\sqrt{1 + (i\zeta)\dfrac{a\gamma + c\alpha - 2b\beta}{ac - b^2} + (i\zeta)^2 \dfrac{\alpha\gamma - \beta^2}{ac - b^2}}} \tag{12}$$

In view of (6), this characteristic function certainly satisfies the Stoka theorem, which thus reveals its right place in the physical theory. Like the Wien displacement law in the case of selection of the right spectrum for blackbody radiation, the Stoka theorem should also serve for the selection of the right probability density in the case of light colors in general. Anyway, the sought for probability density can then be found by a routine Fourier inversion based on tabulated formulas (**Gradshteyn, Ryzhik, 1994; 2007**, the examples 3.384(7); 6.611 (4); 9.215(2)&(3)):

$$p_Z(z \mid a,b,c) = \sqrt{AB}\exp\left(-\frac{A+B}{2}z\right) \cdot I_0\left(\frac{A-B}{2}z\right) \tag{13}$$

Here I_0 is the modified Bessel function of order zero, and A, B are two constants to be calculated from the formulas

$$A + B = \frac{2b\beta - a\gamma - c\alpha}{ac - b^2}; \quad AB = \frac{\alpha\gamma - \beta^2}{ac - b^2}; \quad A > B \tag{14}$$

Again, this probability density obviously satisfies the Stoka theorem, as it is a function of the joint invariants from equation (6). And so do the mean and the standard deviation of the variable Z, for they can be calculated as

$$\langle Z \rangle \equiv \frac{1}{2}\frac{A+B}{AB} = \frac{1}{2}\frac{2b\beta - a\gamma - c\alpha}{ac - b^2}; \quad \mathrm{var}(Z) \equiv \frac{1}{2}\frac{A^2 + B^2}{A^2B^2} = \frac{1}{2}\left(\frac{2b\beta - a\gamma - c\alpha}{ac - b^2}\right)^2 - \frac{\alpha\gamma - \beta^2}{ac - b^2} \tag{15}$$

We thus have the interesting conclusion that the essential statistics related to variable Z do not depend but on the coefficients of the distribution, and the values of the parameters entering the expression of Z. On one hand, this means that the geometry of the color space is dictated by the statistical characteristics of the plane of projection and by the physics describing the color, naturally incorporated in the variable Z. For instance Z can represent the energy of a harmonic oscillator, or even the wavelength of light when described by the wave surface.

Light as a Stochastic Process

One usually insists, and with good reasons at that, upon the fact that the geometry of the color space is not an Euclidean one, but a general Riemannian geometry (see **Schrödinger, 1920**; English translations of these works in **MacAdam, 1970**; see also **Wyszecki, Stiles, 1982** for a pertinent comprehensive review of the theories of colors in all their aspects). In context, the Riemannian metric carries a special statistical significance whereby the components of the metric tensor are covariances of the three color coordinates (**Silberstein, 1938, 1943**). This meaning of the metric does not seem to be secured by anything in the framework of the theory. Yet one works this way, and the results confirm the manner of approach everywhere in the classical theory of color. There should be therefore some fundamental truth there, whose formal expression is not as yet obvious.

The previous statistical theory can help us secure, from a theoretical point of view, this purely statistical connotation in the color space. Assume indeed, that a, b and c are some variations of the parameters α, β and γ, respectively. It thus turns out that this is a variation, dZ say, of the color Z is dictated only by the variations of its coefficients, and it is a process having, according to equation (15), the following expectation and variance:

$$\overline{dZ} \equiv \frac{1}{2}\frac{A+B}{AB} = \frac{1}{2}\frac{2\beta d\beta - \gamma d\alpha - \alpha d\gamma}{\alpha\gamma - \beta^2};$$
$$\overline{[\Delta(dZ)]^2} \equiv \frac{1}{2}\frac{A^2+B^2}{A^2B^2} = \frac{1}{2}\left(\frac{2\beta d\beta - \gamma d\alpha - \alpha d\gamma}{\alpha\gamma - \beta^2}\right)^2 - \frac{d\alpha d\gamma - (d\beta)^2}{\alpha\gamma - \beta^2} \tag{16}$$

