Detecting Fractal Dimensions Via Primordial Gravitational Wave Astronomy.

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Abstract

Lower fractal dimensionality of the early Universe at higher energies is an theoretical possibility as recently pointed out in [1].Gravitational-wave experiments with interferometers and with resonant masses can search for stochastic backgrounds of gravitational waves of cosmological origin.In this paper using cosmological models with fractional action and Calcagni approach to cosmology in fractal spacetime [18], we will examine a number of theoretical aspects of the searches fractal dimensionality from a stochastic fractal background of GWs.

Content

I.Introduction

It has been recently urgently declared [1] that quantum gravity models where the number of dimensions reduces at the ultraviolet (UV) exhibit a potentially observable cutoff in the primordial gravitational wave (GW) spectrum.A new framework was proposed in which the structure of spacetime is *fundamentally (1 + 1)-dimensional* universe, but is "wrapped up" in such a way that it appears even higherdimensional at larger distances.
Furthermore, the problems plaguing (3 + 1)-dimensional quantum gravity quantization programs are solved by virtue of the fact that spacetime is dimensionally-reduced. Indeed, effective models of quantum gravity are plentiful in (2+1) and even (1+1) dimensions.

However as pointed out in paper [2] this claim problematic and even misleading for two distinct reasons.
I.Definition of dimensionality [2]**:** "It is completely ambiguous to which definition of “dimension” is being used when referring to *vanishing dimensions* [1].
The only papers cited there which discuss vanishing dimensions in quantum gravity are [3]-[4],which discuss the “spectral dimension” (**SD**).In particular [2] refers to casual dynamical triangulations [**CDTs**], where it is the **SD** that flows to 2 in
the UV,not the topological [physical] dimension. The latter remains 4 [3]. It is not true that **CDTs** “demonstrate that the four-dimensional spacetime can emerge

from two-dimensional simplicial complexes", as stated in [1]. 4-dimensional **CDTs** by construction arise from 4-dimensional simplices.The **SD** is an analytic feature that provides information about short-distance dispersion relations. The fact that it runs to 2 at short distances does not mean that any physical modes decouple there.

Consider, for instance, Hořava gravity,which is another theory where (Lorentz violating) dispersion relations lead to a **SD** of 2 in the UV.The spin-2 graviton does not decouple at high energies, it remains part of the physical excitation spectrum, albeit with a strongly Lorentz violating dispersion relation.That is, in a wide class of models where some notion of dimension is scale dependent, this is the **SD**.But the **SD** is not the quantity that appears in the Feynman loop integrals (as in the suggestions made in [1]); that is the physical dimension $4,$which is not running".

Remark 2. However as pointed out in [16] for every **SD**$_l$ that flows to 2 or even to 0 in the UV regime,i.e.at short distances $l, l \to 0,$exists effective QGR theory on **SD**$_l$-dimensional fractal spacetime, which imbeded in canonical 4-dimensional spacetime.In this approach in contrast with **CDTs**, the **SD**$_l$ is the quantity which appears generically in the Feynman loop integrals. The spin-2 graviton does not decouple at high energies,it remains part of the physical excitation spectrum,albeit with a strongly Lorentz violating, but in contrast with Hořava gravity,only for any fractional values of **SD**$_l$.

II.Dimensionality and dynamics [2]**:** "Though there are some heuristic models cited in [1] where it is the physical dimension that is running, e.g. [17],these are quantum field theory models which do not include gravity.Let us nevertheless entertain the idea that it is indeed the number of physical dimensions that reduces in the UV in a quantum gravity model and one ends up with a lower-dimensional theory.The argument used in support of the claim that such a theory would have no local degrees of freedom is essentially that $(2+1)$-dimensional general relativity (GR) has this property [1].However, there is no particular reason to believe that a generic quantum gravity model which reduces to a $(2+1)$-dimensional theory at high energies should share this characteristic".

Remark 2. As pointed out in [6],[16],[17] low dimensional general relativity (GR) in fractal spacetime in contrast with classical $(2+1),(1+1)$-dimensional general relativity does not degenerate. For instance S.Vacaru [6] proved that even black holes really exist in low dimensional fractional gravity.Consequently a generic quantum gravity model which reduces to a low dimensional theory on fractal at high energies, does not share specified above degenerative characteristic.

III. Additionally [2]: "There is no reason whatsoever for the theory in question to be close to $2+1$ dimensional GR in the UV. Clearly,if this is a to be a viable gravity theory it should resemble 4-dimensional GR at low energies".

Remark 3. In contemporary GR and cosmology fractal nature of physical spacetime is proposed even declared and argued in many papers [18]-[27].

Thus there is a reason in question: is fractal dimension of the real physical spacetime chainges or not chainges,during the Universe evolution?

Remark 4. From general reasons specified in [2],heuristic model of physical dimension crossover which proposed in [1] and which based on jumping crossover $(1+1) \to (2+1) \to (3+1)$-dimensional spacetime is problematic.

Remark 5.[1] However, exactly at the crossover the description *could be very complicated.*For example,systems whose e ective dimensionality changes with the scale can exhibit fractal behavior, even if they are defined on smooth manifolds. As a good step in that direction,in [18]-[19] a field theory which lives in fractal spacetime and is argued to be Lorentz invariant, power-counting renormalizable, and causal was proposed.

II. Spacetimes with non-integer dimensions.

II.1.Ggeometric formalism with the fractional Caputo derivative.

We assume that $f(x)$ is a derivable function $f: [{}_1x, {}_2x] \to \mathbb{R}$. The fractional Caputo derivatives are defined respectively by formulae

$$\begin{aligned} &\text{left}: {}_{{}_1x}\overset{\alpha}{\underline{\partial}}_x f(x) = \frac{1}{\Gamma(s-\alpha)} \int\limits_{{}_1x}^{x} (x - x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^s f(x')dx', \\ &\text{right}: {}_x\overset{\alpha}{\underline{\partial}}_{{}_2x} f(x) = \frac{1}{\Gamma(s-\alpha)} \int\limits_{x}^{{}_2x} (x' - x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^s f(x')dx' . \end{aligned} \quad (2.1.1)$$

We denote by $\mathcal{L}_x({}_1x, {}_2x)$ the set of those Lesbegue measurable functions $f(x)$ on $[{}_1x, {}_2x]$ for which $\|f\|_z = \left(\int_{{}_1x}^{{}_2x} f(x)dx\right)^{1/z} < \infty$.
For any real–valued function $f(x)$ defined on a closed interval $[{}_1x, {}_2x]$ there is a function $F(x) = {}_{1x}\overset{\alpha}{I}_x f(x)dx$ defined by the fractional Riemann–Liouville integral ${}_{1x}\overset{\alpha}{I}_x f(x)dx = \Gamma^{-1}(\alpha)\int_{{}_1x}^{x} (x-x')^{\alpha-1} f(x')dx'$, when the function $f(x) = {}_{{}_1x}\overset{\alpha}{\underline{\partial}}_x F(x)$, for all $x \in [{}_1x, {}_2x]$ satisfies the conditions

$$
{}_{1^x}\overset{\alpha}{\underline{\partial}}_x \left({}_{1x}\overset{\alpha}{I}_x f(x)dx \right) = f(x), \alpha > 0,
$$

$$
{}_{1^x}\overset{\alpha}{\underline{\partial}}_x \left({}_{1x}\overset{\alpha}{I}_x F(x)dx \right) = F(x) - F({}_1x), \tag{2.1.2}
$$

$$
0 < \alpha < 1.
$$

Definition.2.1.1. A fractional volume integral is a triple fractional integral within a region $X \subseteq \mathbb{R}^3$, for instance, of a scalar field $f(x^k)$:

$$
\overset{\alpha}{I}\left(f\right) = \overset{\alpha}{I}[x^k]f(x^k) = \overset{\alpha}{I}[x^1]\overset{\alpha}{I}[x^2]\overset{\alpha}{I}[x^3]f(x^k). \tag{2.1.3}
$$

For $\alpha = 1$

$$
\overset{\alpha}{I}\left(f\right) = \overset{\alpha}{I}[x^k]f(x^k) = \iiint_V dVf(x_1,x_2,x_3). \tag{2.1.4}
$$

An exterior fractional differential can be defined through the fractional Caputo derivatives which is self–consistent with the definition of the fractional integral considered above. We write the fractional absolute differential $\overset{\alpha}{d}$ in the form

$$
\overset{\alpha}{d} \triangleq (dx^j)^\alpha \, {}_0\overset{\alpha}{\underline{\partial}}_j ,
$$

$$
\overset{\alpha}{d}x^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{2.1.5}
$$

where we consider ${}_1x = 0$.

Definition.2.1.2. An exterior fractional differential is defined via formula

$$\overset{\alpha}{d} = \Gamma(2-\alpha)\sum\nolimits_{j=1}^{n}(x^j)^{1-\alpha}\,\overset{\alpha}{d}\,x^j\;{}_0\overset{\alpha}{\underline{\partial}}_j\ . \tag{2.1.6}$$

Definition.2.1.3. The fractional integration for differential forms on an interval $\delta = [{}_1x, {}_2x]$ is defined

$$_\delta\overset{\alpha}{I}\,[x]\;{}_{_1x}\overset{\alpha}{d}_x\,f(x) = f({}_2x) - f({}_1x). \tag{2.1.7}$$

Definition.2.1.4. The exact fractional differential 0-form is a fractional differential of the an function $f(x)$:

$${}_{_1x}\overset{\alpha}{d}_x\,f(x) = (dx)^{\alpha}\;{}_{_1x}\overset{\alpha}{\underline{\partial}}_{x'}\,f(x'), \tag{2.1.8}$$

where the equation (2.1.7) is considered as the fractional generalization of the integral for a differential 1-form.
Thus the formula for the fractional exterior derivative can be written as

$${}_{_1x}\overset{\alpha}{d}_x = (dx^i)^{\alpha}\;{}_{_1x}\overset{\alpha}{\underline{\partial}}_i\ . \tag{2.1.9}$$

The fractional differential 1-form $\overset{\alpha}{\omega}$ with coefficients $\{\omega_i(x^k)\}$ is

$$\overset{\alpha}{\omega} = (dx^i)^\alpha \omega_i(x^k). \qquad (2.1.10)$$

The exterior fractional derivatives of a fractional 1-form $\overset{\alpha}{\omega}$ is a fractional 2-form

$$ {}_{1x}\overset{\alpha}{d}_x \left(\overset{\alpha}{\omega}\right) = (dx^i)^\alpha \wedge (dx^j)^\alpha \ {}_{1x}\overset{\alpha}{\underline{\partial}}_j \ \omega_i(x^k). \qquad (2.1.11)$$

II.2.Einstein Equations on Fractional Manifolds.

Definition.2.2.1. A real manifold $M,$ with integer dimension $\dim(M) = n,$ can be endowed on charts of a covering atlas with a fractional derivative-integral structure of Caputo type as we explained above. In brief, such a space (of necessary smooth class) $\overset{\alpha}{\underline{M}}$ will be called a fractional manifold.
A tangent bundle TM over a manifold M of integer dimension is canonically defined by its local integer differential structure ∂_i. A fractional generalization can be obtained directly if instead of ∂_i we consider the left Caputo derivatives ${}_{1x^i}\overset{\alpha}{\underline{\partial}}_i$ of type (2.1.1), for every local coordinate x^i.
On $\overset{\alpha}{\underline{T}} M,$ an arbitrary fractional frame basis is

$$\overset{\alpha}{\underline{e}}_\beta = e^{\beta'}_{\ \beta}(u^\beta) \ \overset{\alpha}{\underline{\partial}}_{\beta'} \ , \qquad (2.2.1)$$

where

$$\overset{\alpha}{\underline{\partial}}_{\beta'} = \left(\overset{\alpha}{\underline{\partial}}_{j'} = {}_{1x^{j'}} \ \overset{\alpha}{\partial}_{j'} \ , \overset{\alpha}{\underline{\partial}}_{b'} = {}_{1y^{b'}} \ \overset{\alpha}{\underline{\partial}}_{b'} \ \right) \qquad (2.2.2)$$

when $j' = 1,2,\ldots,n$ and $b' = n+1, n+2,\ldots,n+n.$ There are also fractional co-bases which are dual to (2.2.1)

$$\overset{\alpha}{\underline{e}}{}^{\beta} = e^{\beta}_{\beta'}(u^{\beta}) \overset{\alpha}{d} u^{\beta'}, \tag{2.2.3}$$

where the fractional local coordinate co–basis is

$$\overset{\alpha}{d} u^{\beta'} = \left((dx^{i'})^{\alpha}, (dy^{a'})^{\alpha}\right), \tag{2.2.4}$$

when the h- and v - components, $\left(dx^{i'}\right)^{\alpha}$ and $\left(dx^{a'}\right)^{\alpha}$ are of type (2.1.10). Similarly to $\overset{\alpha}{\underline{T}} M$, we can define a fractional vector bundle $\overset{\alpha}{\underline{E}}$ on $M,$ when the fiber indices of bases run values $a', b', \ldots = n+1, n+2, \ldots, n+m.$

Definition.2.2.2. Let us consider now a "prime" (pseudo) Riemannian manifold V is of integer dimension $\dim(V) = n+m,\ n \geq 2, m \geq 1.$ Its fractional extension is modelled as a fractional nonholonomic manifold $\overset{\alpha}{V}$ defined by a quadruple $\overset{\alpha}{V} = (V, \overset{\alpha}{\mathrm{N}}, \overset{\alpha}{\mathbf{d}}, \overset{\alpha}{\mathbf{I}})$, where $\overset{\alpha}{\mathrm{N}}$-is a nonholonomic distribution defining a nonlinear connection structure, the fractional differential structure $\overset{\alpha}{\mathbf{d}}$ is given by Eq.(2.2.1), Eq.(2.2.3) and the non–integer integral structure $\overset{\alpha}{\mathbf{I}}$.

Definition.2.2.3. A nonlinear connection (N–connection) $\overset{\alpha}{\mathbf{N}}$ for $\overset{\alpha}{V}$ is defined by a nonholonomic distribution (Whitney sum) with conventional h- and v-subspaces, $\underline{h}\overset{\alpha}{V}$ and $\underline{v}\overset{\alpha}{V}$,

$$\overset{\alpha}{\underline{T}}\, \overset{\alpha}{V} = \underline{h}\, \overset{\alpha}{V} \bigoplus \underline{v}\, \overset{\alpha}{V} \tag{2.2.5}$$

Nonholonomic manifolds with a nonlinear connection $\overset{\alpha}{\mathbf{N}}$ are called, in brief, N–anholonomic fractional manifolds. Locally, a fractional N–connection is defined by its coefficients, $\overset{\alpha}{\mathbf{N}} = \{ {}^{\alpha}N^{a}_{i} \}$, when

$$\overset{\alpha}{\mathbf{N}} = {}^{\alpha}N_i^a(u)(dx^i)^{\alpha} \otimes \overset{\alpha}{\underline{\partial}}_a \ . \qquad (2.2.6)$$

For a N–connection $\overset{\alpha}{\mathbf{N}}$ we can always a class of fractional (co) frames linearly depending on ${}^{\alpha}N_i^a$

$$\begin{aligned} {}^{\alpha}\mathbf{e}_{\beta} &= \left[{}^{\alpha}\mathbf{e}_j = \overset{\alpha}{\underline{\partial}}_j - {}^{\alpha}N_j^a \overset{\alpha}{\underline{\partial}}_a \ , \ {}^{\alpha}e_b = \overset{\alpha}{\underline{\partial}}_b \right], \\ {}^{\alpha}\mathbf{e}^{\beta} &= [\ {}^{\alpha}e^j = (dx^j)^{\alpha},\ {}^{\alpha}\mathbf{e}^b = (dy^b)^{\alpha} + {}^{\alpha}N_k^b(dx^k)^{\alpha}]. \end{aligned} \qquad (2.2.7)$$

The nontrivial nonholonomy coefficients are computed ${}^{\alpha}W_{ib}^a = \overset{\alpha}{\underline{\partial}}_b\ {}^{\alpha}N_i^a$ and ${}^{\alpha}W_{ij}^a = {}^{\alpha}\Omega_{ji}^a = {}^{\alpha}\mathbf{e}_i\ {}^{\alpha}N_j^a - {}^{\alpha}\mathbf{e}_j\ {}^{\alpha}N_i^a$ (where ${}^{\alpha}\Omega_{ji}^a$ are the coefficients of the N-connection curvature) for $[{}^{\alpha}\mathbf{e}_{\alpha}, {}^{\alpha}\mathbf{e}_{\beta}] = {}^{\alpha}\mathbf{e}_{\alpha}\ {}^{\alpha}\mathbf{e}_{\beta} - {}^{\alpha}\mathbf{e}_{\beta}\ {}^{\alpha}\mathbf{e}_{\alpha} = {}^{\alpha}W_{\alpha\beta}^{\alpha}\ {}^{\alpha}\mathbf{e}_{\beta}$.

Definition.2.2.4. A fractional metric structure $\overset{\alpha}{\mathbf{g}} = \left\{ {}^{\alpha}g_{\underline{\alpha}\underline{\beta}} \right\}$ is defined on a $\overset{\alpha}{V}$ by a symmetric second rank tensor

$$\overset{\alpha}{\mathbf{g}} = {}^{\alpha}g_{\underline{\gamma}\underline{\beta}}(u)(du^{\underline{\gamma}})^{\alpha} \otimes (du^{\underline{\beta}})^{\alpha}. \qquad (2.2.8)$$

For N-adapted constructions, it is important to use the property that any fractional metric $\overset{\alpha}{\mathbf{g}}$ can be represented equivalently as a distinguished metric (d–metric), $\overset{\alpha}{\mathbf{g}} = [{}^{\alpha}g_{kj}, {}^{\alpha}g_{cb}]$, where

$$\overset{\alpha}{\mathbf{g}} = {}^{\alpha}g_{kj}(x,y)\,{}^{\alpha}e^{k} \otimes {}^{\alpha}e^{j} + {}^{\alpha}g_{cb}(x,y)\,{}^{\alpha}\mathbf{e}^{c} \otimes {}^{\alpha}\mathbf{e}^{b} = \eta_{k'j'}\,{}^{\alpha}e^{k'} \otimes {}^{\alpha}e^{j'} + \eta_{c'b'}\,{}^{\alpha}e^{c'} \otimes {}^{\alpha}e^{b'}, \qquad (2.2.8')$$

where matrices $\eta_{k'j'} = diag[\pm1,\pm1,\ldots,\pm1]$ and $\eta_{c'b'} = diag[\pm1,\pm1,\ldots,\pm1]$ are obtained by frame transforms

$$\eta_{k'j'} = e^{k}{}_{k'}e^{j}{}_{j'}\,{}^{\alpha}g_{kj},\ \eta_{c'b'} = e^{a}{}_{a'}e^{b}{}_{b'}\,{}^{\alpha}g_{ab}.$$

Definition.2.2.5. A distinguished connection (d-connection) $\overset{\alpha}{\mathbf{D}}$ on a $\overset{\alpha}{V}$ is a linear connection preser- ving under parallel transports the Whitney sum (2.2.5).

