# PRIMES IN THE INTERVALS $[kn, (k+1)n]$

S. SAMBASIVARAO[1]

ABSTRACT. In this paper, we prove: (a) for every integer $n > 1$ and a fixed integer $k \leq n$, there exists a prime number $p$ such that $kn \leq p \leq (k+1)n$, and (b) conjectures of Legendre, Oppermann, Andrica, Brocard, and Improved version of Legendre conjecture as a particular case of (a).

## 1. INTRODUCTION

In 1845, J. Bertrand conjectured that for every positive integer $n$, there is always atleast one prime $p$ such that $n < p \leq 2n$. This was first proved by P. Chebyshev in 1850, therefore it is also called the Bertrand-Chebyshev Theorem(B.C.T). S. Ramanujan provided a very simple proof to B.C.T using elementary properties of Gamma function(see [7]). In conformity with S. Ramanujan, J. Nagura established the following:

**Theorem 1.1** ([5]). *There is atleast one prime number between $n$ and $\frac{6n}{5}$ for $n \geq 25$.*

Recently an interesting generalization of B.C.T was proposed by M.El.Bachraoui as an open problem: "Is it true for all integer $n > 1$ and a fixed integer $k \leq n$, there exists a prime number $p$ such that $kn \leq p \leq (k+1)n$?", and proved that this is true for $k = 2$, whereas B.C.T answers this question affirmatively for $k = 1$. Latter, he concluded that a positive answer to this problem for every positive integer $k$ with $k = n$ would prove Legendre conjecture( see [6]).
The purpose of this note is to provide a positive answer to the problem posed by M.El.Bachraoui. Consequently, we show that the conjectures of Legendre, Oppermann, Andrica, Brocard and Improved Version of Legendre conjecture are true.

## 2. MAIN RESULTS

In what follows $\mathbb{Z}_+$ denote the set of postive integers.
Let $\pi(n)$ denote the number of prime numbers less than or equal to $n$, known as prime-counting function. Then one can restate B.C.T in terms of $\pi$ as "For every $n \in \mathbb{Z}_+, \pi(2n) - \pi(n) \geq 1$".

**Lemma 2.1** ([3],pp.427). *The function*

$$f(z) = \frac{e^{\frac{2\pi i \Gamma(z)}{z}} - 1}{e^{-\frac{2\pi i}{z}} - 1} \tag{2.1}$$

*equals 0 or 1 according as z is composite or prime.*

[1]Address for correspondence: Plot No.118, Vaidehi Nagar, Vanasthalipuram, Hyderabad-500070, Andhra Pradesh, India. Email Id. ssrao.siginam@gmail.com.

**Lemma 2.2** ([4]). *For $n \geq 5$ and $n \in \mathbb{Z}_+$, then*

$$\pi(n) = 2 + \sum_{q=5}^{n} \frac{e^{\frac{2\pi i \Gamma(q)}{q}} - 1}{e^{-\frac{2\pi i}{q}} - 1}. \tag{2.2}$$

The following theorem proves the claim made in (a).

**Theorem 2.3.** *For every integer $n > 1$ and a fixed integer $k \leq n$, there exists a prime number $p$ such that $kn \leq p \leq (k+1)n$.*

*Proof.* We prove this theorem in two cases.
**Case1:** Let $k \in \mathbb{Z}_+$ and $1 \leq k \leq 5$.
It is clear that for all integers $n \geq 25$, $(k+1)n - \frac{6}{5}(kn) \geq 0$. By Theorem1.1, there exists atleast one prime in between $kn$ and $(k+1)n$ for all $n \geq 25$.
By actual verification, we find that it is true for smaller values of $n$ with $k \leq n$.
**Case2:** Let $k \in \mathbb{Z}_+$ and $k > 5$.

For each $q \in \mathbb{Z}_+$, we write $u_q = \frac{e^{\frac{2\pi i \Gamma(q)}{q}} - 1}{e^{-\frac{2\pi i}{q}} - 1}$. By Lemma2.2, for all $n \geq k$

$$\pi((k+1)n) = 2 + \sum_{q=5}^{(k+1)n} u_q$$

$$= 2 + \sum_{q=5}^{2k} u_q + \sum_{q=2k+1}^{(k+1)n} u_q \tag{2.3}$$

$$= \pi(2k) + \sum_{q=2k+1}^{(k+1)n} u_q,$$

and

$$\pi(kn) = \pi(k) + \sum_{q=k+1}^{kn} u_q. \tag{2.4}$$

Therefore,

$$\pi((k+1)n) - \pi(kn) = \pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} u_q - \sum_{q=k+1}^{2k} u_q. \tag{2.5}$$

Now,

$\pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} \min_q u_q - \sum_{q=k+1}^{2k} \min_q u_q \leq \pi((k+1)n) - \pi(kn) \leq \pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} \max_q u_q - \sum_{q=k+1}^{2k} \max_q u_q$ for all $n \geq k$.