Here a bar over the symbol means average using the probability density given by equation (13). From these formulas we get

$$\overline{[\Delta(dZ)]^2} - \overline{dZ}^2 \equiv \overline{dZ^2} - 2\overline{dZ}^2 = \frac{1}{4}\left(\frac{2\beta d\beta - \gamma d\alpha - \alpha d\gamma}{\alpha\gamma - \beta^2}\right)^2 - \frac{d\alpha d\gamma - (d\beta)^2}{\alpha\gamma - \beta^2} \tag{17}$$

The right hand side of this formula carries a special meaning: it is the Riemannian metric which can be built by the methods of absolute geometry for the space of the 2×2 matrices having the singular matrices as points of the absolute quadric (**Mazilu, Agop, 2012**). In fact, one can prove, and we will show this immediately, that the quadratic form (17) is just the Cartan-Killing metric of the certain action of the 2×2 real matrices. This is indeed of the quadratic form

$$\frac{1}{4}(\omega_2{}^2 - 4\omega_1\omega_3) \tag{18}$$

where $\omega_{1,2,3}$ are three 1-forms representing three conservation laws of the SL(2,R), and has the exquisite interpretation already mentioned. Meanwhile, let's notice that, from a stochastic point of view, the process of physical variation of the parameters of the quadratic form is 'almost' a Lévy-type process with three parameters (**Lévy, 1965**), in the sense that the elementary distance is decided by the variance function. This validates indeed the statistical interpretation of the metric of the space of colors, but raises instead another problem related to the coordinates representing the colors. This problem indicates, in turn, the feasibility of another, more special, approach of the geometry of colors.

Resnikoff's Special Theory

Notice indeed that, as a matter of fact it is not the variable dZ we are after, but the parameters dα, dβ and dγ, and they can be assumed to have zero averages, without any problem. Equations (16) and (17) are then just *control* equations, related to a space coherence of light for instance. Indeed, we usually measure the wavelength in order to get the characteristics of light, and the wavelength is a quadratic form in the parameters of the plane of dichromatic measurements. Howard Resnikoff introduced as representative for what he calls the 'perceptual lights' a set of 2×2 symmetric matrices (**Resnikoff, 1972**):

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{19}$$

The determinant of this symmetric matrix is taken as the *brightness variable* of the light, to be constructed from the three basic color perceptions. Resnikoff suggests that the entries of the matrix (19) are to be taken as color coordinates. In that case the coordinate β can be chosen to be the regular 'B' – the quantifier for the 'blue' color in an RGB color scheme – of course, in the cases where the brightness of light thus calculated is positive. For a certain situation β has therefore to play the part of a correlation when statistically considered in the case of dichromatic basic variables. The choice is not unique, for there are three manners of calculating this brightness on a certain range of the color parameters RGB, in order to satisfy the positivity requirement, but let us go with it just for the sake of illustration. Thus, if we take, in the manner of Resnikoff:

$$\xi = \sqrt{\alpha\gamma - \beta^2}; \quad u = \frac{\beta}{\alpha}; \quad v = \frac{\sqrt{\alpha\gamma - \beta^2}}{\alpha} \tag{20}$$

the matrix (19) becomes

$$\boldsymbol{\alpha} = (\xi/v)\begin{pmatrix} 1 & u \\ u & u^2 + v^2 \end{pmatrix} \tag{21}$$

In this case we have by direct calculation:

$$\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha} = d\ln(\xi/v)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1/v^2)\begin{pmatrix} -udu & -(u^2 - v^2)du - 2uvdv \\ du & udu + 2vdv \end{pmatrix} \tag{22}$$

and the Resnikoff metric is just the Cartan-Killing metric of this group of matrices, given by:

$$\mathrm{tr}[(\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha}) \cdot (\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha})] = 2\left\{\left(\frac{d\xi}{\xi}\right)^2 + \frac{du^2 + dv^2}{v^2}\right\} \tag{23}$$

Now, the matrix (22) has the general form:

$$\boldsymbol{\alpha}^{-1}d\boldsymbol{\alpha} = d(\ln\xi)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1/v^2)\begin{pmatrix} -udu - vdv & -(u^2 - v^2)du - 2uvdv \\ du & udu + vdv \end{pmatrix} \tag{24}$$

and carries a special meaning in the geometrical theory of color. In order to reveal this meaning let's consider the quadratic forms in their most generality, from the general standpoint that their coefficients do represent lights or color coordinates, as suggested by Resnikoff.