To a fractional d-connection $\overset{\alpha}{\mathbf{D}}$ we can associate a N-adapted differential 1-form of type (2.1.10)

$$ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} = {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta\gamma}\,{}^{\alpha}\mathbf{e}^{\gamma}, \qquad (2.2.9)$$

where the coefficients are computed with respect to Eqs.(2.2.7) and parametrized the form

$$ {}^{\alpha}\mathbf{\Gamma}^{\gamma}{}_{\tau\beta} = \left({}^{\alpha}L^{i}_{jk},\ {}^{\alpha}L^{a}_{bk},\ {}^{\alpha}C^{i}_{jc},\ {}^{\alpha}C^{a}_{bc} \right), \qquad (2.2.10)$$

On fractional forms on $\overset{\alpha}{V}$, one can act with the absolute fractional differential ${}^{\alpha}\mathbf{d} = {}_{1x}\overset{\alpha}{d}_{x} + {}_{1y}\overset{\alpha}{d}_{y}$. In N-adapted fractional form, the value ${}^{\alpha}\mathbf{d} \triangleq {}^{\alpha}\mathbf{e}^{\beta}\,{}^{\alpha}\mathbf{e}_{\beta}$ consists from exterior h-and v-derivatives of type (2.1.9), i.e.

$$ {}_{1x}\overset{\alpha}{d}_x \triangleq (dx^i)^\alpha, \quad {}_{1x}\overset{\alpha}{\underline{\partial}}_i = {}^\alpha\mathbf{e}^j\, {}^\alpha e_j, $$
$$ {}_{1y}\overset{\alpha}{d}_y \triangleq (dx^a)^\alpha, \quad {}_{1x}\overset{\alpha}{\underline{\partial}}_a = {}^\alpha\mathbf{e}^b\, {}^\alpha e_b, \qquad (2.2.11) $$

Definition.2.2.6. The torsion and curvature of a fractional d–connection $\overset{\alpha}{\mathbf{D}} = \{ {}^\alpha\mathbf{\Gamma}^\tau_{\beta\gamma} \}$ are computed, respectively, as fractional 2-forms,

$$ {}^\alpha\mathrm{T}^\tau = \overset{\alpha}{\mathbf{D}}\, {}^\alpha\mathbf{e}^\tau = {}^\alpha\mathbf{d}\, {}^\alpha\mathbf{e}^\tau + {}^\alpha\mathbf{\Gamma}^\tau_{\ \beta} \wedge {}^\alpha\mathbf{e}^\beta, $$
$$ {}^\alpha\mathrm{R}^\tau_{\ \beta} = \overset{\alpha}{\mathbf{D}}\, {}^\alpha\mathbf{\Gamma}^\tau_{\ \beta} = {}^\alpha\mathbf{d}\, {}^\alpha\mathbf{\Gamma}^\tau_{\ \beta} - {}^\alpha\mathbf{\Gamma}^\gamma_{\ \beta} \wedge {}^\alpha\mathbf{\Gamma}^\tau_{\ \gamma} = {}^\alpha\mathbf{R}^\tau_{\ \beta\gamma\delta}\, {}^\alpha\mathbf{e}^\gamma \wedge {}^\alpha\mathbf{e}, \qquad (2.2.12) $$
$$ {}^\alpha\mathbf{d} = {}^\alpha\mathbf{e}^\beta\, {}^\alpha\mathbf{e}_\beta. $$

Definition.2.2.7. The fractional Ricci tensor ${}^\alpha\mathrm{R}ic = \{ {}^\alpha\mathbf{R}_{\alpha\beta} = {}^\alpha\mathbf{R}^\tau_{\ \alpha\beta\tau} \}$ is

$$ {}^\alpha R_{ij} = {}^\alpha R^k_{\ ijk}, \quad {}^\alpha R_{ia} = -{}^\alpha R^k_{\ ika}, \quad {}^\alpha R_{ai} = {}^\alpha R^b_{\ aib}, \quad {}^\alpha R_{ab} = {}^\alpha R^c_{\ abc}. \qquad (2.2.13) $$

Definition.2.2.8. The scalar curvature of a fractional d-connection $\overset{\alpha}{\mathbf{D}}$ is

$$ {}^\alpha_s\mathbf{R} = {}^\alpha\mathbf{g}^{\tau\beta}\, {}^\alpha\mathbf{R}_{\tau\beta} = {}^\alpha R + {}^\alpha S, \ {}^\alpha R = {}^\alpha g^{ij}\, {}^\alpha R_{ij}, \ \ {}^\alpha S = {}^\alpha g^{ab}\, {}^\alpha R_{ab}, \qquad (2.2.14) $$

defined by a sum the h- and v-components of (2.2.13) and contractions with the inverse coefficients to a d-metric (2.2.8′).

Definition.2.2.9. We introduce the Einstein tensor ${}^{\alpha}\mathbf{G}_{\alpha\beta}$

$$ {}^{\alpha}\mathbf{G}_{\alpha\beta} \triangleq {}^{\alpha}\mathbf{R}_{\alpha\beta} - \frac{1}{2}\,{}^{\alpha}\mathbf{g}_{\alpha\beta}\ {}^{\alpha}_{s}\mathbf{R}. \tag{2.2.15} $$

Note that for applications in geometry and physics, there are considered more special classes of d-connections:

1.On a fractional nonholonomic $\overset{\alpha}{\mathbf{V}}$, there is a unique canonical fractional d-connection

$$ {}^{\alpha}\widehat{\mathbf{D}} = \left\{ {}^{\alpha}\widehat{\boldsymbol{\Gamma}}^{\gamma}_{\alpha\beta} = \left({}^{\alpha}\widehat{L}^{i}_{jk},\ {}^{\alpha}\widehat{L}^{a}_{bk},\ {}^{\alpha}\widehat{C}^{i}_{jc},\ {}^{\alpha}\widehat{C}^{a}_{bc} \right) \right\} \tag{2.2.16} $$

which is compatible with the metric structure,i.e.

$$ {}^{\alpha}\widehat{\mathbf{D}}\left({}^{\alpha}\mathbf{g} \right) = 0, \tag{2.2.17} $$

and satisfies the conditions

$$ {}^{\alpha}\widehat{T}^{i}_{\ jk} = 0, {}^{\alpha}\widehat{T}^{a}_{\ bc} = 0. \tag{2.2.18} $$

2.The Levi–Civita connection $\overset{\alpha}{\nabla} = \{ {}^{\alpha}\Gamma'^{\gamma}_{\alpha\beta} \}$ can be defined in standard from but by using the fractional Caputo left derivatives acting the coefficients of a fractional metric (2.2.8).

The coefficients of the fractional Levi–Civita and canonical d-connection satisfy the distorting relation

$$ {}^{\alpha}\Gamma^{\gamma}_{\alpha\beta} = {}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta} + {}^{\alpha}Z^{\gamma}_{\alpha\beta}. \qquad (2.2.19) $$

The Einstein equations on a spacetime manifold V of integer dimension,for anenergy-momentum source of matter $T_{\alpha\beta}$, are written in the form

$$ E_{\beta\delta} = R_{\beta\delta} - \frac{1}{2} g_{\beta\delta} R = \varkappa T_{\beta\delta}. \qquad (2.2.20) $$

where $\varkappa$= const and the Einstein tensor is computed for the Levi–Civita connection ∇. The Einstein equations (2.2.20) can be rewritten equivalently using the canonical d-connection $\widehat{\mathbf{D}} = \left\{\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}\right\}$,

$$ \widehat{\mathbf{E}}_{\beta\delta} = \widehat{\mathbf{R}}_{\beta\delta} - \frac{1}{2} \mathbf{g}_{\beta\delta} \left({}^{s}R\right) = \pounds_{\beta\delta}, \qquad (2.2.21) $$

$$ \widehat{L}^{c}_{aj} = e_a\left(N^{c}_{j}\right), \widehat{C}^{i}_{jb} = 0, \Omega^{a}_{ji} = 0, \qquad (2.2.22) $$

where $\widehat{\mathbf{R}}_{\beta\delta}$ is the Ricci tensor for $\widehat{\mathbf{R}}_{\beta\delta}$, ${}^{s}R = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\beta\delta}$ and $\pounds_{\beta\delta}$ is such a way constructed that $\pounds_{\beta\delta}$ reduced to $\varkappa T_{\beta\delta}$ when $\widehat{\mathbf{D}} \to \nabla$.

Remark 2.2.1.[6].There are two possibilities to make equivalent two different systems of equations for ∇ and, respectively, for $\widehat{\mathbf{D}}$. In the first case, we can include the contributions of distortion tensor $Z^{\gamma}_{\alpha\beta}$ from Eq.(2.2.19) into the source $\pounds_{\beta\delta} \sim \varkappa T_{\beta\delta} + {}^{z}\pounds_{\beta\delta}[Z^{\gamma}_{\alpha\beta}]$ in such a form that the system (2.2.21).The second case that is $\pounds_{\beta\delta} = \varkappa T_{\beta\delta}$
but in order to keep fundamental the Einstein equations for ∇.

Introducing the fractional canonical d-connection ${}^{\alpha}\widehat{\mathbf{D}}$ (2.2.15) into the Einstein d-tensor,following the same principle of constructing the matter source ${}^{\alpha}\pounds_{\beta\delta}$ as in general relativity but for fractional d-connections, one derive geometrically a fractional generalization of N-adapted Einstein equations

$$ {}^{\alpha}\widehat{\mathbf{E}}_{\beta\delta} ={}^{\alpha} \pounds_{\beta\delta}. \quad (2.2.23)$$

Such a system can be restricted to fractional nonholonomic configurations for ${}^{\alpha}\nabla$ if we impose a fractional analog of constraints (2.2.22).

$$ {}^{\alpha}\widehat{L}^{c}_{aj} = ({}^{\alpha}e_a)\left({}^{\alpha}N^{c}_{j}\right), {}^{\alpha}\widehat{C}^{i}_{jb} = 0, {}^{\alpha}\Omega^{a}_{ji} = 0. \quad (2.2.24)$$

Let us consider a fractional metric

$$\begin{aligned} {}^{\alpha}_{\eta}\mathbf{g} &= ({}^{\alpha}\eta_i(x^k,v))({}^{\alpha}_{\circ}g_i(x^k,t))[({}^{\alpha}dx^i)\bigotimes({}^{\alpha}dx^i)] + \\ &({}^{\alpha}\eta_a(x^k,v))({}^{\alpha}_{\circ}h_a(x^k,v))[({}^{\alpha}\mathbf{e}^a)\bigotimes({}^{\alpha}\mathbf{e}^a)] \\ {}^{\alpha}\mathbf{e}_3 &= ({}^{\alpha}dv) + ({}^{\alpha}\eta^3_i(x^k,v))({}^{\alpha}_{\circ}w_i(x^k,v))({}^{\alpha}dx^i), \\ {}^{\alpha}\mathbf{e}_4 &= ({}^{\alpha}dy^4) + ({}^{\alpha}\eta^4_i(x^k,v))({}^{\alpha}_{\circ}n_i(x^k,v))({}^{\alpha}dx^i), \end{aligned} \quad (2.2.25)$$

where the coefficients will be defined below and shall work with the ”prime” dimension splitting of type $2+2$ when coordinated are labeled in the form $u^{\beta} = (x^j, y^3 = v, y^4)$, for $i,j,\ldots = 1,2$.
Remark 2.2.2.[6].The solutions of Einstein equations will be constructed for a general source of type

$$ {}^{\alpha}\pounds^{\alpha}{}_{\beta} = diag\left[{}^{\alpha}\pounds_{\gamma};\ {}^{\alpha}\pounds_1 = {}^{\alpha}\pounds_2 = {}^{\alpha}\pounds_2(x^k, v); {}^{\alpha}\pounds_3 = {}^{\alpha}\pounds_4 = {}^{\alpha}\pounds_4(x^k) \right]. \qquad (2.2.26) $$

A straightforward computation of the components of the Ricci and Einstein d-tensors corresponding to ansatz (2.2.25) reduces the Einstein equations (2.2.23) to system of partial differential equations [6]:

$$ (a) \quad {}^{\alpha}\hat{R}^1_1 = {}^{\alpha}\hat{R}^2_2 = $$

$$ -\frac{1}{2({}^{\alpha}g_1)({}^{\alpha}g_2)}\left[{}^{\alpha}g_2^{\bullet\bullet} - \frac{({}^{\alpha}g_1^{\bullet})({}^{\alpha}g_2^{\bullet})}{2({}^{\alpha}g_1)} - \frac{({}^{\alpha}g_2^{\bullet})^2}{2({}^{\alpha}g_2)} + ({}^{\alpha}g_1^{\prime\prime}) - \frac{({}^{\alpha}g_1^{\prime})^2}{2({}^{\alpha}g_1)} \right] = $$

$$ -({}^{\alpha}\pounds_4), $$

$$ (b) \quad {}^{\alpha}\hat{R}^3_3 = {}^{\alpha}\hat{R}^4_4 = $$

$$ -\frac{1}{2({}^{\alpha}h_3)({}^{\alpha}h_4)}\left[({}^{\alpha}h_4^{**}) - \frac{({}^{\alpha}h_4^{*})^2}{2{}^{\alpha}h_4} - \frac{{}^{\alpha}h_3^{*}\,({}^{\alpha}h_4^{*})}{2{}^{\alpha}h_3} \right] = -({}^{\alpha}\pounds_2), \qquad (2.2.27) $$

$$ {}^{\alpha}\hat{R}_{3k} = \frac{{}^{\alpha}w_k}{2{}^{\alpha}h_4}\left[{}^{\alpha}h_4^{**} - \frac{({}^{\alpha}h_4^{*})^2}{2{}^{\alpha}h_4} - \frac{{}^{\alpha}h_3^{*}\,({}^{\alpha}h_4^{*})}{2{}^{\alpha}h_3} \right] + $$

$$ (c) \quad \frac{{}^{\alpha}h_4^{*}}{3{}^{\alpha}h_4}\left(\frac{{}_{1x^i}\overset{\alpha}{\partial}_{x^i}}{{}^{\alpha}h_3} + \frac{\partial_k({}^{\alpha}h_4)}{{}^{\alpha}h_4} \right) - \frac{{}_{1x^k}\overset{\alpha}{\partial}_{x^k}({}^{\alpha}h_4^{*})}{2{}^{\alpha}h_4} = 0, $$

$$ (d) \quad {}^{\alpha}\hat{R}_{4k} = \frac{{}^{\alpha}h_4}{2{}^{\alpha}h_3}({}^{\alpha}n_k^{**}) + \left(\frac{{}^{\alpha}h_4}{{}^{\alpha}h_3}({}^{\alpha}h_3^{*}) - \frac{3}{2}({}^{\alpha}h_4^{*}) \right)\frac{{}^{\alpha}n_k^{*}}{2{}^{\alpha}h_3} = 0, $$

where we wrote the partial derivatives in the brief form:

$$
\begin{aligned}
{}^{\alpha}a^{\bullet} &= \overset{\alpha}{\underline{\partial}}_1 a =_{1x^1} \overset{\alpha}{\underline{\partial}}_{x^1}({}^{\alpha}a),\\
{}^{\alpha}a' &= \overset{\alpha}{\underline{\partial}}_2 a =_{1x^2} \overset{\alpha}{\underline{\partial}}_{x^2}({}^{\alpha}a),\\
{}^{\alpha}a^{*} &= \overset{\alpha}{\underline{\partial}}_v a =_{1v} \overset{\alpha}{\underline{\partial}}_v({}^{\alpha}a).
\end{aligned}
\tag{2.2.28}
$$

Configurations with fractional Levi–Civita connection ${}^{\alpha}\nabla$ can be extracted by imposing additional constraints

$$
\begin{aligned}
{}^{\alpha}w_i^{*} &= ({}^{\alpha}\mathbf{e}_i)\ln|{}^{\alpha}h_4|,\ {}^{\alpha}\mathbf{e}_k({}^{\alpha}w_i) = ({}^{\alpha}w_k)({}^{\alpha}\mathbf{e}_i),\\
{}^{\alpha}n_i^{*} &= 0,\ \overset{\alpha}{\underline{\partial}}_i({}^{\alpha}n_k) = \overset{\alpha}{\underline{\partial}}_k({}^{\alpha}n_i).
\end{aligned}
\tag{2.2.29}
$$

satisfying the conditions (2.2.24).