In view of Lemma2.1, we have $\pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} 0 - \sum_{q=k+1}^{2k} 0 \leq \pi((k+1)n) - \pi(kn) \leq \pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} 1 - \sum_{q=k+1}^{2k} 1$
$\Rightarrow \pi(2k) - \pi(k) \leq \pi((k+1)n) - \pi(kn) \leq \pi(2k) - \pi(k) + (n-k)$.
Since $n \geq k$ and $\pi(2k) - \pi(k) \geq 1$ for all $k > 5$, $\pi((k+1)n) - \pi(kn) \geq 1$. This completes the proof.                                                                    □

**Corollary 2.4.** *For each $n \in \mathbb{Z}_+$ and $n > 5$, $\pi(2n) - \pi(n) = \pi(n^2 + n) - \pi(n^2)$.*

*Proof.* Follows from case 2. of Theorem2.3 by taking $k = n$.                      □

Now we can prove a few well-known conjectures in number theory as a special case of Theorem2.3. Most of them are still unsolved.

**Corollary 2.5.** *(Oppermann's Conjecture is true) For each $n \in \mathbb{Z}_+$ and $n > 1$, $\pi(n^2) - \pi(n^2 - n) \geq 1$ and $\pi(n^2 + n) - \pi(n^2) \geq 1$.*

*Proof.* Follows from Theorem2.3. □

**Corollary 2.6.** *For each $n \in \mathbb{Z}_+$, $\pi((n+1)^2) - \pi(n^2) \geq 2$*
*(This conjecture is due to Adway Mitra et.al [1], called the Improved version of Legendre Conjecture).*

*Proof.* Let $n \in \mathbb{Z}_+$. Then there exist primes $p$ and $q$ such that $n^2 \leq p \leq n(n+1)$ and $n(n+1) \leq q \leq (n+1)^2$. (By Theorem2.3)
Hence $\pi((n+1)^2) - \pi(n^2) \geq 2$. □

**Corollary 2.7.** *(Legendre's Conjecture is true) For each $n \in \mathbb{Z}_+$, $\pi((n+1)^2) - \pi(n^2) \geq 1$.*

*Proof.* Follows from Corollary2.6 □

**Corollary 2.8.** *(Brocard's Conjecture is true) For each integer $n > 1$, $\pi((p_{n+1})^2) - \pi((p_n)^2) \geq 4$ where $p_n$ is the nth prime number.*

*Proof.* Let $p_n, p_{n+1}$ be consecutive primes with $n > 1$.
Then $(p_n + 2)^2 \leq (p_{n+1})^2$ since the minimum gap between consecutive primes is 2. By applying Theorem2.3 repeatedly, there exist primes $p$, $q$, $r$, $s$ such that $(p_n)^2 < p < p_n(p_n + 1)$, $p_n(p_n + 1) < q < (p_n + 1)(p_n + 1)$, $(p_n + 1)(p_n + 1) < r < (p_n + 2)(p_n + 1)$, $(p_n + 1)(p_n + 2) < s < (p_n + 2)^2$.
Hence $\pi((p_{n+1})^2) - \pi((p_n)^2) \geq 4$. □

**Corollary 2.9.** *(Andrica's Conjecture is true) The inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all $n$, where $p_n$ is the nth prime number.*

*Proof.* Let $p_n, p_{n+1}$ be two consecutive prime numbers.
In view of Theorem 2.3 we can find a $k \in \mathbb{Z}_+$ such that $p_n$ is in any one of these intervals $I_1 = [k(k-1), k^2]$, $I_2 = [k^2, k(k+1)]$, $I_3 = [k(k+1), (k+1)^2]$.
**Case 1:** Suppose $p_n, p_{n+1} \in I_2 \cup I_3$ then $k^2 < p_n < p_{n+1} < (k+1)^2$
$\Rightarrow \sqrt{p_{n+1}} - \sqrt{p_n} < 1$.
**Case 2:** Suppose $p_n, p_{n+1} \in I_1 \cup I_2$ then $k^2 - k + \frac{1}{4} < p_n < p_{n+1} < k^2 + k + \frac{1}{4}$
$\Rightarrow \sqrt{p_{n+1}} - \sqrt{p_n} < 1$.
**Case3:** Suppose $p_n \in I_3$ and $p_{n+1} \notin I_1 \cup I_2 \cup I_3$
Since $p_n, p_{n+1}$ are consecutive primes,we have $(k+1)^2 < p_{n+1} < (k+1)(k+2)$ (in view of Theorem2.3),
$\Rightarrow k^2 + k + \frac{1}{4} < p_n < p_{n+1} < k^2 + 3k + \frac{9}{4}$
$\Rightarrow \sqrt{p_{n+1}} - \sqrt{p_n} < 1$. This completes the proof. □

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