Differential Dichromacy: the MacAdam Ellipses

The general equation of a conic section is a quadratic equation of the form

$$f(x, y) \equiv \alpha x^2 + 2\beta xy + \gamma y^2 + 2ax + 2by + c = 0 \tag{25}$$

This time in the quadratic form we have included the possibility of an arbitrary center – not just the origin – whose coordinates are related to the coefficients a, b through a linear homogeneous relation determined by α, β and γ. There is a merit, given by handling simplicity among others, in using the 'notation of Dirac'. This also allows for a suggestive interpretation of the final geometrical results. In broad lines this notation amounts to representing the position vector either by a "ket" or by a "bra" vector, according to its position in the matrix multiplication product. These are given by the matrices

$$\left|x\right\rangle \equiv \begin{pmatrix} x \\ y \end{pmatrix} \quad \therefore \quad \left\langle x\right| \equiv \begin{pmatrix} x & y \end{pmatrix}$$

As known, in such a notation, the distance from the origin of the coordinate system depicts a "bracket", i.e. the dot product of the two instances of the position vector:

$$\left\langle x | x \right\rangle = x^2 + y^2$$

On the other hand, one knows that a translation comes to a sum of vectors, so that we can write:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \quad \leftrightarrow \quad \left|x'\right\rangle = \left|x\right\rangle + \left|u\right\rangle$$

Further on, a general homogeneous transformation boils down to either a left multiplication of the "ket" with a 2×2 matrix:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \leftrightarrow \quad \left|x'\right\rangle = \boldsymbol{\alpha}\left|x\right\rangle$$

or to a right multiplication of the corresponding "bra" with such a matrix:

$$\begin{pmatrix} x' & y' \end{pmatrix} = \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \quad \leftrightarrow \quad \left\langle x'\right| = \left\langle x\right|\boldsymbol{\alpha}$$

In common calculations, the difference between these two last equations is simply a transposition of the matrices.

In these notations the equation (25) can be written as

$$f(x,y) \equiv \left\langle x|\boldsymbol{\alpha}|x\right\rangle + 2\left\langle a|x\right\rangle + c = 0 \tag{26}$$

where we used the following identification:

$$\left|a\right\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} \quad \therefore \quad \left\langle a\right| \equiv \begin{pmatrix} a \\ b \end{pmatrix}^t = \begin{pmatrix} a & b \end{pmatrix}$$

This vector represents the relative position of the center of the conic in the known geometrical sense:

$$\boldsymbol{\alpha}\left|x_c\right\rangle + \left|a\right\rangle = \left|0\right\rangle \quad \therefore \quad \left|x_c\right\rangle = -\boldsymbol{\alpha}^{-1}\left|a\right\rangle \tag{27}$$

If we refer the conic to this center, by means of the translation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \leftrightarrow \quad \left|x\right\rangle = \left|x_c\right\rangle + \left|\xi\right\rangle$$

the equation (26) becomes purely quadratic in coordinates, although otherwise inhomogeneous:

$$\left\langle \xi|\boldsymbol{\alpha}|\xi\right\rangle - \left\langle x_c|\boldsymbol{\alpha}|x_c\right\rangle + c = 0 \tag{28}$$

This algebra is now used in constructing an argument for matrix representation of colors.

Within the framework of Resnikoff representation as above, the problem of identification of a center of color in a plane of measurement – what we would like to call the MacAdam's problem (**MacAdam, 1942**) – has an explicit algebraical expression. Indeed, we can simply represent targeting "the same geometrical color center" by the differential equations $dx_c = dy_c = 0$. Then the condition (27) comes formally down to the following matrix differential equation:

$$\left|0\right\rangle \equiv d\left(\boldsymbol{\alpha}^{-1}\left|a\right\rangle\right) = (d\boldsymbol{\alpha}^{-1})\left|a\right\rangle + \boldsymbol{\alpha}^{-1}\left|da\right\rangle \tag{29}$$

Obviously this equation limits the set of possible conics having the same geometric center. Using the definition of the inverse of a matrix, to the effect that $\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\alpha}$ is the identity matrix, one can easily prove by direct differentiation the matrix differential relation $d\boldsymbol{\alpha}^{-1} = -\boldsymbol{\alpha}^{-1}\cdot d\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^{-1}$, so that from equation (27) we must have

$$\left|da\right\rangle = (d\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^{-1})\left|a\right\rangle \tag{30}$$