Remark 2.2.3.[6].One can construct 'non-Killing' general fractional solutions depending on all coordinates when:

$$
\begin{aligned}
{}^{\alpha}\mathbf{g} &= [{}^{\alpha}g_i(x^k)][({}^{\alpha}dx^i)\bigotimes({}^{\alpha}dx^i)] + \left({}^{\alpha}\omega^2(x^j,v,y^4)\right)\left({}^{\alpha}h_a(x^k,v)\right)({}^{\alpha}\mathbf{e}^a)\bigotimes({}^{\alpha}\mathbf{e}^a),\\
{}^{\alpha}\mathbf{e}^3 &= ({}^{\alpha}dy^3) + [{}^{\alpha}w_i(x^k,v)]({}^{\alpha}dx^i),\\
{}^{\alpha}\mathbf{e}^4 &= ({}^{\alpha}dy^4) + [{}^{\alpha}n_i(x^k,v)]({}^{\alpha}dx^i),
\end{aligned}
\tag{2.2.30}
$$

for any ${}^{\alpha}\omega$ for which

$$
{}^{\alpha}\mathbf{e}_k({}^{\alpha}\omega) = \overset{\alpha}{\underline{\partial}}_k\ ({}^{\alpha}\omega) + ({}^{\alpha}\omega_k)({}^{\alpha}\omega^{*}) + ({}^{\alpha}n_k)\ \overset{\alpha}{\underline{\partial}}_4\ ({}^{\alpha}\omega) = 0,
\tag{2.2.31}
$$

where ${}^{\alpha}\omega^2 = 1.$

I.Solutions with ${}^{\alpha}\mathbf{h}^{*}_{3,4} \neq 0$ and ${}^{\alpha}£_{2,4} \neq 0$.
Such metrics are defined by ansatz:

$$
\begin{aligned}
{}^{\alpha}\mathbf{g} &= (\exp[{}^{\alpha}\psi(x^{k})])[({}^{\alpha}dx^{i})\bigotimes({}^{\alpha}dx^{i})] + \left({}^{\alpha}h_{3}(x^{k},v)\right)({}^{\alpha}\mathbf{e}^{3})\bigotimes({}^{\alpha}\mathbf{e}^{3}) + \\
&\left({}^{\alpha}h_{4}(x^{k},v)\right)({}^{\alpha}\mathbf{e}^{4})\bigotimes({}^{\alpha}\mathbf{e}^{4}), \\
{}^{\alpha}\mathbf{e}^{3} &= ({}^{\alpha}dv) + \left({}^{\alpha}w_{i}\left(x^{k},v\right)\right)({}^{\alpha}dx^{i}), \\
{}^{\alpha}\mathbf{e}^{4} &= ({}^{\alpha}dy^{4}) + ({}^{\alpha}n_{i}(x^{k},v))({}^{\alpha}dx^{i}),
\end{aligned}
\qquad (2.2.32)
$$

with the coefficients which is solutions of the system

$$
\begin{aligned}
&(a)\ \ {}^{\alpha}\ddot{\psi} + ({}^{\alpha}\psi'') = 2\,{}^{\alpha}£_{4}(x^{k}), \\
&(b)\ \ {}^{\alpha}h_{4}^{*} = \frac{2\,{}^{\alpha}h_{3}({}^{\alpha}h_{4})\,{}^{\alpha}£_{2}(x^{i},v)}{{}^{\alpha}\phi^{*}},
\end{aligned}
\qquad (2.2.33)
$$

$$
\begin{aligned}
&(a)\ \ {}^{\alpha}\beta({}^{\alpha}w_{i}) + ({}^{\alpha}\alpha_{i}) = 0, \\
&(b)\ \ {}^{\alpha}n_{i}^{**} + ({}^{\alpha}\gamma)({}^{\alpha}n_{i}^{*}) = 0,
\end{aligned}
\qquad (2.2.34)
$$

where

$$(a)\quad {}^{\alpha}\phi = \ln\left| \frac{{}^{\alpha}h_4^*}{\sqrt{|{}^{\alpha}h_3({}^{\alpha}h_4)|}} \right|, {}^{\alpha}\gamma = \left(\ln\left(\frac{|{}^{\alpha}h_4|^{3/2}}{|{}^{\alpha}h_3|}\right)\right)^*, \tag{2.2.35}$$

$$(b)\quad {}^{\alpha}\alpha_i = ({}^{\alpha}h_4^*)\ \overset{\alpha}{\underline{\partial}}_k\ ({}^{\alpha}\phi), {}^{\alpha}\beta = ({}^{\alpha}h_4^*)({}^{\alpha}\phi^*).$$

For $({}^{\alpha}h_4^* \neq 0) \wedge ({}^{\alpha}£_2 \neq 0)$ we have also ${}^{\alpha}\phi^* \neq 0.$The exponential function $\exp[{}^{\alpha}\psi(x^k)]$
in (2.2.32) is the fractional analog of the ”integer” exponential functions and called the Mittag–Leffer function $E_{\alpha}\left[\left(x - {}^{1}x\right)^{\alpha}\right]$.We shall write usual symbols for functions as in the case of integer calculus,but providing a label α considering such fractional construction.
It is possible to consider any nonconstant ${}^{\alpha}\phi = {}^{\alpha}\phi(x^i, v)$ as a generating function, we can construct exact solutions of Eq.(2.2.33)-Eq.(2.2.34).One have to solve respec- tively the two dimensional fractional Laplace equation, for $({}^{\alpha}g_1) = ({}^{\alpha}g_2) = = \exp[{}^{\alpha}\psi(x^k)]$ Then one integrate on v, in order to determine ${}^{\alpha}h_3, {}^{\alpha}h_4, {}^{\alpha}n_i,$and solving algebraic equations, for ${}^{\alpha}w_i$. Thus one obtain:

$$({}^{\alpha}g_1) = ({}^{\alpha}g_2) = \exp[{}^{\alpha}\psi(x^k)], {}^{\alpha}h_3 = \pm\frac{|{}^{\alpha}\psi(x^k, v)|}{{}^{\alpha}£_2},$$

$${}^{\alpha}h_4 = ({}^{\alpha}_{0}h_4(x^k)) \pm 2 \times {}_{{}_{1}v}\overset{\alpha}{I}\left[{}_{v}\frac{(\exp[2{}^{\alpha}\psi(x^k, v)])^*}{{}^{\alpha}£_2}\right], \tag{2.2.36}$$

$${}^{\alpha}w_i = -\overset{\alpha}{\underline{\partial}}_i\ ({}^{\alpha}\phi)/{}^{\alpha}\phi^*,$$

$${}^{\alpha}n_i = ({}^{\alpha}_{1}n_k(x^i)) + ({}^{\alpha}_{2}n_k(x^i)) \times {}_{{}_{1}v}\overset{\alpha}{I}\left[\frac{{}^{\alpha}h_3}{\left(\sqrt{|{}^{\alpha}h_4|}\right)^3}\right],$$

where ${}^{\alpha}_{0}h_4(x^k), {}^{\alpha}_{1}n_k(x^i)$ and ${}^{\alpha}_{2}n_k(x^i)$ are integration functions, and ${}_{{}_{1}v}\overset{\alpha}{I}\ [\cdot]$ is the fractional integral on variables v.To construct exact solutions for the Levi-Civita connection ${}^{\alpha}\nabla$, we have to constrain the coefficients (2.2.36) to satisfy the conditions (2.2.29). For instance, we can fix a nonholonomic distribution when ${}^{\alpha}_{2}n_k(x^i) = 0$ and ${}^{\alpha}_{1}n_k(x^i)$ are any functions satisfying the conditions

$$\overset{\alpha}{\underline{\partial}}_i({}^{\alpha}_{1}n_k(x^j)) = \overset{\alpha}{\underline{\partial}}_k({}^{\alpha}_{1}n_i(x^j)). \tag{2.2.37}$$

The constraints on ${}^{\alpha}\phi(x^k,v)$ are related to the N-connection coefficients ${}^{\alpha}w_i = -\overset{\alpha}{\underline{\partial}}_i\ ({}^{\alpha}\phi)/{}^{\alpha}\phi^{*}$ following relations

$$\begin{aligned} &({}^{\alpha}w_i[{}^{\alpha}\phi])^{*} + ({}^{\alpha}w_i[{}^{\alpha}\phi])({}^{\alpha}h_4[{}^{\alpha}\phi])^{*} + \overset{\alpha}{\underline{\partial}}_i({}^{\alpha}h_4[{}^{\alpha}\phi]) = 0, \\ &\overset{\alpha}{\underline{\partial}}_i({}^{\alpha}_{1}w_k(x^j)) = \overset{\alpha}{\underline{\partial}}_k({}^{\alpha}_{1}w_i(x^j)), \end{aligned} \tag{2.2.38}$$

where we denoted by ${}^{\alpha}w_i[{}^{\alpha}\phi]$ and ${}^{\alpha}h_4[{}^{\alpha}\phi]$ the functional dependence on ${}^{\alpha}\phi$. Such conditions are always satisfied for ${}^{\alpha}\phi = {}^{\alpha}\phi(v)$ or ${}^{\alpha}\phi = const$ where ${}^{\alpha}w_i(x^k,v)$ can be any functions.

II.Solutions with ${}^{\alpha}\mathbf{h}_4^{*} = 0$

The equation (2.2.27.b) can be solved for such a case ${}^{\alpha}h_4^{*} = 0$, only iff ${}^{\alpha}£_2 = 0$. Any set of functions ${}^{\alpha}w_i(x^k,v)$ obviously define a solution of Eq.(2.2.27.c),and its equivalent (2.2.34.a),because the coefficients ${}^{\alpha}\beta$, ${}^{\alpha}\alpha_i$ are zero.The coefficients ${}^{\alpha}n_i$ are determined from Eq.(2.2.34.b) ${}^{\alpha}h_4^{*} = 0$ and any given ${}^{\alpha}h_3$ which results in

$${}^{\alpha}n_k = ({}^{\alpha}_{1}n_k) + ({}^{\alpha}_{2}n_k)\ {}_{1v}\overset{\alpha}{I}_v\ [{}^{\alpha}h_3]. \tag{2.2.39}$$

It is possible to choose $({}^{\alpha}g_1) = ({}^{\alpha}g_2) = \exp[{}^{\alpha}\psi(x^k)]$ with ${}^{\alpha}\psi(x^k)$ determined by Eq.(2.2.33.a) for any given ${}^{\alpha}£_4(x^k)$. This class of solutions is given by ansatz [6]:

$$\begin{aligned} {}^{\alpha}\mathbf{g} &= \exp[{}^{\alpha}\psi(x^k)]({}^{\alpha}dx^i)\bigotimes({}^{\alpha}dx^i) + {}^{\alpha}h_3(x^k,v)({}^{\alpha}\mathbf{e}^3)\bigotimes({}^{\alpha}\mathbf{e}^3) + \\ &\quad {}^{\alpha}_{0}h_4(x^k)({}^{\alpha}\mathbf{e}^4)\bigotimes({}^{\alpha}\mathbf{e}^4), \\ {}^{\alpha}\mathbf{e}^3 &= ({}^{\alpha}dv) + ({}^{\alpha}w_i(x^k,v))({}^{\alpha}dx^i), \\ {}^{\alpha}\mathbf{e}^4 &= ({}^{\alpha}dy^4) + \Big[({}^{\alpha}_{1}n_k) + ({}^{\alpha}_{2}n_k)\ {}_{1v}\overset{\alpha}{I}_v\ [{}^{\alpha}h_3]\Big]({}^{\alpha}dx^i), \end{aligned} \tag{2.2.40}$$

for every fractional functions ${}^{\alpha}h_3(x^k,v), {}_0^{\alpha}h_4(x^k), {}^{\alpha}w_i(x^k,v)$ and integration fractional functions ${}_1^{\alpha}n_k, {}_2^{\alpha}n_k$. A subclass of solutions for the Levi–Civita connection can be obtained from (2.2.40) by using the conditions

$$
\begin{aligned}
{}_2^{\alpha}n_k(x^i) = 0,\ \overset{\alpha}{\underline{\partial}}_i({}_1^{\alpha}n_k) = \overset{\alpha}{\underline{\partial}}_k({}_1^{\alpha}n_i), \\
{}^{\alpha}w_i^* + \overset{\alpha}{\underline{\partial}}_i({}^0h_4) = 0,\ \overset{\alpha}{\underline{\partial}}_i({}_1^{\alpha}w_k) = \overset{\alpha}{\underline{\partial}}_k({}_1^{\alpha}w_i),
\end{aligned}
\tag{2.2.41}
$$

for any such ${}^{\alpha}w_i(x^k,v)$ and ${}_0^{\alpha}h_4(x^k)$

III.Solutions with ${}^{\alpha}\mathbf{h}_3^* = 0$ and ${}^{\alpha}\mathbf{h}_4^* \neq 0$.

The ansatz for metric is of the type

$$
\begin{aligned}
{}^{\alpha}\mathbf{g} = &\exp[{}^{\alpha}\psi(x^k)]({}^{\alpha}dx^i)\bigotimes({}^{\alpha}dx^i) - {}_0^{\alpha}h_3(x^k)({}^{\alpha}\mathbf{e}^3)\bigotimes({}^{\alpha}\mathbf{e}^3) + \\
&{}^{\alpha}h_4(x^k,v)({}^{\alpha}\mathbf{e}^4)\bigotimes({}^{\alpha}\mathbf{e}^4), \\
{}^{\alpha}\mathbf{e}^3 = &({}^{\alpha}dv) + ({}^{\alpha}w_i(x^k,v))({}^{\alpha}dx^i), \\
{}^{\alpha}\mathbf{e}^4 = &({}^{\alpha}dy^4) + ({}^{\alpha}n_i(x^k,v))({}^{\alpha}dx^i),
\end{aligned}
\tag{2.2.42}
$$

where $({}^{\alpha}g_1) = ({}^{\alpha}g_2) = \exp[{}^{\alpha}\psi(x^k)]$ with ${}^{\alpha}\psi(x^k)$ determined from Eq.(2.2.33.a) for any given ${}^{\alpha}£_4(x^k)$. A function ${}^{\alpha}h_4(x^k,v)$ solves the equation (2.2.33.b) for ${}^{\alpha}h_3^* = 0$ which can be represented in the form

$$
{}^{\alpha}h_4^{**} - \frac{({}^{\alpha}h_4^*)^2}{2\,{}^{\alpha}h_4} - 2\,{}_0^{\alpha}h_3({}^{\alpha}h_4)({}^{\alpha}£_2(x^k,v)) = 0. \tag{2.2.43}
$$

The solutions for the N-connection coefficients are

$$
\begin{aligned}
{}^{\alpha}w_i &= -\overset{\alpha}{\underline{\partial}}_i\left({}^{\alpha}\widetilde{\phi}\right)/\left({}^{\alpha}\widetilde{\phi}^{*}\right),\\
{}^{\alpha}n_i &= \left({}_{1}^{\alpha}n_k(x^i)\right) + \left({}_{2}^{\alpha}n_k(x^i)\right)\ {}_{1v}\overset{\alpha}{I}_v\left[1/\left(\sqrt{|{}^{\alpha}h_4|}\right)^3\right],\\
{}^{\alpha}\widetilde{\phi} &= \ln\left|\frac{{}^{\alpha}h_4^{*}}{\sqrt{|{}_{0}^{\alpha}h_3({}^{\alpha}h_4)|}}\right|.
\end{aligned}
\tag{2.2.44}
$$

The Levi–Civita conditions for ansatz (2.2.42) is

$$
\begin{aligned}
&{}_{1}^{\alpha}n_k(x^i) = 0,\ \ \overset{\alpha}{\underline{\partial}}_i({}_{1}^{\alpha}n_k) = \overset{\alpha}{\underline{\partial}}_k({}_{1}^{\alpha}n_i),\\
&\left({}^{\alpha}w_i\left[{}^{\alpha}\widetilde{\phi}\right]\right)^{*} + \left({}^{\alpha}w_i\left[{}^{\alpha}\widetilde{\phi}\right]\right)\left({}^{\alpha}h_4\left[{}^{\alpha}\widetilde{\phi}\right]\right)^{*} + \overset{\alpha}{\underline{\partial}}_i\left({}^{\alpha}h_4\left[{}^{\alpha}\widetilde{\phi}\right]\right) = 0,\\
&\overset{\alpha}{\underline{\partial}}_i\left({}_{1}^{\alpha}w_k\left[{}^{\alpha}\widetilde{\phi}\right]\right) = \overset{\alpha}{\underline{\partial}}_k\left({}_{1}^{\alpha}w_i\left[{}^{\alpha}\widetilde{\phi}\right]\right).
\end{aligned}
\tag{2.2.45}
$$

Note that for small fractional deformations, it is not obligatory to impose such conditions.One can consider integer Levi-Civita configurations and then to transform them nonholonomically into certain d-connection ones.

IV.Solutions with ${}^{\alpha}\phi = const.$

Fixing in (2.2.35.a) ${}^{\alpha}\phi = {}^{\alpha}\phi_0 = const$ and considering ${}^{\alpha}h_3^{*} \neq 0$ and ${}^{\alpha}h_4^{*} \neq 0$, we get that the general solutions of Eq.(2.2.33)-Eq.(2.2.34) are

$$
\begin{aligned}
{}^{\alpha}\mathbf{g} &= \exp[{}^{\alpha}\psi(x^k)]({}^{\alpha}dx^i)\bigotimes({}^{\alpha}dx^i) - \\
&{}_0^{\alpha}h^2[{}^{\alpha}f^*(x^i,v)]^2({}^{\alpha_\varsigma}\pounds(x^i,v))({}^{\alpha}\mathbf{e}^3)\bigotimes({}^{\alpha}\mathbf{e}^3) + ({}^{\alpha}f^2(x^i,v))({}^{\alpha}\mathbf{e}^4)\bigotimes({}^{\alpha}\mathbf{e}^4), \\
{}^{\alpha}\mathbf{e}^3 &= ({}^{\alpha}dv) + ({}^{\alpha}w_i(x^i,v))({}^{\alpha}dx^i), \\
{}^{\alpha}\mathbf{e}^4 &= ({}^{\alpha}dy^4) + ({}^{\alpha}n_k(x^i,v))({}^{\alpha}dx^i),
\end{aligned}
\tag{2.2.46}
$$

where ${}_0^{\alpha}h = const$ and $({}^{\alpha}g_1) = ({}^{\alpha}g_2) = \exp[{}^{\alpha}\psi(x^k)]$ with ${}^{\alpha}\psi(x^k)$ determined from Eq.(2.2.33.a) for any given ${}^{\alpha}\pounds_4(x^k)$.
By using the fractional function

$$
{}^{\alpha_\varsigma}\pounds(x^i,v) = \alpha_{\varsigma 4[0]}(x^i) - \frac{{}_0^{\alpha}h^2}{16}\ {}_{1v}\overset{\alpha}{I}_v\left[{}^{\alpha}\pounds_2(x^k,v)\left({}^{\alpha}f^2(x^i,v)\right)^2\right] \tag{2.2.47}
$$

we write the fractional solutions for N-connection coefficients ${}^{\alpha}N_i^3 = {}^{\alpha}w_i$ and ${}^{\alpha}N_i^4 = {}^{\alpha}n_i$ in the next form

$$
{}^{\alpha}w_i = \overset{\alpha\,\alpha_\varsigma}{\underline{\partial}_i}({}^{\alpha_\varsigma}\pounds(x^k,v))/({}^{\alpha_\varsigma}\pounds(x^k,v))^* \tag{2.2.48}
$$

and

$$
{}^{\alpha}n_k = ({}_1^{\alpha}n_k(x^i)) + ({}_2^{\alpha}n_k(x^i))\ {}_{1v}\overset{\alpha}{I}_v\left[\frac{\left({}^{\alpha}f^*(x^i,v)\right)^2}{{}^{\alpha}f(x^i,v)}\right] \tag{2.2.49}
$$

If ${}^{\alpha_\varsigma}\pounds(x^i,v) = \pm 1$ for ${}^{\alpha}\pounds_2 \to 0$, we take $\alpha_{\varsigma 4[0]}(x^i) = \pm 1$. For such conditions, the functions
${}^{\alpha}h_3 = -{}_0^{\alpha}h^2\left({}^{\alpha}f^*(x^i,v)\right)^2$ and ${}^{\alpha}h_4 = {}^{\alpha}f^2(x^i,v)$ satisfy the equation (2.2.33.b), when

$$\sqrt{|{}^{\alpha}h_3|} = {}^{0}h\left(\sqrt{|{}^{\alpha}h_4|}\right)^{*} \tag{2.2.50}$$

is compatible with the condition ${}^{\alpha}\phi = {}^{\alpha}\phi_0 = const.$

II.3.Fractional Spacetimes and Black Holes. Fractional deformations of the Schwarzschild spacetime.

In the paper [6], was proved that black holes really exist in fractional gravity contrary to the hope that involving a new type of derivative calculus,and changing respectively the differential spacetime structure, we may eliminate ”ambiguities” with singularities etc.The concepts of black hole,singularity and horizon seem to be fundamental ones for various types of holonomic and nonholonomic, commutative and noncommutative, pseudo–Riemanann and Finlser like,fractional and integer etc. theories of gravity.
We consider a diagonal integer dimensional metric ${}^{\varepsilon}\mathbf{g}$ depending on a small real parameter $1 > \varepsilon > 0$,

$${}^{\varepsilon}\mathbf{g} = -d\xi \bigotimes d\xi - r^2(\xi) d\vartheta \bigotimes d\vartheta - r^2(\xi)\sin^2\vartheta d\varphi \bigotimes d\varphi + \varpi^2(\xi) dt \bigotimes dt \tag{2.3.1}$$

The local coordinates and nontrivial metric coefficients are parametrized:

$$x^1 = \xi, x^2 = \vartheta, y^3 = \upsilon = \varphi, y^4 = t,$$

$$\check{g}_1 = -1, \check{g}_2 = -r^2(\xi), \check{h}_3 = -r^2(\xi)\sin^2\vartheta, \check{h}_4 = \varpi^2(\xi),$$

$$\xi = \int dr \left|1 - \frac{2m_0}{r} + \frac{\varepsilon}{r^2}\right|^{1/2}, \tag{2.3.2}$$

$$\varpi^2(r) = 1 - \frac{2m_0}{r} + \frac{\varepsilon}{r^2}.$$

For $\varepsilon = 0$ in variable $\xi(r)$ and coefficients, the metric (2.3.1) is just the the Schwarzschild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$ with a point mass m_0.
In paper [6] was introduced a class of exact fractional vacuum solutions of type ( ) when the fractional metrics are generated by nonholonomic deformations

$$ {}^{\alpha}g_i = {}^{\alpha}\eta_i \breve{g}_i, h_a = {}^{\alpha}\eta_a \breve{h}_a. \tag{2.3.4} $$

and some nontrivial ${}^{\alpha}w_i$, ${}^{\alpha}n_i$ [where $\left(\breve{g}_i, \breve{h}_a\right)$ are given via Eqs.(2.3.2)] and parametrized by ansatz

$$ \begin{gathered} {}^{\varepsilon}_{\eta}\mathbf{g} = -[{}^{\alpha}\eta_1(\xi,\vartheta,\theta)]\left({}^{\alpha}d\xi\right)\otimes({}^{\alpha}d\xi) - [{}^{\alpha}\eta_2(\xi,\vartheta,\theta)]r^2(\xi)({}^{\alpha}d\vartheta)\otimes({}^{\alpha}d\vartheta) - \\ [{}^{\alpha}\eta_3(\xi,\vartheta,\varphi,\theta)]r^2(\xi)\sin^2\vartheta(\delta\varphi)\otimes(\delta\varphi) + [{}^{\alpha}\eta_4(\xi,\vartheta,\varphi,\theta)]\varpi^2(\xi)({}^{\alpha}\delta t)\otimes({}^{\alpha}\delta t) \\ {}^{\alpha}\delta\varphi = ({}^{\alpha}d\varphi) + [{}^{\alpha}w_1(\xi,\vartheta,\varphi,\theta)]\left({}^{\alpha}d\xi\right) + [{}^{\alpha}w_2(\xi,\vartheta,\varphi,\theta)]\left({}^{\alpha}d\vartheta\right), \\ {}^{\alpha}\delta t = ({}^{\alpha}dt) + [{}^{\alpha}n_1(\xi,\vartheta,\theta)]\left({}^{\alpha}d\xi\right) + [{}^{\alpha}n_2(\xi,\vartheta,\theta)]\left({}^{\alpha}d\vartheta\right). \end{gathered} \tag{2.3.5} $$

where the coefficients will be constructed determine solutions of the system of equations ()–() with ${}^{\alpha}\pounds_{\beta} = 0.$ The equation () for ${}^{\alpha}\pounds_2 = 0$ is solved via formulae

$$ \begin{gathered} {}^{\alpha}h_3 = -({}^{\alpha}_{0}h)^2({}^{\alpha}b^*)^2 = [{}^{\alpha}\eta_3(\xi,\vartheta,\varphi,\theta)]r^2(\xi)\sin^2\vartheta(\delta\varphi), \\ {}^{\alpha}h_4 = ({}^{\alpha}b^2)\,[{}^{\alpha}\eta_4(\xi,\vartheta,\varphi,\theta)]\varpi^2(\xi), \\ |{}^{\alpha}\eta_3| = ({}^{\alpha}_{0}h)^2\left|\frac{\breve{h}_4}{\breve{h}_3}\right|\left[\left(\sqrt{|{}^{\alpha}\eta_4|}\right)^*\right]^2. \end{gathered} \tag{2.3.6} $$

Assume ${}^{\alpha}_{0}h = const.$ It must be ${}^{\alpha}_{0}h = 2$ in order to satisfy the condition () with zero source, where ${}^{\alpha}\eta_4$ can be any function satisfying the condition ${}^{\alpha}\eta_4^* \neq 0.$ This way, it is possible to generate a class of solutions for any function ${}^{\alpha}b(\xi,\vartheta,\varphi,\theta)$ with

${}^{\alpha}b^{*} \neq 0.$

Remark 2.3.1.[6] Note that for classes of solutions with nontrivial sources, it is more convenient to work directly with fractional polarizations ${}^{\alpha}\eta_4$ with ${}^{\alpha}\eta_4^{*} \neq 0.$
In another turn, for vacuum configurations, it is better to chose as a generating function, for instance,${}^{\alpha}h_4$ with ${}^{\alpha}h_4^{*} \neq 0.$
The fractional polarizations $({}^{\alpha}\eta_1)$ and $({}^{\alpha}\eta_2)$,when $({}^{\alpha}\eta_1) = ({}^{\alpha}\eta_2 r^2) = \exp[{}^{\alpha}\psi(\xi,\vartheta)]$, from () with ${}^{\alpha}£_2 = 0,$ i.e. ${}^{\alpha}\psi^{**} + {}^{\alpha}\psi'' = 0.$
Putting the above coefficient in Eq.(2.3.5), we construct a class of exact vacuum solutions in fractional gravity defining stationary fractional nonholonomic deformations on a small parameter ε of the Schwarzschild metric,

$$
\begin{aligned}
{}^{\alpha}_{\varepsilon}\mathbf{g} = &- \exp[{}^{\alpha}\psi(\xi,\vartheta,\theta)] \left\{ \left({}^{\alpha}d\xi\right) \otimes ({}^{\alpha}d\xi) - ({}^{\alpha}d\vartheta) \otimes ({}^{\alpha}d\vartheta) \right\} - \\
&4\left[\sqrt{|{}^{\alpha}\eta_4(\xi,\vartheta,\varphi,\theta)|} \right]^2 \varpi^2(\xi)({}^{\alpha}\delta\varphi) \otimes ({}^{\alpha}\delta\varphi) + \\
&[{}^{\alpha}\eta_4(\xi,\vartheta,\varphi,\theta)]\varpi^2(\xi)({}^{\alpha}\delta t) \otimes ({}^{\alpha}\delta t),
\end{aligned}
\tag{2.3.7}
$$

$$
{}^{\alpha}\delta\varphi = ({}^{\alpha}d\varphi) + [{}^{\alpha}w_1(\xi,\vartheta,\varphi,\theta)]\left({}^{\alpha}d\xi\right) + [{}^{\alpha}w_2(\xi,\vartheta,\varphi,\theta)]\left({}^{\alpha}d\vartheta\right),
$$

$$
{}^{\alpha}\delta t = ({}^{\alpha}dt) + [{}^{\alpha}n_1(\xi,\vartheta,\theta)]\left({}^{\alpha}d\xi\right) + [{}^{\alpha}n_2(\xi,\vartheta,\theta)]\left({}^{\alpha}d\vartheta\right).
$$

III.COSMOLOGICAL MODELS WITH FRACTIONAL ACTION FUNCTIONAL

III.1.Friedmann-Robertson-Walker cosmology with a fractional time dimensions.

Let's consider a smooth manifold M and denote $\mathcal{L} : \mathbb{R} \times TM \to \mathbb{R}$ be the smooth Lagrangian function. For any piecewise smooth path $\gamma : [0, t_1] \to \mathbb{R}$ we define the generalized fractional action $\left(S^{\varepsilon}_{\alpha\in[0,\infty)}[\gamma]\right)_{\varepsilon\in(0,1]}$ as corresponding Colombeau generalized function via formula [11] :

$$\left(S^{\varepsilon}_{\alpha\in(-\infty,1]}[\gamma]\right)_{\varepsilon\in(0,1]} = \frac{1}{\Gamma(\alpha)}\left(\int_0^t \mathcal{L}(\dot{\gamma}(\tau),\gamma(\tau),\tau)\frac{d\tau}{(t-\tau)^{1-\alpha}+i\varepsilon}\right)_{\varepsilon} =$$

$$\left(S^{\varepsilon}_{\beta\in[0,\infty)}[\gamma]\right)_{\varepsilon\in(0,1]} = \left(\int_0^t \mathcal{L}(\dot{\gamma}(\tau),\gamma(\tau),\tau)\frac{d\tau}{(t-\tau)^{\beta}+i\varepsilon}\right)_{\varepsilon}, \tag{3.1.1}$$

$$\alpha \in (-\infty, 1],$$

where $\beta = 1-\alpha$. For $\alpha \in (0,1]$ generalized fractional action (3.1) can be rewritten as the strictly singular Riemann-Liouville type fractional derivative Lagrangian

$$S_{\alpha\in(0,1]}[\gamma] = \frac{1}{\Gamma(\alpha)}\int_0^t \mathcal{L}(\dot{\gamma}(\tau),\gamma(\tau),\tau)(t-\tau)^{\alpha-1}d\tau =$$

$$S_{\beta\in[0,1)}[\gamma] = \int_0^t \mathcal{L}(\dot{\gamma}(\tau),\gamma(\tau),\tau)\frac{d\tau}{(t-\tau)^{\beta}}, \tag{3.1.2}$$

where $\beta = 1-\alpha$. Let $\mathcal{L} : \mathbb{R} \times TM \to \mathbb{R}$ be the Lagrangian map, $\{p_0, \}$ are two fixed points and smooth path $\gamma : [0, t_1] \to \mathbb{R}$ be a smooth path such that $\gamma(t_i) = p_i, i = 0,1$

and $S_{\mathcal{L}}[\gamma] \leq S_{\mathcal{L}}[\widetilde{\gamma}]$ for any smooth path $\widetilde{\gamma}$ joining from p_0 to p_1. Then, γ satisfies the fractional or modified Euler-Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\gamma} - \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{\gamma}}\right) = \frac{1-\alpha}{(t-\tau+i\varepsilon)_{\varepsilon}}\left(\frac{\partial\mathcal{L}}{\partial\dot{\gamma}}\right). \tag{3.1.3}$$

Let’s consider Lagrangian

$$\mathcal{L}(\dot{x}(\lambda), x(\lambda), \lambda) = g_{\mu\nu}(\dot{x}, x)\dot{x}^{\mu}\dot{x}^{\nu}. \tag{3.1.4}$$

Then corresponding geodesic equation is

$$\ddot{x}^{\mu} + \frac{\alpha - 1}{(t + i\varepsilon)_{\varepsilon}}\dot{x}^{\mu} + \Gamma^{\mu}_{\nu\delta}\dot{x}^{\nu}x^{\delta} = 0 \tag{3.1.5}$$

where $\Gamma^{\mu}_{\nu\delta}$ is the Christoffel symbol.It was showed in [12] that equation (3.5) will modify the General Relativity by perturbing the gravitational constant G by a certain decaying factor given by:

$$\Delta G = \frac{3(1 - \alpha)}{4\pi G\rho t}\frac{\dot{R}}{R}, \tag{3.1.6}$$

where ρ being the fluid density and $R(t)$ is the scale factor of the universe.It is easy to see that in Friedmann-Robertson-Walker cosmology (FRW cosmology), the term ΔG modifies the Friedman equations in the absence of the cosmological constant as follows

$$\begin{aligned} \left(\frac{\dot{R}}{R}\right)^2 + \frac{2(\alpha - 1)}{t}\frac{\dot{R}}{R} + \frac{k}{R^2} &= \frac{8\pi G\rho}{3}, \\ \frac{\ddot{R}}{R} + \frac{(\alpha - 1)}{t}\frac{\dot{R}}{R} &= -\frac{4\pi G}{3}(\rho + 3p), \end{aligned} \tag{3.1.7 – 3.1.8}$$

Where $k = -1, 0, +1$ for open, flat and closed fractal spacetime respectively.For zero pressure, while in case of radiation ($\rho = 3p$), from Eq.(3.1.7) one obtain

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2(\alpha-1)}{t}\frac{\dot{R}}{R} + \frac{k}{R^2} = \frac{8\pi G\rho}{3},$$

$$\frac{\ddot{R}}{R} + \frac{2(\alpha-1)}{t}\frac{\dot{R}}{R} = -\frac{8\pi G\rho}{3}. \qquad (3.1.9-3.1.10)$$

Equations (3.1.7) and (3.1.9) can be rewritten like:

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\left[\rho + \frac{(1-\alpha)}{t}\frac{3}{4\pi G}\frac{\dot{R}}{R}\right] =$$

$$\frac{8\pi G}{3}[\rho + \tilde{\rho}], \qquad (3.1.11)$$

where

$$\tilde{\rho} = \frac{3(1-\alpha)\dot{R}}{4\pi G R t}. \qquad (3.1.12)$$

This is to say that the density is perturbed.

III.2. (3+α) DIMENSIONAL FRACTAL UNIVERSE.THE VACUUM CASE.

Let us now consider the very early universe and choose for simplicity the spatially flat solution ($k = 0$). Eq.(3.1.9) with $p = -\rho$ gives:

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2(\alpha-1)}{t}\frac{\dot{R}}{R} = 0. \qquad (3.2.1)$$

Solution is

$$R(t) \propto t^{2(1-\alpha)}. \qquad (3.2.2)$$

Solution (3.2.2) corresponds to an accelerated expansion for $0 < \alpha < 1/2,$ to an eternal expansion for $1/2 < \alpha < 1$ and to a decelerating expansion for $\alpha > 1.$

III.3.(3 +α) DIMENSIONAL UNIVERSE.THE RADIATION-DOMINATED EPOCH.

This is characterized by the equation of state $p = \rho/3$ and is modeled by equations (8) and (9) combined in the following form ($k = 0$) :

$$\frac{\ddot{R}}{R} + \frac{4(\alpha-1)}{t}\frac{\dot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 = 0. \qquad (3.3.1)$$

A possible solution is given also by the power-law $R(t) \propto t^p, p = (5-4\alpha)/2.$
$p > 1$ for $-\infty < \alpha \leq 1$ and the acceleration of the universe may be attributed of the fractional dissipative force. For $\alpha = 1$ the usual (3+1)-dimensional behavior is permitted:

$$R(t) \propto \sqrt{t}\,. \tag{3.3.2}$$

III.4.(3 + α) DIMENSIONAL UNIVERSE.THE INFLATIONARY EPOCH.

We consider the spatially flat solution ($k = 0$). From equation (3.1.7), we obtain:

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2(\alpha - 1)}{t}\frac{\dot{R}}{R} = \frac{8\pi G\rho}{3}. \tag{3.4.1}$$

In the absence of the gravity perturbations ($\alpha = 1$), the solution of Eq.(3.4.1) is given by the classical de-Sitter inflationary solution:

$$R(t) \propto \exp[Ht], \qquad H = \sqrt{\frac{8\pi G\rho}{3}} = const, \tag{3.4.2}$$

where ρ and G are constants. In the presence of the perturbed gravity, a possible inflationary solution is given via formula [11] :

$$R(t) = H\frac{e^{-\beta}t^{\beta}}{2(1-\alpha)} \times$$

$$\exp\left\{\beta\left[\frac{\beta^2 + Ht^2 + \beta\sqrt{\beta^2 + Ht^2}}{\beta^2 - Ht^2 + \beta\sqrt{\beta^2 + Ht^2}} - \ln\left(\frac{\beta}{Ht} + \sqrt{\frac{\beta^2}{Ht^2} + 1}\right)\right]\right\} \quad (3.4.3)$$

where $\beta = 1 - \alpha$ and $-\infty < \alpha < 1$.

(3.4.4)

III.5. (3 +α) dimensional Universe. Cosmological models of scalar field with fractional action.