Thus the condition of fixed center comes actually down to a certain evolution of the vector $|a\rangle$, dictated by the matrix of the quadratic form from the equation of the conic section and its variation. In detail, the equation (30) can be written as

$$\begin{pmatrix} da \\ db \end{pmatrix} = \frac{1}{\alpha\gamma - \beta^2}\begin{pmatrix} \gamma d\alpha - \beta d\beta & \alpha d\beta - \beta d\alpha \\ \gamma d\beta - \beta d\gamma & \alpha d\gamma - \beta d\beta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \tag{31}$$

The matrix governing the evolution in the right hand side of this equation can be further adjusted to a special form:

$$\boldsymbol{\Omega} \equiv d(\ln\sqrt{\Delta})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\omega_2/2 & \omega_1 \\ -\omega_3 & \omega_2/2 \end{pmatrix}; \quad \Delta \equiv \alpha\gamma - \beta^2 \tag{32}$$

Δ is therefore the determinant of $\boldsymbol{\alpha}$, i.e. Resnikoff's brightness squared, and we denoted

$$\omega_1 = \frac{\alpha d\beta - \beta d\alpha}{\Delta}; \quad \omega_2 = \frac{\alpha d\gamma - \gamma d\alpha}{\Delta}; \quad \omega_3 = \frac{\beta d\gamma - \gamma d\beta}{\Delta} \tag{33}$$

three differential forms generated by the elements of the matrix of quadratic form representing the family of local colors, and their differentials. When calculated in the coordinates from equation (20) these differential forms are

$$\omega_1 = \frac{du}{v^2}; \quad \omega_2 = 2\frac{udu + vdv}{v^2}; \quad \omega_3 = \frac{(u^2 - v^2)du + 2uvdv}{v^2} \tag{34}$$

showing explicitly that the matrix from equation (32) is the transposed of that from equation (24)

Thus, the proposed representation of Resnikoff's has actually a firm physical basis, in relation to MacAdam's ellipses. Indeed, assume that we are to identify a certain center, as in MacAdam experiments. The center is the one position satisfying equation (27), and therefore asking for the differential correlation (30), which turns out to be an equation of motion for the vector $|a\rangle$. When the Resnikoff's matrix is taken as described, i.e. representing an ellipse, then the motion of the center $|a\rangle$ itself is along an ellipse, which is the real case with MacAdam results. Therefore, the MacAdam's ellipse gives a statistical interpretation to differentials of the elements of color in Resnikoff's representation.

Now, a few algebraical relations among the differential forms (34) are in order. They form a basis of a sl(2,R) algebra. The following differential relations can be directly calculated:

$$d\wedge\omega_1 = \frac{\alpha}{\sqrt{\Delta}}\Theta; \quad d\wedge\omega_2 = \frac{2\beta}{\sqrt{\Delta}}\Theta; \quad d\wedge\omega_3 = \frac{\gamma}{\sqrt{\Delta}}\Theta \tag{35}$$

where Θ is the differential 2-form

$$\Theta \equiv \frac{\alpha d\beta\wedge d\gamma + \beta d\gamma\wedge d\alpha + \gamma d\alpha\wedge d\beta}{\Delta^{3/2}} \tag{36}$$

The 2-form Θ is closed because it is the exterior differential of a 1-form:

$$\Theta \equiv d\wedge\psi; \quad \psi \equiv \frac{\alpha+\gamma}{\sqrt{\Delta}}d\left(\tan^{-1}\frac{2\beta}{\alpha-\gamma}\right) \tag{37}$$

representing the *Hannay angle* of this problem. In our context it gives a way to 'objectify', so to speak, the subjective experimental evaluations of colors, and has certainly everything in common with the original angle (**Hannay, 1985**; **Berry, 1985**).

On the other hand, we can verify the following relations:

$$\omega_1 \wedge \omega_2 = \frac{\alpha}{\sqrt{\Delta}}\Theta; \quad \omega_2 \wedge \omega_3 = \frac{\gamma}{\sqrt{\Delta}}\Theta; \quad \omega_3 \wedge \omega_1 = -\frac{\beta}{\sqrt{\Delta}}\Theta \tag{38}$$

Thus, from (35) and (38) we have the characteristic equations of a sl(2,R) structure:

$$d \wedge \omega_1 - \omega_1 \wedge \omega_2 = 0; \quad d \wedge \omega_3 - \omega_2 \wedge \omega_3 = 0; \quad d \wedge \omega_2 + 2(\omega_3 \wedge \omega_1) = 0 \tag{39}$$

Using these relations we can draw an important conclusion: the quadratic forms associated with the matrix in Resnikoff representation of light perceptuals are actually fluxes of color in the color space, induced by the 'subjective' uncertainty in determining a color. Indeed, the quadratic form conserved along MacAdam's evolution can be written as $\langle a|\boldsymbol{\omega}|a\rangle$, where $\boldsymbol{\omega}$ is a symmetric matrix of 1-form in Resnikoff's perceptuals. One can construct the 2-form

$$\langle a|d \wedge \boldsymbol{\omega}|a\rangle = \langle a|\boldsymbol{\alpha}|a\rangle \frac{\Theta}{\sqrt{\Delta}}; \quad \boldsymbol{\omega} \equiv \begin{pmatrix} \omega_1 & \omega_2/2 \\ \omega_2/2 & \omega_3 \end{pmatrix} \tag{40}$$

where we have used the equations (35). As the 2-form Θ, is a flux, the analogous of the solid angle in the usual Euclidean space, the quadratic form $\langle a|\boldsymbol{\alpha}|a\rangle$ is indeed the intensity of a flux of colors in the color space thus defined. One might say that the human eye is driven, in evaluating the light, by a flux of color as represented by Hannay's angle.

Conclusions and Outlook

The fact that, as far as the color is involved, the physical theory of light should be a statistical theory seems today beyond any reasonable doubt. So is the fact that the color space should be a general Riemannian space. The usual tenet is that the Riemann metric of the color space reflects the statistics related to color. In our opinion, the theoretical effort should then concentrate in finding representations of color that naturally satisfy these requirements.

The measurements of light are usually performed in cross sections of beams of light, in general positions with respect to the beams. This fact allows some geometrical considerations amounting to special choice of color variables, which entitle quite naturally both the dichromacy and trichromacy considerations for the theory of light colors. Along this line of thought, the Riemannian structure arises naturally in connection with the very statistical theory of measurement. The supporting manifold of colors is a Riemannian one having constant negative curvature. The 'subjective' uncertainty in color decision can then be described by a geometry whose basis is the usual Hannay angle. This time however, the Hannay angle is connected to a flux in the space of colors, representing the adaptability of the human eye to the color of light. The classical subjectivity of establishing a color has this way an 'objective' mathematical counterpart, confirmed by experimental practice.

But the implications of a theory that uses a Resnikoff's representation of colors, whereby they are quantitatively given by the entries of a 2×2 symmetric real matrix, are far more intricate from physical theoretical point of view. Indeed, such a representation has an outstanding theoretical meaning. A matrix is obviously an element of a noncommutative algebra, which can be simply a Yang-Mills field. It turns out that this theory of colors is plainly a Yang-Mills theory. It completes the classical theory of light in a natural way, by including the color in it. The classical electromagnetic theory, even though undoubtedly a gauge theory, is not a Yang-Mills theory yet. The present work shows that it takes considerations of color of light in order to render to the theory of light a plain Yang-Mills character. From this point of view, the light itself actually enters the realm of quantum chromodynamics, as it should naturally do, for the everyday color is related to light. But there is more to it: if the mechanism of color is the one explaining the strong interactions, then this color should be classical too. Thus one might figure out why the noncommutativity is the essential ingredient allowing asymptotic freedom in the case of strong interactions: after all, the light is a model of interaction everywhere in the universe, at any level!

These conclusions may seem momentarily only conjectures, but we think that even by now they can be securely promoted to “educated guesses” at least. For the purpose of a more thorough validation, one may need to correlate them with an electromagnetic theory of light. This will be the object of a future work. Specifically, we intend to report how the Resnikoff-type theory of color relates to the classical ideas of light ray and wave surface, and therefore to the classical electromagnetic theory of light. This, in our opinion, will give more credibility to the theory of colors as a legitimate classical counterpart of the QCD, much in the same manner in which the classical mechanics offers a limit to quantum mechanics.

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