The classical Einstein-Hilbert action-like functional for FRW model of the $(3+1)$ dimensional Universe $ds^2 = N^2(t)dt^2 - a^2(t)(dr^2 + f(r)d\Omega^2)$, where N is a laps function, filled with a real homogeneous scalar field $\varphi(t)$, is:

$$S_{EH} = \frac{1}{\Gamma(\alpha)}\int_0^t N(t) \times$$

$$\times\left[\frac{3}{8\pi G}\left(\frac{a^2(t)\ddot{a}(t)}{N^2(t)} + \frac{a(t)\dot{a}(t)}{N^2(t)} - \frac{a^2(t)\dot{a}(t)\dot{N}(t)}{N^2(t)} + \mathbf{k}a(t) - \frac{\Lambda a^3(t)}{3}\right)\right. \quad (3.5.1)$$

$$\left. + a^3\left(\epsilon\frac{\dot{\varphi}^2(t)}{2N^2(t)} - V(\varphi(t))\right)\right]dt.$$

where $V(\phi)$ is a potential of the field. By variation over $a(t), \varphi(t)$ and $N(t)$ (with the subsequent choice of the gauge $N = 1$) in the action (3.5.1), one obtains the following standard Friedmann and scalar field equations:

$$(a)\ \ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \epsilon\frac{dV(\varphi)}{d\varphi} = 0,$$

$$(b)\ 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\mathbf{k}}{a^2} = -\frac{8\pi G}{3}(\epsilon\dot{\varphi}^2 - V(\varphi)) + \frac{\Lambda}{3}, \tag{3.5.2}$$

$$(c)\ \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right) + \frac{\mathbf{k}}{a^2} = \frac{8\pi G}{3}\left(\epsilon\frac{\dot{\varphi}^2}{2} + V(\varphi)\right) + \frac{\Lambda}{3}.$$

Remark 3.5.1. Besides, instead of Eq.(3.5.2.b) one frequently uses the following equation:

$$\frac{\ddot{a}}{a} + \frac{\mathbf{k}}{a^2} = -\frac{8\pi G}{3}(\epsilon\dot{\varphi}^2 - V(\varphi)) + \frac{\Lambda}{3}, \tag{3.5.3}$$

which follows from Eq.(3.5.2.b)-Eq.(3.5.2.c).One can rewrite Eq.(3.5.2.a) - Eq.(3.5.3.c) in terms of effective energy density $\rho(t)$ and pressure $p(t)$, taking into account standard expressions:

$$\rho = \epsilon\frac{\dot{\varphi}^2}{2} + V(\varphi), p = \epsilon\frac{\dot{\varphi}^2}{2} - V(\varphi). \tag{3.5.4}$$

From Eq.(3.5.2)-Eq.(3.5.4) one obtains

$$(a)\ \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0,$$

$$(b)\ \frac{\ddot{a}}{a} = -\frac{8\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{3.5.5}$$

$$(c)\ \left(\frac{\dot{a}}{a}\right)^2 + \frac{\mathbf{k}}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}.$$

Let’s remind how these equations play out in the simplest universe,the Einstein-de Sitter universe. This is a universe that is spatially flat and consists only of matter ($w = 0$).It is NOT the real universe because it doesn’t have a Λ ($\Lambda = 0$).Density today is given by the Friedmann equation in terms of the Hubble constant is:

$$H_0^2 = \frac{8}{3}\pi G\rho_0,$$
$$\rho_0 = \frac{3H_0^2}{8\pi G}. \tag{3.5.6}$$

The scale factor $a(t)$ as a function of time t is

$$a(t) = \left(\frac{3}{2}H_0 t\right)^{3/2}. \tag{3.5.7}$$

The time t as a function of scale factor is

$$t = \frac{2}{3H_0}a^{3/2}(t). \tag{3.5.8}$$

The Hubble constant $H(t)$ as a function of scale factor is

$$H(t) = H_0 a^{-3/2}(t). \tag{3.5.9}$$

The conformal time $\tau = \tau(t)$ is

$$\tau(t) = \frac{2}{H_0} a^{1/2}(t). \qquad (3.5.10)$$

From Eq.(3.5.9)-Eq.(3.5.10) one obtain

$$H(t)\tau(t) = 2a^{-1}(t), \qquad (3.5.11)$$

and

$$\tau(t) = \frac{2}{a(t)H(t)}. \qquad (3.5.12)$$

We consider now the generalized cosmological model of a scalar field,which follows from the variational principle for the fractional action (3.1.2). In this section we have use the modified Einstein-Hilbert action [15]:

$$S^{\alpha}_{EH} = \int_0^t \mathcal{L}^{\alpha}_{EH} d\tau \qquad (3.5.13)$$

as the following fractional integral:

$$S^{\alpha}_{EH} = \frac{1}{\Gamma(\alpha)} \int_0^t N(t) \times$$

$$\times \left[\frac{3}{8\pi G} \left(\frac{a^2(t)\ddot{a}}{N^2(t)} + \frac{a(t)\dot{a}(t)}{N^2(t)} - \frac{a^2\dot{a}\dot{N}}{N^2(t)} + \mathbf{k}a(t) - \frac{\Lambda a^3(t)}{3} \right) \right. \qquad (3.5.14)$$

$$\left. + a^3 \left(\epsilon \frac{\dot{\varphi}^2(t)}{2N^2} - V(\varphi(t)) \right) \right] (t-t')^{\alpha-1} d\tau,$$

where all functions in $\mathcal{L}_{EH}^{\alpha}$ depend on the intrinsic time t, and $\epsilon = +1, -1$ for the usual and phantom scalar fields respectively.Varying the action (3.5.14) over $\varphi(t)$, $a(t)$ and $N(t)$ with the subsequent choice of the gauge $N = 1$, we obtain the following equations:

$$
\begin{aligned}
&(a)\ \ddot{\varphi} + 3\left(\frac{\dot{a}}{a} + \frac{1-\alpha}{3t}\right)\dot{\varphi} + \epsilon\frac{dV(\varphi)}{d\varphi} = 0,\\
&(b)\ \frac{\ddot{a}}{a} + \frac{1-\alpha}{2t}\left(\frac{\dot{a}}{a}\right) + \frac{(1-\alpha)(2-\alpha)}{2t^2} = -\frac{8\pi G}{3}(\epsilon\dot{\varphi}^2 - V(\varphi)) + \frac{\Lambda}{3},\\
&(c)\ \left(\frac{\dot{a}}{a}\right)^2 + \frac{(1-\alpha)}{t}\left(\frac{\dot{a}}{a}\right) + \frac{\mathbf{k}}{a^2} = \frac{8\pi G}{3}\left(\epsilon\frac{\dot{\varphi}^2}{2} + V(\varphi)\right) + \frac{\Lambda}{3}.
\end{aligned}
\tag{3.5.15}
$$

One can rewrite equations (3.5.15.a) - (3.5.15.c) in terms of effective energy density $\rho(t)$ and pressure $p(t)$, taking into account the well known expressions:

$$
\rho = \epsilon\frac{\dot{\varphi}^2}{2} + V(\varphi), p = \epsilon\frac{\dot{\varphi}^2}{2} - V(\varphi). \tag{3.5.16}
$$

From Eq.(3.5.15)-Eq.(3.5.16) one obtain

$$
\begin{aligned}
&(a)\ \dot{\rho} + 3\left(\frac{\dot{a}}{a} + \frac{1-\alpha}{3t}\right)(\rho + p) = 0,\\
&(b)\ \frac{\ddot{a}}{a} + \frac{1-\alpha}{2t}\left(\frac{\dot{a}}{a}\right) + \frac{(1-\alpha)(2-\alpha)}{2t^2} = -\frac{8\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3},\\
&(c)\ \left(\frac{\dot{a}}{a}\right)^2 + \frac{(1-\alpha)}{t}\left(\frac{\dot{a}}{a}\right) + \frac{\mathbf{k}}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}.
\end{aligned}
\tag{3.5.17}
$$

It is easy to integrate equation (3.5.17.a) for the perfect fluid with equation of state $p = \gamma\rho$:

$$\rho(t) = \frac{\rho_0}{a^{3(1+\gamma)} t^{[(1+\gamma)(1-\alpha)]}}. \tag{3.5.18}$$

Let us consider an example of an exact solution for the flat model ($\mathbf{k} = 0$) and for the quasivacuum state of matter: $\gamma = -1$. From (3.5.18) it follows that $\rho(t) = \rho_0 = = constant$. Then, the remaining equations of (3.5.17) for the Hubble parameter and Λ-term can be rewritten as follows:

$$\begin{aligned} &(a)\ \dot{H} + \frac{1-\alpha}{2t} H + \frac{(1-\alpha)(2-\alpha)}{2t^2} = 0, \\ &(b)\ H^2 + \frac{(1-\alpha)}{t} H + \frac{\mathbf{k}}{a^2} = \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3}. \end{aligned} \tag{3.5.19}$$

From Eq.(3.5.19.a) one obtain

$$\begin{aligned} &H(t,\alpha) = \frac{c_\alpha}{t} + H_0 t^{\frac{1-\alpha}{2}}, c_\alpha = \frac{(1-\alpha)(2-\alpha)}{(3-\alpha)}, \\ &H_0 = const. \end{aligned} \tag{3.5.20}$$

The scale factor $a(t,\alpha)$ as a function of time t is

$$a(t,\alpha) = a_0 t^{c_\alpha} \exp\Big[\Big(\frac{3-\alpha}{2} H_0\Big) t^{\frac{3-\alpha}{2}}.\Big] \tag{3.5.21}$$

while the cosmological $\Lambda(t,\alpha)$-term as a function of time t is

$$\Lambda(t,\alpha) = 3H_0^2 t^{1-\alpha} + 3H_0 \frac{(1-\alpha)(2-\alpha)}{(3-\alpha)} t^{-\frac{1-\alpha}{2}} +$$

$$\frac{3(1-\alpha)^2(2-\alpha)(5-2\alpha)}{(3-\alpha)^2} t^{-2} - 8\pi G\rho_0. \tag{3.5.22}$$

The conformal time $\tau(t)$ is

$$\tau(t) = \int_{t_0}^{t} \frac{dt}{a(t,\alpha)} = \frac{1}{a_0} \int_{t_0}^{t} t^{-c_\alpha} \exp\Big[\Big(-\frac{3-\alpha}{2} H_0\Big) t^{\frac{3-\alpha}{2}}\Big]. \tag{3.5.23}$$

It is obvious that in the limit $\alpha \to 1$, the solutions (3.5.20) - (3.5.23) reduce to the well known exponential expansion of the $(3+1)$-dimensional Universe :

$$a(t) = a_0 \exp(H_0 t), H(t) = H_0 = const,$$

$$\Lambda(t,\alpha) = 3H_0^2 - 8\pi G\rho_0. \tag{3.5.24}$$

Let us consider now the dynamics of the flat model of the Universe (**k** $= 0$) filled by a scalar field φ.It is convenient to rewrite equations (3.5.17.a) - (3.5.17.c) in terms of the Hubble parameter $H = \frac{\dot{a}}{a}$ in the following form:

$$(a)\ \ddot{\varphi} + 3\Big(H + \frac{1-\alpha}{3t}\Big)\dot{\varphi} + \epsilon \frac{dV(\varphi)}{d\varphi} = 0,$$

$$(b)\ \dot{H} + \frac{1-\alpha}{2t} H + \frac{(1-\alpha)(2-\alpha)}{2t^2} = -4\pi G\epsilon\dot{\varphi}^2, \tag{3.5.25}$$

$$(c)\ H^2 + \frac{(1-\alpha)}{t} H + \frac{\mathbf{k}}{a^2} = \frac{8\pi G}{3}\Big(\epsilon\frac{\dot{\varphi}^2}{2} + V(\varphi)\Big) + \frac{\Lambda}{3}.$$

It is easy to see that the given set of the independent equations can contain some arbitrariness in a choice of unknown functions, for instance $H(t)$ or $V(t)$, only if the cosmological Λ-term depends on time t. However, it is possible to proceed from some dependence $\Lambda(t)$. Let us rewrite Eqs.(3.5.25) in the form [15]:

$$\dot{H}+3H^2-\frac{2(4-\alpha)}{t}H-\frac{(1-\alpha)(2-\alpha)}{2t^2}=\frac{t\dot{\Lambda}}{1-\alpha},$$

$$3H^2-\frac{3(5-\alpha)}{2t}H-\frac{3(1-\alpha)(2-\alpha)}{2t^2}-4\pi G\epsilon\dot{\varphi}^2=\frac{t\dot{\Lambda}}{1-\alpha}, \tag{3.5.26}$$

$$8\pi GV(\varphi)+\Lambda=\frac{3(7-3\alpha)}{2t}H+\frac{3(1-\alpha)(2-\alpha)}{2t^2},$$

$$\alpha\neq 1.$$

IV.Crossover from low dimensional to $(3+1)$-dimensional universe.

IV.1. Mureika and Stojkovic crossover from (2 + 1)- dimensional fractal universe to standard (3 + 1)-dimensional universe.

In order to determine an approximate value for the frequency of the PGWs,we revisit the current state of PGW detection. Standard cosmological theory generalizad on the case of fractal spacetime $M^{(D_{\mathbf{t}},D_{\mathbf{f}})}$, predicts that gravitational waves in fractal spacetime $M^{(D_{\mathbf{t}},D_{\mathbf{f}})}$ will be generated in the pre/postinflationary regime due to quantum fluctuations of the fractal spacetime manifold $M^{(D_{\mathbf{t}},D_{\mathbf{f}})}$.
At temperatures below the $(D_{\mathbf{t}}=3,D_{\mathbf{f}}=2)\to(D_{\mathbf{t}}=3,D_{\mathbf{f}}=3)$ cross-over scale, a standard $3D_{\mathbf{t}}$ FRW cosmology is assumed, with the usual radiation- and matter-dominated eras.Gravity waves can be produced at different times $t_*<t_0=H^{-1}$, when the temperature of the universe was T. The co-moving entropy per volume of

the universe at temperature T_* can be expressed as a function of the scale factor $R(t)$ as

$$S \sim g_S(T)R^3(t)T^3, \tag{4.1.1}$$

where the factor g_S represents the effective number of degrees of freedom at temperature T in terms of entropy by formula

$$g_S(T) = \sum_i \left(\frac{T_i}{T} \right) + \frac{7}{8} \sum_j \left(\frac{T_j}{T} \right). \tag{4.1.2}$$

The parameters i,j runs over all particle species.In the standard model, this assumes a constant value for $T = 300\ GeV$, with $g_S(T) = 106.75$ due to the fact that all species were thermalized to a common temperature. Assuming that entropy is generally conserved over the evolution of the universe, one can write

$$g_S(T_*)R^3(t_*)T_*^3 = g_S(T_0)R^3(t_0)T_0^3. \tag{4.1.3}$$

The characteristic frequency of a gravitational wave produced at some time t in the past is thus redshifted to its present-day value

$$f_0 = f_* \left(\frac{R(t_*)}{R(t_0)} \right) \tag{4.1.4}$$

by the factor [13]

$$f_0 = 9.37 \times 10^{-5} Hz (H \times 1\mathbf{mm})^{1/2} \times g_*^{-1/12} \left(\frac{g_*}{g_{*S}} \right)^{1/3} T_{2.728}, \tag{4.1.5}$$

where the original production frequency f_0 is bounded by the horizon size of the universe at time t_*,i.e. $f_* \sim \lambda_*^{-1} \sim H_*^{-1}$.Note that this is an upper bound,and the actual value may be smaller by a factor ϵH_*^{-1}although the final result is weakly sensitive to the value $\epsilon \leq 1$.This quantity can be related to the temperature T_* by noting that, during the radiation-dominated phase, the scale is

$$H_*^2 = \frac{8\pi^3 g_* T_*^4}{90 M_{Pl}^4}. \tag{4.1.6}$$

Remark 4.1.1.[1].Note that above was used equations valid only in the $(3+1)$-dimen- sional regime.Without the details of an underlying lower dimensional cosmology we do not know the size of a lower dimensional Hubble volume as a function of the temperature. However, in order to estimate the frequency cut-off , we are approaching the dimensional cross-over from the known $(3+1)$-dimensional regime. Thus, while Eq.(4.1.6) is not valid in a lower dimensional regime, it is valid a few Hubble times after the dimensional crossover. Since most of the $3D$ volume of the universe comes from the last few Hubble times, this will be a reasonable
estimate of the size of the $3D$ Hubble volume after the dimensional cross-over.

If we plug $T_* = 1$TeV,we see that H^{-1}~1mm, which is much larger than TeV^{-1}.This is not in contradiction with our assumption that the cross-over happened at $T_* = 1$ TeV since the size of a 2-dimensional plane/universe could be arbitrarily large before the cross-over.Since the size of the $2D$ universe does not matter (no gravity waves), the crucial thing here is that the highest frequency that PGWs can carry is limited by the size of a $3D$ Hubble volume right after the dimensional cross-over, which is given by Eq.(4.1.6).With above assumptions, combining Eq.(4.1.5) and Eq.(4.1.6), the frequency of PGWs that would be detectable is

$$f_\Lambda = 7.655 \times 10^{-5} g_*^{1/6} \left(\frac{T_*}{\text{TeV}} \right) \text{ Hz} \approx 1.67 \times 10^{-4} \text{ Hz}, \tag{4.1.7}$$

where the latter equality holds for $g_* \sim 10^2$. When $T_* = 1$ TeV, the frequency is $f_\Lambda \sim 10^{-4}$ Hz. This is well below the seismic limit of $f \sim 40$ Hz on ground-based gravity wave interferometer experiments like LIGO or VIRGO [1], but sits precisely at the threshold of LISA's sensitivity range.

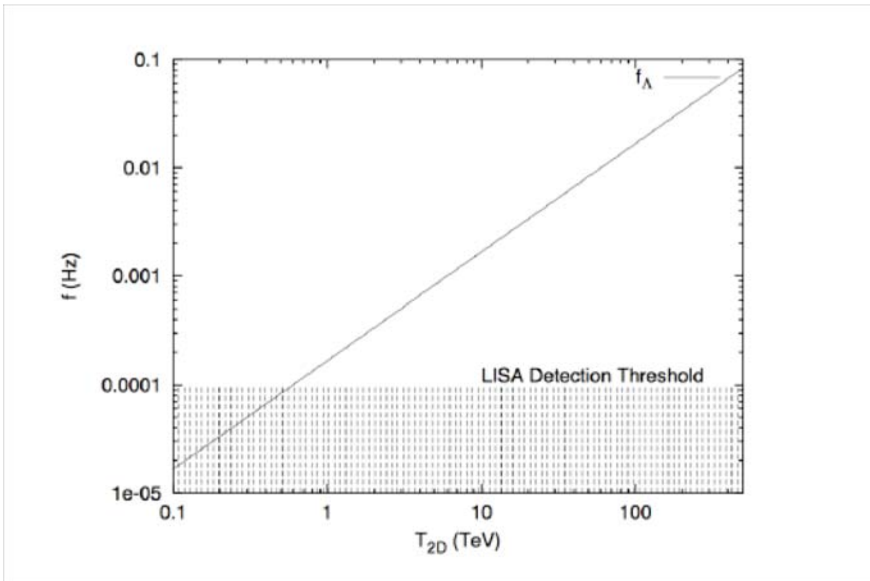

FIG.4.1. Frequency threshold for primordial gravitational waves produced when the universe was at temperature T_* LISA.[1].

A $(2+1)$-dimensional FRW metric is

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right). \tag{4.1.8}$$

where $R(t)$ is the scale factor and $k = 1, 0, +1$. The Einstein's equations for this metric are

$$\left(\frac{\dot{R}}{R}\right)^2 = 2\pi G\rho - \frac{k}{R^2},$$
$$\frac{d}{dt}(\rho R^2) + p\frac{d}{dt}(R^2) = 0 \tag{4.1.9}$$

where G is the (2+1) dimensional gravitational constant, p is the pressure and is the energy density. In a radiation dominated universe $p = 1/2\rho$ and $\rho R^3 = \rho_0 R_0^3 = const.$ For $k = 0$ the solution to these equations is

$$R(t) = \left(\frac{9}{2}\pi G\rho_0 R_0^3\right)t^{2/3}. \tag{4.1.10}$$

Remark 4.1.2. Note that three-dimensional solution $R(t) \propto t^{2/3}$ is different from the usual four-dimensional behavior $R(t) \propto \sqrt{t}$ in radiation dominated era.
Assume that the crossover from (2 + 1)-dimensional to (3 + 1)-dimensional universe happened when the temperature of the universe was $T_{2D\to3D} \sim 1$ TeV [1]. Working backwards,we can estimate the size of the Universe at the transition from the ratio of scale sizes at various epochs, speci cally between present day ($t_{\textbf{today}} = 10^{17}$ sec), the radiation/matter-dominated era ($t_{\textbf{RM}} = 10^{10}$ sec) and the TeV-era ($t_{\textbf{TeV}} = 10^{-12}$ sec).

$$\frac{R_{\textbf{TeV}}}{R_{\textbf{today}}} = \left(\frac{t_{\textbf{TeV}}}{t_{\textbf{RM}}}\right)^{1/2}\left(\frac{t_{\textbf{RM}}}{t_{\textbf{today}}}\right)^{2/3} =$$
$$\left(\frac{10^{-12}}{10^{10}}\right)^{1/2}\left(\frac{10^{10}}{10^{17}}\right)^{2/3} = 10^{-11} \times 10^{-14/3} \simeq 10^{-15.6}. \tag{4.1.11}$$

The scale factor at the latter epoch is thus $R_{\textbf{TeV}} = 10^{-15.6}R_{\textbf{today}}$.

This value may also be obtained by noting that conservation of entropy requires the product $R(t)T_t$ to be constant,and so $R_{\textbf{TeV}} = 10^{-15.6}R_{\textbf{today}}$ (since $T_{\textbf{today}} \sim 10^{-3}$eV). Eq.(4.1.11) implies that the size of the currently visible universe (10^{28}cm) at $T = 1$ TeV was $10^{13.6}$cm. This distance is *macroscopic* but it is not in contrast with assumption [1] that the crossover from $(2+1)$-dimensional to $(3+1)$-dimensional universe happened when the temperature of the universe was $T \sim 1$ TeV, since the causally connected universe today contains many causally connected regions of some earlier time.
Going towards even higher temperatures, the spacetime becomes $(1+1)$-dimensio-
nal [1]. To avoid large hierarchy in the standard model, the crossover from an $(1+1)$-dimensional to $(2+1)$-dimensional universe needs to happen when the temperature of the universe was $T_{\textbf{1D}\to\textbf{2D}} = 100$ TeV. Conservation of entropy (if between $T = 1$ TeV and $T = 100$ TeV nothing nonadiabatic happened) requires $R(t)T_t = const.$ This implies

$$\frac{R_{2D\to 3D}}{R_{1D\to 2D}} = \frac{T_{1D\to 2D}}{T_{2D\to 3D}} \sim 100 \tag{4.1.12}$$

and $R_{1D\to 2D} = 10^{-2} R_{2D\to 3D} = 10^{11.6}$ cm.

Similarly one obtain

$$\frac{R_{1D\to 2D}}{R_{\#}} = \frac{T_{\#}}{T_{1D\to 2D}} \tag{4.1.13}$$

where $R_{\#}$ and $T_{\#}$ are the scale factor and temperature of the universe the first time it appears classically. It is tempting to set $R_{\#} = M_{Pl}^{-1}$ and $T_{\#} = M_{Pl}$ .However $M_{Pl} = 10^{19}$ GeV is inherently $(3+1)$-dimensional quantity whose meaning is not quite clear in the context of evolving dimensions.

A pure $(1+1)$-dimensional FRW metric is

$$ds^2 = dt^2 - R^2(t)\frac{dx^2}{1 - \mathbf{k}x^2}. \tag{4.1.14}$$

The denominator in the second term in Eq.(4.1.14) can be absorbed into a definition of the spatial coordinate x. Moreover, all $(1+1)$-dimensional spaces are conformally flat,i.e. one can always use coordinate transformations (independently of the dynamics) and put the metric in the form $g = \exp(\phi)\eta_{\mu\nu}$.

Remark 4.1.3. [1] Einstein's action in a twodimensional spacetime is just the Euler characteristics of the manifold in question, so the the theory does not have any dynamics, unless the scalar eld is promoted into a dynamical eld by adding a kinetic term for it. Even in this case there are no gravitons in theory, so *there are no gravity waves* and the threshold of importance remains the $(1+1) \to (2+1)$ transition.

Remark 4.1.4.[1] However, exactly at the crossover the description *could be very complicated.*For example,systems whose effective dimensionality changes with the scale can exhibit fractal behavior, even if they are defined on smooth manifolds. As a good step in that direction,in [18-19] a field theory which lives in fractal spacetime and is argued to be Lorentz invariant, power-counting renormalizable, and causal was proposed.

Since entropy $S(T,D_{\mathbf{f}})$ is generally conserved over the evolution of the$(D_{\mathbf{f}}+1)$-dimensional fractal universe, we obtain

$$s(T_*,D_{\mathbf{f}})R^{D_{\mathbf{f}}}(t_*,D_{\mathbf{f}})T^{D_{\mathbf{f}}} = s(T_0,3)R^3(t_0)T_0^3 \tag{4.1.15}$$

[compare with standard case given by Eq.(4.1.3)]. From Eq.(4.1.15) one obtain

$$R(t_*,D_{\mathbf{f}}) = \left(\frac{s(T_0,3)}{s(T_*,D_{\mathbf{f}})}\right)^{1/D_{\mathbf{f}}} \frac{R^{3/D_{\mathbf{f}}}(t_0)T_0^{3/D_{\mathbf{f}}}}{T_*} \sim$$
$$\sim \frac{R^{3/D_{\mathbf{f}}}(t_0)T^{\,3/D_{\mathbf{f}}}}{T_*}. \tag{4.1.16}$$

Let us consider dimensional crossover from $((D_{\mathbf{f}}=2)+1)$-dimensional fractal Universe to (3+1)-dimensional standard Universe by using Eq.(4.1.15). Assume that $T_0 == T_{\mathbf{today}} = 10^{-3}$eV, $T_* = 1$TeV $= 10^{12}$eV, $R(t_0) = R_{\mathbf{today}} = 10^{28}$ cm. Thus

$$R(t_*,D_{\mathbf{f}}=2) = T_*^{-1}R^{3/2}(t_0)T_0^{3/2} =$$
$$10^{-13}R_{\mathbf{today}}^{3/2} = 10^{29} \text{ cm}. \tag{4.1.17}$$

IV.2.Cross-over from (3 + α)- to (3 +1)-dimensional fractal universe using α-FRW cosmology.

In the case of the RD epoch the α-FRW solution is given also by the power-law $R(t) \propto t^p, p = (5-4\alpha)/2$ i.e.,

$$R(t) \propto t^{\frac{5-4\alpha}{2}} \tag{4.2.1}$$

for $-\infty < \alpha \leq 1$. For $\alpha = 1$ the usual $(3+1)$-dimensional behavior is permitted:

$$R(t) \propto \sqrt{t}. \tag{4.2.2}$$

Remark 4.2.1. We note that in $(3+\alpha)$-dimensional fractal universe standard conservation law is permitted:

$$g_S(T_{3+\alpha})R(t_{3+\alpha})T_{3+\alpha} = const. \tag{4.2.3}$$

Let us consider crossover from $(3+\alpha_1)$-dimensional fractal universe to $(3+\alpha_2)$-dimensional universe with $\alpha_1 < \alpha_2 \leq 1$. From Eq.(4.2.1) and Eq.(4.2.3) one obtain

$$\frac{R_{\mathbf{TeV}}(t_{3+\alpha_1})}{R_{\mathbf{RM}}(t_{3+\alpha_2})} = \frac{g_S(T_{3+\alpha_2})}{g_S(T_{3+\alpha_1})}\left(\frac{T_{3+\alpha_2}}{T_{3+\alpha_1}}\right), \tag{4.2.4}$$

where

$$R_{\mathbf{TeV}}(t_{3+\alpha_1}) \propto t_{3+\alpha_1}^{\frac{5-4\alpha_1}{2}}, R_{\mathbf{RM}}(t_{3+\alpha_2}) \propto t_{3+\alpha_2}^{\frac{6-4\alpha_2}{3}},$$
$$T_{3+\alpha_1} \triangleq T(t_{3+\alpha_1}), T_{3+\alpha_2} \triangleq T(t_{3+\alpha_2}). \tag{4.2.5}$$

From Eq.(4.2.4)-Eq.(4.2.5) we obtain

$$\frac{(5-4\alpha_1)}{2}\ln t_{3+\alpha_1} - \frac{(6-4\alpha_2)}{3}\ln t_{3+\alpha_2} = \ln\left(\frac{g_S(T_{3+\alpha_2})}{g_S(T_{3+\alpha_1})}\frac{T_{3+\alpha_2}}{T_{3+\alpha_1}}\right). \qquad (4.2.6)$$

Assume for instance that crossover from $(3+\alpha_1)$-dimensional fractal universe to $(3+\alpha_2)$-dimensional universe with $\alpha_2 \simeq 1$ happened when the temperature of the universe was $T_{(3+\alpha_1)\to(3+\alpha_2)} = 1$TeV. The Universe at the transition from the TeV-era: $t_{(3+\alpha_1)} = t_{\mathbf{TeV}} = 10^{-12}$ sec, $T_{(3+\alpha_1)} = 1\text{TeV} = 10^{12}$eV to radiation/matter-dominated era: $t_{(3+\alpha_2)} = t_{\mathbf{RM}} = 10^{10}$ sec, $T_{(3+\alpha_2)} \sim 1\text{MeV} - 0.7\text{MeV}$. Note that at the energy scales above ~ 1 TeV,$g_S^{\mathbf{SM}}(T) = 106.75$ and $g_S^{\mathbf{MSSM}} \simeq 220$ [34]. From Eq.(4.2.6) we obtain $\alpha_1 = 30/24$.

IV.3.Crossover from $D_{\mathbf{f}} = D_{\mathbf{t}}(1 - \alpha)$- to (3 +1)-dimensional universe by using G.Colgany cosmology.

The Ansatz for the gravitational action of the G.Colgany gravity [18]-[19] in fractal spacetime is

$$S_g = \frac{1}{2\kappa^2}\int d^{D_{\mathrm{t}}}x\left[v(x)\sqrt{-g(x)}\,R(x) - 2\Lambda - \omega(\partial_\mu v(x)\partial^\mu v(x))\right] \qquad (4.3.1)$$

where $g(x)$ is the determinant of the metric tensor $g_{\mu\nu}(x), \kappa^2 = 8\pi G$ is Newton's constant, Λ is a bare cosmological constant, and the term $\omega(\partial_\mu v(x)\partial^\mu v(x))$ proportional to ω has been added, because $v(x)$ is now dynamical variable. Assuming that matter is minimally coupled with gravity, the total action is

$$S = S_g + S_m,$$
$$S_m = \int v(x)\mathcal{L}_m d^4x. \quad (4.3.2)$$

If $\rho + p \neq 0$ from (4.3.1) one get a purely gravitational constraint [18]:

$$\dot{H} + (D_{\mathbf{t}} - 1)H^2 + H\frac{\dot{v}}{v} + \frac{\Box v}{v} + \omega(v\Box v - \dot{v}^2) = 0. \quad (4.3.3)$$

The continuity equation $\nabla_\mu(v(x)T^\mu_\lambda) - \partial_\lambda v(x)\mathcal{L}_m = 0$ gives

$$\dot{\rho} + \left[(D_{\mathbf{t}} - 1)H + \frac{\dot{v}}{v}\right](\rho + p) = 0. \quad (4.3.4)$$

Substitution $\rho = \dot{\varphi}^2/2 + V(\varphi), p = \dot{\varphi}^2/2 - V(\varphi)$ gives

$$\ddot{\varphi} + \left[(D_{\mathbf{t}} - 1)H + \frac{\dot{v}}{v}\right]\dot{\varphi} + V'(\varphi) = 0. \quad (4.3.5)$$

For the case $v(x) = t^{-D_{\mathbf{f}}}, D_{\mathbf{f}} = D_{\mathbf{t}}(1 - \alpha)$ one obtain formulae

$$H\frac{\dot{v}}{v} = -H\frac{\beta}{t}, \frac{\Box v}{v} = \frac{\beta}{t}\left[(D_{\mathbf{t}} - 1)H - \frac{1 + \beta}{t}\right]. \quad (4.3.6)$$

Let us consider the cases $\omega = 0$ and $\omega \neq 0$ separately.

1. We assume: $D_{\mathbf{t}} = 4, D_{\mathbf{f}} = 2,\ v(x) = t^{-D_{\mathbf{f}}},\ \omega = 0.$ Solution is [18]:

$$a(t) = \frac{(t^9 + c)^{1/3}}{t^2},$$

$$H(t) = \frac{t^9 - 2c}{t(t^9 + c)},$$

$$\epsilon = -\frac{\dot{H}}{H^2} = \qquad (4.3.7)$$

$$\frac{\left[t^9 - \left(14 + 3\sqrt{22}\right)c\right]\left[t^9 - \left(14 - 3\sqrt{22}\right)c\right]}{(t^9 - 2c)^2},$$

where c is an integration constant. The energy density and pressure is

$$\rho(t) = -\frac{3}{\kappa^2}\frac{(t^9 + 4c)(t^9 - 2c)}{(\quad)},$$

$$p(t) = -\frac{3}{\kappa^2}\frac{\left[t^9 + \left(14 - 3\sqrt{22}\right)c\right]\left[t^9 + \left(14 + 3\sqrt{22}\right)c\right]}{(t^9 - 2c)^2} \qquad (4.3.7)$$

These expressions are sufficient to characterize three cases:
(**a**) $c > 0$: From $t = t_* = (2c)^{1/9}$ to $t = t_1 = [(14 + 3\sqrt{22})c]^{1/9}$, the universe expands in superacceleration ($\epsilon < 0$), while for $t > t_1$ the expansion is only accelerated. The energy density ρ is negative for $t > t_*$ while the pressure p is always negative.
(**b**) $c = 0$: Linear (decelerating) expansion, $a = t$, while $\rho = p < 0$ always.
(**c**)
All these scenarios need a matter component with non-positive definite energy density.
2. We assume: $D_{\mathbf{t}} = 4, D_{\mathbf{f}} = 2$, $v(x) = t^{-D_{\mathbf{f}}}$, $\omega \neq 0$. There is only one real solution to the gravitational constraint, namely [18]:

$$a(t) = \frac{1}{t^2}\Phi\left(\frac{11}{4}, \frac{13}{4}, \frac{3\omega}{2t^4}\right),$$

$$H(t) = -\frac{2}{t} - \frac{22\omega}{13t^5}\frac{\Phi\left(\frac{15}{4}, \frac{17}{4}, \frac{3\omega}{2t^4}\right)}{\Phi\left(\frac{11}{4}, \frac{13}{4}, \frac{3\omega}{2t^4}\right)}, \qquad (4.3.8)$$

$$\Phi(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{+\infty}\frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{z^n}{n!}.$$

The formulae for $\rho(t)$ and $p(t)$ are

$$\rho(t) = \frac{2(2\omega+3t^4)(3\omega+4t^4)}{t^{10}} + \frac{48\omega^2}{13^2t^{10}}\frac{\Phi\left(\frac{11}{4}, \frac{17}{4}, \frac{3\omega}{2t^4}\right)}{\Phi\left(\frac{11}{4}, \frac{13}{4}, \frac{3\omega}{2t^4}\right)} -$$

$$-\frac{24\omega(2\omega+3t^4)}{13t^{10}}\frac{\Phi^2\left(\frac{11}{4}, \frac{17}{4}, \frac{3\omega}{2t^4}\right)}{\Phi^2\left(\frac{11}{4}, \frac{13}{4}, \frac{3\omega}{2t^4}\right)}, \qquad (4.3.9)$$

$$p(t) = \frac{2(2\omega+3t^4)(6\omega+5t^4)}{t^{10}} + \frac{48\omega^2}{13^2t^{10}}\frac{\Phi\left(\frac{11}{4}, \frac{17}{4}, \frac{3\omega}{2t^4}\right)}{\Phi\left(\frac{11}{4}, \frac{13}{4}, \frac{3\omega}{2t^4}\right)}.$$

At early times one must distinguish between positive and negative ω. For $\omega > 0$, the
universe is contracting and the fluid behaves like an effective cosmological constant:

$$a(t) \underset{t\to 0}{\propto} const \times \frac{\exp\left(\frac{\omega}{2t^4}\right)}{t^{4/3}}, \quad H(t) \underset{t\to 0}{\propto} -\frac{2\omega}{t^5},$$

$$\epsilon \underset{t\to 0}{\propto} -\frac{5t^4}{2\omega}, w \underset{t\to 0}{\propto} -1, \tag{4.3.10}$$

$$\rho(t) \underset{t\to 0}{\propto} \frac{12\omega^2}{t^{10}}, \; p(t) \underset{t\to 0}{\propto} -\frac{12\omega^2}{t^{10}}.$$

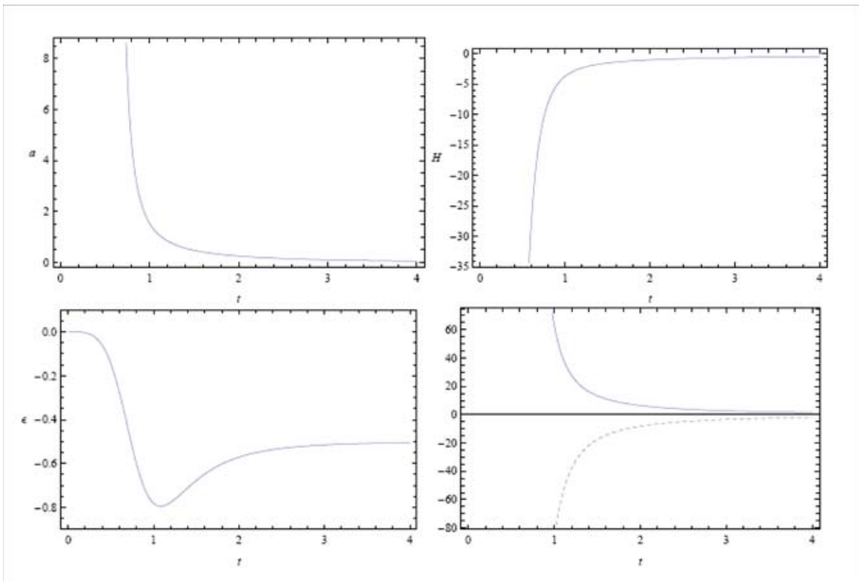

Figure1. The scale factor $a(t)$,Hubble parameter H,slow-rollparameter ϵ, energy density and pressure $p(t)$ (dashed line)for $\omega = +1$. [18]

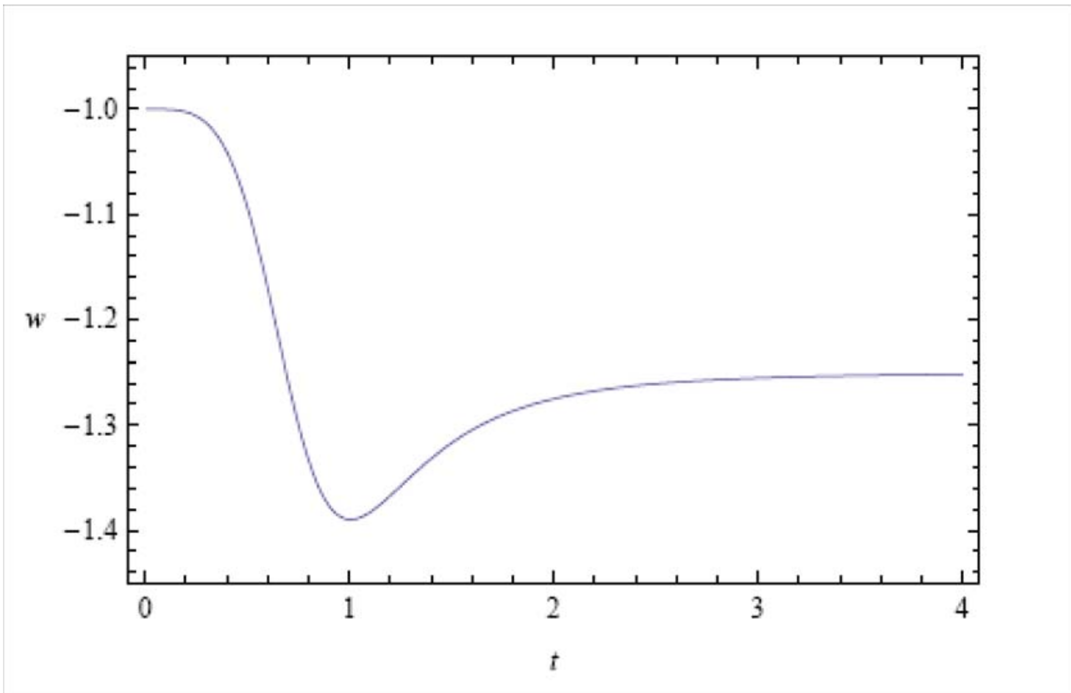

Figure 2.The equation of state $w = \frac{p}{\rho}$ for $\omega = +1.$ [18]

For $\omega < 0,$ at early times the universe expands and accelerates, even if the perfect fluid is stiff:

$$a(t) \underset{t \to 0}{\propto} const \times t^{5/3}, H(t) \underset{t \to 0}{\propto} \frac{5}{3t},$$

$$\epsilon \underset{t \to 0}{\propto} \frac{3}{5}, \rho(t) \underset{t \to 0}{\propto} \frac{2|\omega|}{t^6}, \qquad (4.3.11)$$

$$p(t) \underset{t \to 0}{\propto} \frac{2|\omega|}{t^6}, w \underset{t \to 0}{\propto} 1.$$

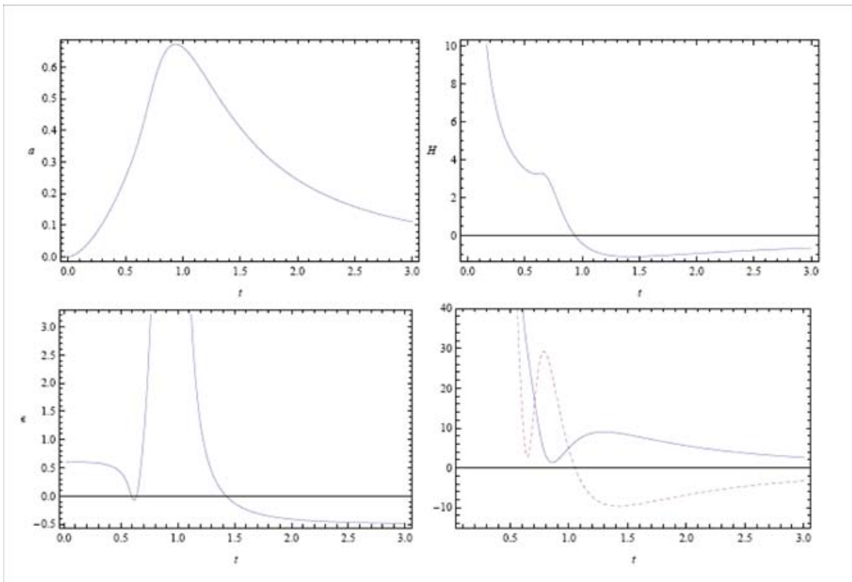

Figure 3. The scale factor $a(t)$,Hubble parameter H,slow-roll parameter ϵ, energy density and pressure $p(t)$ (dashed line)for $\omega = -1$. [18]

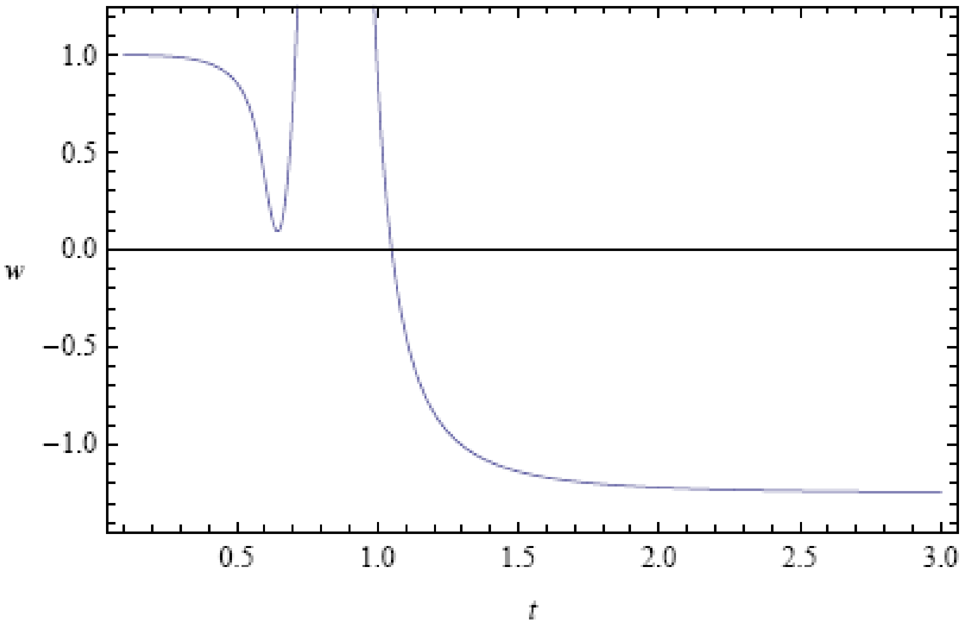

Figure 4.The equation of state $w = \frac{p}{\rho}$ for $\omega = -1$. [18]

The most natural possibility is that a classical FRW background,either exact or linearly perturbed, is not realistic. Then, one would have to treat the UV limit as highly inhomogeneous.This is not at all unexpected,as we are dealing with quantum scales where the minisuperspace equations (maximal symmetry) are likely to fail.

V.Calculation of the primordial gravitational wave Spectrum in fractal cosmology.

V.1.The power spectrum, $\Delta_h^2(k,\alpha)$, and relative spectral energy density, $\Omega_h(k,\alpha)$, of the fractional gravitational wave background.

In this section we define the power spectrum, $\Delta_h^2(k,\alpha)$, and relative spectral energy density, $\Omega_h(k,\alpha)$, of the fractional gravitational wave background.Units are chosen as $c = \hbar = k_B = 1$ and $\sqrt{8\pi G}$ is retained.Indices $\lambda,\mu,\nu,\ldots$ run from 0 to 3, and $i,j,k,\ldots$ run from 1 to 3. Over-dots are used for derivatives with respect to coordinate time t throughout the paper. In this section for instance we will be used the perturbed α-FRW metric, $ds^2 = -dt^2 + R^2(t,\alpha)[dr^2 + f(r)(d\theta^2 + \sin^2\theta d\phi^2)]$. We define the confor- mal time τ by $\tau(t) = \int_{t_0}^{t} \frac{dt}{R(t)}$. When we do perturbation theory it will be useful to write the metric ds^2 with τ instead of t. Since $dt = a(\tau)d\tau$ (where we denote by $a(\tau,\alpha)$ the scale factor $R(t,\alpha)$ as a function of the conformal time τ) we have
$ds^2 = a^2(\tau)[-d\tau^2 + dr^2 + f(r)(d\theta^2 + \sin^2\theta d\phi^2)]$.
For tensor perturbations on an isotropic, uniform and flat background spacetime, the metric is given by

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j],$$

$$g_{\mu\nu} = a^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}),$$

$$\eta_{\mu\nu} = diag(-1,1,1,1), \qquad (5.1.1\text{-}5.1.3)$$

$$h_{00} = h_{0i} = 0, |h_{ij}| \ll 1.$$

We assume that $h_{ij,j} = 0, h^i_{\ i} = 0.$ We shall denote the two independent polarization states of the perturbation as $\lambda = +,\times$ and sometimes suppress.We decompose h_{ij} into plane waves with the comoving wave number, $|\mathbf{k}| \equiv k,$ as

$$h_{ij}(\tau,x) = \sum\nolimits_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3} h_\lambda(\tau;\mathbf{k})\epsilon^\lambda_{ij} \exp[i\mathbf{k}\cdot\mathbf{x}], \qquad (5.1.4)$$

where ϵ^λ_{ij} is the polarization tensor.The equation for the wave amplitude, $h_\lambda(\tau;\mathbf{k}) \triangleq$ $\triangleq h_{\lambda,\mathbf{k}}$ in the linear order is

$$-\frac{1}{2} h_{ij;\nu}^{\ \ ;\nu} = 8\pi\widehat{G}\Pi_{ij},$$

$$\widehat{G} = G + \Delta G \qquad (5.1.5)$$

where ΔG given by Eq.(3.1.6) and Π_{ij} is the anisotropic part of the stress tensor, defined by writing the spatial part of the perturbed energy-momentum tensor as

$$T_{ij} = pg_{ij} + a^2\Pi_{ij} \tag{5.1.6}$$

where p is pressure. For a perfect fluid $\Pi_{ij} = 0.$ In the cosmological context, the amplitude of gravitational waves is affected by anisotropic stress when neutrinos are freely streaming (less than ~ 10^{10}**K**) As we only deal with tensor perturbations,h_{ij},

we may treat each component as a scalar quantity under general coordinate transformation, which means e.g. $h_{ij;\mu} = h_{ij,\mu}.$ The left-hand side of Eq. (5.1.5) becomes

$$\begin{aligned} h_{ij;\nu}^{\;;\nu} &= \bar{g}^{\mu\nu}(h_{ij,\mu\nu} - \Gamma^{\alpha}_{\mu\nu}h_{ij,\alpha}) = \\ &-\ddot{h}_{ij} + \left(\frac{\nabla^2}{a^2}\right)h_{ij} - 3\left(\frac{\dot{a}}{a}\right)\dot{h}_{ij}, \end{aligned} \tag{5.1.7}$$

where equalities $\Gamma^0_{0\nu} = \Gamma^0_{\mu 0}, \Gamma^0_{ij} = \delta_{ij}\dot{a}a, \bar{g}^{ij} = \delta_{ij}a^{-2}$ have been used. Commas denote partial derivatives, while semicolons denote covariant derivatives in Eqs. (5.1.5) and (5.1.7). Transforming this equation into Fourier space, we obtain

$$\ddot{h}_{\lambda,\mathbf{k}} + 3\left(\frac{\dot{a}}{a}\right)\dot{h}_{\lambda,\mathbf{k}} + \left(\frac{k^2}{a^2}\right)h_{\lambda,\mathbf{k}} = 16\pi\widehat{G}\Pi_{\lambda,\mathbf{k}}. \tag{5.1.8}$$

Using conformal time derivative $' \triangleq \frac{\partial}{\partial\tau}$ one obtain

$$h''_{\lambda,\mathbf{k}} + 2\left(\frac{a'}{a}\right)h'_{\lambda,\mathbf{k}} + k^2h_{\lambda,\mathbf{k}} = 16\pi\widehat{G}a^2\Pi_{\lambda,\mathbf{k}}. \tag{5.1.9}$$

This is just the massless Klein-Gordon equation for a plane wave in an expanding space with a source term. Thus, each polarization state of the wave behaves as a massless, minimally coupled, real scalar field.Let us consider the time evolution of the spectrum. After the fluctuations left the horizon, $k \ll aH,$ equation (5.1.9) would become

$$\frac{h''_{\lambda,\mathbf{k}}(\tau,\alpha)}{h'_{\lambda,\mathbf{k}}(\tau,\alpha)} \simeq -\frac{2a'(\tau,\alpha)}{a(\tau,\alpha)}. \tag{5.1.10}$$

Hence

$$h_{\lambda,\mathbf{k}}(\tau,\alpha) = c_1 + c_2 \int_{\tau_0}^{\tau} \frac{d\tau'}{a^2(\tau',\alpha)}, \tag{5.1.11}$$

where c_1 and c_2 are integration constants.Ignoring the second term that is a decaying mode, one finds that $h_{\lambda,\mathbf{k}}(\tau,\alpha)$ remains constant outside the horizon.Note that we have ignored the effect of anisotropic stress outside the horizon, as this term is usually given by causal mechanism which must vanish outside the horizon. Therefore, one may write a general solution of $h_{\lambda,\mathbf{k}}(\tau,\alpha)$ at any time as

$$h_{\lambda,\mathbf{k}}(\tau,\alpha) = h^{\mathbf{prim}}_{\lambda,\mathbf{k}}(\tau,\alpha)\Im(\tau,k,\alpha), \tag{5.1.12}$$

where $h^{\mathbf{prim}}_{\lambda,\mathbf{k}}(\tau,\alpha)$is the primordial gravitational wave mode in fractal spacetime that left the horizon during inflation. The transfer function,$\Im(\tau,k,\alpha)$, then describes the sub-horizon evolution of gravitational wave modes in fractal spacetime after the modes entered the horizon.The transfer function is normalized such that$\Im(\tau,k,\alpha)$ $\to 1$ as $k \to 0$. The power spectrum of gravitational waves in fractal spacetime, $\Delta_h^2(k,\alpha)$,may be defined as

$$\langle h_{ij}(\tau,\mathbf{x},\alpha)h^{ij}(\tau,\mathbf{x},\alpha)\rangle = \int \frac{dk}{k}\Delta_h^2(\tau,k,\alpha). \tag{5.1.13}$$

From Eq.(5.1.13) one obtain

$$\Delta_h^2(\tau, k, \alpha) = \frac{2k^3}{2\pi^2} \sum_\lambda \left\langle |h_{\lambda,\mathbf{k}}(\tau, \alpha)|^2 \right\rangle. \quad (5.1.14)$$

From Eq.(5.1.14) and Eq.(5.1.12) one obtain the time evolution of the power spectrum via formula

$$\Delta_h^2(\tau, k, \alpha) = \Delta_{h,\mathbf{prim}}^2 |\Im(\tau, k, \alpha)|^2, \quad (5.1.15)$$

where

$$\Delta_{h,\mathbf{prim}}^2(\alpha) = \frac{2k^3}{2\pi^2} \sum_\lambda \left\langle |h_{\lambda,\mathbf{k}}|^2 \right\rangle \sim \left(\frac{H_{\mathbf{inf}}(\alpha)}{M_{Pl}} \right). \quad (5.1.16)$$

We have used the prediction for the amplitude of gravitational waves from de-Sitter inflation in fractal spacetime at the last equality, and $H_{\mathbf{inf}}(\alpha)$ is the perturbed Hubble constant during inflation. One may easily extend this result to slow-roll inflation models in fractal spacetime.The energy density of gravitational waves in fractal spacetime is given by the $0-0$ component of stress-energy tensor of gravitational waves:

$$\rho(\tau, \alpha) = \frac{\langle h_{ij}'(\tau, \mathbf{x}, \alpha) h^{ij\prime}(\tau, \mathbf{x}, \alpha) \rangle}{32 \widehat{G} \pi a^2(\tau, \alpha)} \quad (5.1.17)$$

The relative spectral energy density,$\Omega_h(\tau, k, \alpha)$, is then given by the Fourier transform of energy density,

$$\widetilde{\rho}(\tau,k,\alpha) \triangleq \frac{d\rho_h(\tau,k,\alpha)}{d\ln k} \tag{5.1.18}$$

divided by the critical density of the fractal universe, $\rho_{\mathbf{cr}}(\tau,\alpha)$

$$\Omega_h(\tau,k,\alpha) = \frac{\widetilde{\rho}(\tau,k,\alpha)}{\rho_{\mathbf{cr}}(\tau,\alpha)} = \frac{\Delta^2_{h,\mathbf{prim}}}{a^2(\tau,\alpha)H^2(\tau,\alpha)}|\Im'(\tau,k,\alpha)|^2. \tag{5.1.19}$$

V.2.THE EFFECTIVE RELATIVISTIC DEGREES OF FREEDOM: g$_*$(T).

During the radiation era many kinds of particles interacted with photons frequently so that they were in thermal equilibrium. In an adiabatic system, the entropy $S(T)$ per unit comoving volume in (3+1)-dimensional spacetime must be conserved [32],[34]:

$$S(T) = s(T)a^3(T,\alpha) = const.$$

$$s(T) = \frac{2\pi^2}{45}g_{*s}(T)T^3. \tag{5.2.1.a}$$

In an adiabatic system in fractal spacetime, the entropy $S(T,D_{\mathbf{f}})$ per unit comoving volume $V^{(D_{\mathbf{t}},D_{\mathbf{f}})}$ in ($D_{\mathbf{f}}$+1)-dimensional fractal spacetime must be conserved:

$$S(T,D_{\mathbf{f}}) = s(T,D_{\mathbf{f}})a^{D_{\mathbf{f}}}(T,\alpha) = const.$$

$$s(T,D_{\mathbf{f}}) = c(D_{\mathbf{f}})g_{*s}(T,D_{\mathbf{f}})T^3, \tag{5.2.1.b}$$

see **Appendix I.**
The entropy density, $s(T,D_{\mathbf{f}})$, is given by the energy density and pressure

$$s(T,D_{\mathbf{f}}) = \frac{\rho(T,D_{\mathbf{f}}) + p(T,D_{\mathbf{f}})}{T}. \tag{5.2.2}$$

The energy density and pressure in such a plasma-dominant (3+1)-dimensional universe are given by

$$\rho(T) = \frac{\pi^2}{30}g_*(T)T^4,$$

$$p(T) = \frac{1}{3}\rho(T) \tag{5.2.3.a}$$

respectively, where we have defined the “effective number of relativistic degrees of freedom”, g_* and g_{*s}, following [14].
The energy density $\rho(T,D_{\mathbf{f}})$ and pressure $p(T,D_{\mathbf{f}})$ in such a plasma-dominant fractal universe are given by

$$\rho(T,D_{\mathbf{f}}) = g_*(T,D_{\mathbf{f}})T^{1+D_{\mathbf{f}}},$$

$$p(T,D_{\mathbf{f}}) = \frac{1}{3}\rho(T,D_{\mathbf{f}}), \tag{5.2.3.b}$$

see **Appendix I.**Equation (5.2.2.a) and (5.2.3.a) immediately imply that energy density of the (3+1)-dimensional universe during the radiation era should evolve as

$$\rho \propto g_* g_{*s}^{-4/3} a^{-4}. \quad (5.2.4)$$

Therefore, unless g_* and g_{*s} are independent of time, the evolution of ρ would deviate from $\rho \propto a^{-4}$. In other words,the evolution of ρ during the radiation era is sensitive to how many relativistic species the universe had at a given epoch. As the wave equation of gravitational waves contains $(a'(\tau,D_{\mathbf{f}})/a(\tau,D_{\mathbf{f}}))h'_{\lambda,k},$ the solution of $h_{\lambda,k}$ would be affected by $g_*$and g_{*s} via the perturbed fractional Friedman equation (3.3.1).Although the interaction rate among particles and antiparticles is assumed to be fast enough (compared with the expansion rate) to keep them in thermal equilib-
rium, the interaction is assumed to be weak enough for them to be treated as ideal gases. In the case of the (3+1)-dimensional universe and ideal gas at temperature $T,$ each particle species of a given mass, $m_i = x_i T,$would contribute to g_* and g_{*s} the amount given by formulae

$$g_{*i} = g_i \frac{15}{\pi^4} \int_{x_i}^{\infty} du \frac{\sqrt{(u^2 - x_i^2)}\, u^2}{e^u \pm 1},$$

$$g_{*s,i} = g_i \frac{15}{\pi^4} \int_{x_i}^{\infty} du \frac{\sqrt{(u^2 - x_i^2)}\, (u^2 - x_i^2/4)}{e^u \pm 1}, \quad (5.2.5)$$

where the sign is $(+)$ for bosons and $(-)$ for fermions and g_i is the number of helicity states of the particle and antiparticle.Note that an integral variable is defined as
$u \equiv E/T,$where $E = \sqrt{|\mathbf{p}|^2 + m^2}$ .We assume that the chemical potential,μ_i,is negligible. One might also define a similar quantity for the number density,

$$n(T) = \frac{\zeta(3)}{\pi^2} g_{*n} T^3, \quad (5.2.6)$$

where $\zeta(3) \simeq 1.20206$ is the Riemann zeta function at $3.$ Each species in case of the (3+1)-dimensional universe would contribute to g_{*n} by

$$g_{*n,i} = \frac{1}{2\zeta(3)} \int_{x_i}^{\infty} du \frac{\sqrt{(u^2 - x_i^2)}\, u}{e^u \pm 1} \tag{5.2.7}$$

The effective number relativistic degrees of freedom is then given by thetempera-ture-weighted sum of all particles contributions:

$$\begin{aligned} g_*(T) &= \sum_i g_{*i}(T)\left(\frac{T_i}{T}\right)^3, \\ g_{*s}(T) &= \sum_i g_{*s,i}(T)\left(\frac{T_i}{T}\right)^3, \\ g_{*n}(T) &= \sum_i g_{*n,i}(T)\left(\frac{T_i}{T}\right)^3, \end{aligned} \tag{5.2.8.a}$$

where we have taken into account the possibility that each species i may have a thermal distribution with a different temperature from that of photons.
For the case of the ($D_{\mathbf{f}}$+1)-dimensional fractal universe we obtain

$$\begin{aligned} g_*(T, D_{\mathbf{f}}) &= \sum_i g_{*i}(T)\left(\frac{T_i}{T}\right)^{D_{\mathbf{f}}}, \\ g_{*s}(T) &= \sum_i g_{*s,i}(T)\left(\frac{T_i}{T}\right)^{D_{\mathbf{f}}}, \\ g_{*n}(T) &= \sum_i g_{*n,i}(T)\left(\frac{T_i}{T}\right)^{D_{\mathbf{f}}}, \end{aligned} \tag{5.2.8.b}$$

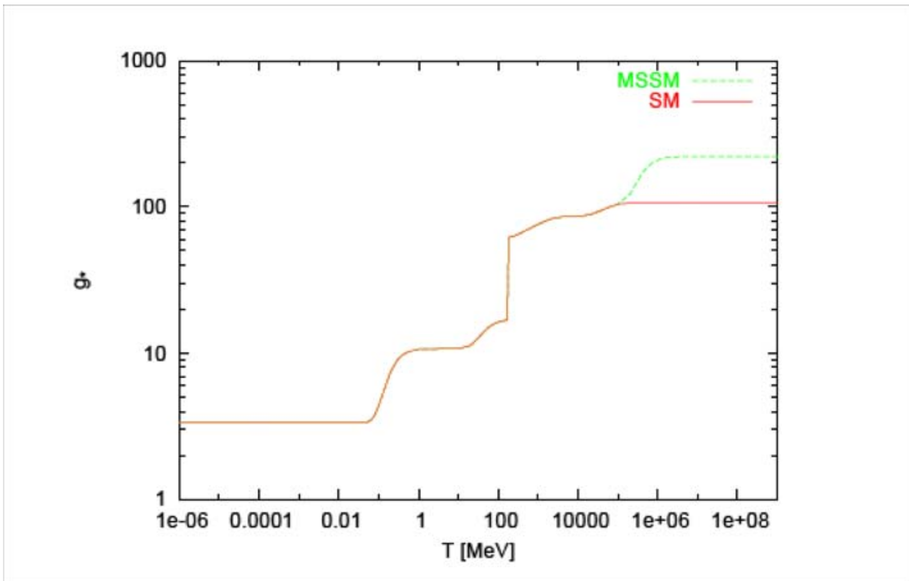

FIG. 5. 2. 1. [34].Evolution of the effective number of relativistic degrees of freedom contributing to energy density,g_*,as a function of temperature T. At the energy scales above ~ 1TeV, $g_*^{SM} = 106.75$ and $g_*^{MSSM} \simeq 220$.

V.3.ANALYTICAL SOLUTIONS OF PERTURBED WAVE EQUATIONS.

In this section we shall discuss solutions of the equation of motion Eq.(5.1.9).While we assume $\Pi_{ij} = 0$ in this section. Imposing appropriate boundary conditions

$$(a)\ h_{\mathbf{k}}(\tau,\alpha) = \frac{\sqrt{16\pi G}}{a(\tau,\alpha)\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)\exp[-ik\tau]\beta(\mathbf{k}) =$$

$$-\frac{\tau}{a(\tau,\alpha)}\sqrt{8\pi Gk}\,h_1^{(2)}(k\tau)\beta(\mathbf{k}) \text{ - Inflation,}$$

$$(b)\ \ h_{\mathbf{k}}(\tau,\alpha) = [j_0(k\tau)]h_{\mathbf{k}}^{\text{prim}}(\alpha) \text{ - RD,}$$

$$(c)\ h_{\mathbf{k}}(\tau,\alpha) = \left[\frac{3j_1(k\tau)}{k\tau}\right]h_{\mathbf{k}}^{\text{prim}}(\alpha) \text{ - MD,} \qquad (5.3.1)$$

$$j_0(k\tau) = \frac{\sin(k\tau)}{k\tau},\ j_1(k\tau) = \frac{1}{k\tau}\left[\frac{\sin(k\tau)}{k\tau} - \cos(k\tau)\right],$$

$$h_1^{(2)}(k\tau) = -\frac{1}{k\tau}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau}.$$

where $\beta(\mathbf{k})$ is a generalized stochastic variable satisfying

$$\langle\alpha(\mathbf{k})\alpha^*(\mathbf{k}')\rangle = \delta(\mathbf{k} - \mathbf{k}'). \qquad (5.3.2)$$

We classify wave modes by their horizon crossing time, $\tau_{\mathbf{hc}}$:

$$|\mathbf{k}| = k\begin{cases} > k_{\mathbf{eq}}\text{the modes that entered the horizon during RD:}\tau_{\mathbf{hc}} < \tau_{\mathbf{eq}} \\ < k_{\mathbf{eq}}\text{the modes that entered the horizon during MD:}\tau_{\mathbf{hc}} > \tau_{\mathbf{eq}} \end{cases} \qquad (5.3.3)$$

where $\tau_{\mathbf{eq}}$ denotes the time at the matter-radiation equality, and $\tau_{\mathbf{hc}}$ denotes the time when fluctuation modes crossed the horizon, $k_{\tau\mathbf{h}}c = 1.$ Notice that $|h_k(\tau)|^2$ for each solution (5.3.1.a) - (5.3.1.c) does not depend on time ($\equiv |h_k^{\text{prim}}|^2$) at the

super-horizon scale, $|k\tau| \ll 1$.
The tensor mode fluctuations from the inflationary universe left the horizon and froze out. Its dimensionless spectrum is given from Eq.(5.3.1.a) is

$$
\begin{aligned}
&\Delta_h^2(k,\alpha) = 4k^3 \frac{\left|h_{\mathbf{k}}^{\text{inf}}(\alpha)\right|^2}{2\pi^2} = \\
&\frac{4k^3}{2\pi^2}\frac{16\pi G}{2ka^2(\tau,\alpha)}\left(1+\frac{1}{k^2\tau^2}\right) = \\
&64\pi G\frac{1}{a^2(\tau,\alpha)}\left(\frac{k\tau}{2\pi}\right)^2\left(1+\frac{1}{k^2\tau^2}\right) = \\
&64\pi G\left(\frac{H_{\text{inf}}^{\textbf{eff}}(\alpha)\times k\tau}{2\pi}\right)^2\left(1+\frac{1}{k^2\tau^2}\right) = \\
&\simeq \frac{16}{\pi}\left(\frac{H_{\text{inf}}^{\textbf{eff}}(\alpha)}{M_{Pl}}\right)^2 = 4k^3\frac{\left|h_k^{\textbf{prim}}(\alpha)\right|^2}{2\pi^2}, \\
&|k\tau| \ll 1.
\end{aligned}
\tag{5.3.4}
$$

where $H_{\text{inf}}(\alpha)$ is the effective Hubble parameter during inflation and

$$\tau \simeq -1/a(\tau,\alpha)H_{\text{inf}}^{\textbf{eff}}(\alpha) \tag{5.3.5}$$

is used in the fourth equality of the (5.3.4). Note that the conventional factor 4 is from equality

$$\int \frac{dk}{k}\Delta_h^2(k,\alpha) = \langle h_{ij}h^{ij}\rangle = 2[\langle|h_+|^2\rangle + \langle|h_\times|^2\rangle] \\ = 4h^2, \tag{5.3.6}$$

where $|h_+| = |h_\times| = h.$ The dimensionless spectrum (5.3.4) is nearly independent $k.$ This is the famous prediction well known from the standard inflationary scenario in (3+1)spacetime known as a nearly scale invariant spectrum.
From the fractional Friedman equation (3.5.3) during inflation,one obtains

$$H_{\inf}^{\mathbf{eff}}(\alpha) \simeq \frac{8\pi}{3M_{Pl}^2}V(\varphi), \\ \Delta_{h,\mathbf{prim}}^2(\alpha) \simeq \frac{10}{M_{Pl}^4}V(\varphi). \tag{5.3.7}$$

Using the transfer function $\Im(\tau,k,\alpha)$ [Eq.(5.1.12)], we obtain the time evolution of the amplitude of gravitational waves as

$$(a)\ \Im(\tau < \tau_{\mathbf{eq}},k < k_{\mathbf{eq}}) = j_0(k\tau), \\ (b)\ \Im(\tau > \tau_{\mathbf{eq}},k > k_{\mathbf{eq}}) = \frac{\tau_{\mathbf{eq}}}{\tau}[A(k,\alpha)j_1(k\tau) + B(k,\alpha)y_1(k\tau)], \\ (c)\ \Im(\tau,k < k_{\mathbf{eq}}) = \frac{3j_1(k\tau)}{k\tau}, \\ y_1(k\tau) = -\frac{1}{k\tau}\left[\frac{1}{k\tau}\cos(k\tau) + \sin(k\tau)\right]. \tag{5.3.8}$$

Their conformal time derivatives are given as

$$(a)\ \Im'(\tau < \tau_{\mathbf{eq}}, k < k_{\mathbf{eq}}) = -j_1(k\tau),$$

$$(b)\ \Im'(\tau > \tau_{\mathbf{eq}}, k > k_{\mathbf{eq}}) = -\frac{\tau_{\mathbf{eq}}}{\tau}[A(k,\alpha)j_2(k\tau) + B(k,\alpha)y_2(k\tau)],$$

$$(c)\ \Im'(\tau, k < k_{\mathbf{eq}}) = -\frac{3j_2(k\tau)}{k\tau}, \qquad (5.3.9)$$

$$j_2(k\tau) = \frac{1}{k\tau}\left[\left(\frac{3}{k^2\tau^2} - 1\right)\sin(k\tau) - \frac{3}{k\tau}\cos(k\tau)\right],$$

$$y_2(k\tau) = -\frac{1}{k\tau}\left[\left(\frac{3}{k^2\tau^2} - 1\right)\cos(k\tau) + \frac{3}{k\tau}\sin(k\tau)\right].$$

Eqs. (5.3.9.a) and (5.3.9.b) are the evolution of modes which entered the horizon during the radiation era, while Eq. (B8) is the evolution of modes which entered the horizon during the matter era. Coefficients A(k) and B(k) are obtained by equating a solution (B6) with (B7) and their first derivatives [(B11) and (B12)] at the corespon-
ding matter-radiation equality.

Appendix I.Fractal equilibrium thermodynamics.

We assume the second law of thermodynamics in fractal spacetime with integer fractal dimensions $D_{\mathbf{f}} \leq D_{\mathbf{t}}$ in the form:

$$TdS = d\left(\rho V^{(D_{\mathbf{t}},D_{\mathbf{f}})}\right) + pdV^{(D_{\mathbf{t}},D_{\mathbf{f}})} = d\left[(\rho + p)V^{(D_{\mathbf{t}},D_{\mathbf{f}})}\right] - V^{(D_{\mathbf{t}},D_{\mathbf{f}})}dp \qquad (1.1)$$

where ρ and p are the equilibrium energy density density and pressure. Moreover, the integrability condition,

$$\frac{\partial^2 S}{\partial T \partial V^{(D_t,D_f)}} = \frac{\partial^2 S}{\partial V^{(D_t,D_f)} \partial T} \tag{1.2}$$

gives

$$T\frac{dp(T)}{dT} = \rho(T) + p(T) \tag{1.3}$$

or

$$dp(T) = \frac{[\rho(T) + p(T)]}{T} dT. \tag{1.4}$$

Substitution Eq.(1.4) into Eq.(1.1) gives

$$\begin{gathered} dS = \frac{1}{T} d\left[(\rho(T) + p(T)) V^{(D_t,D_f)} \right] - (\rho(T) + p(T)) V^{(D_t,D_f)} \frac{dT}{T^2} = \\ d\left[\frac{(\rho(T) + p(T)) V^{(D_t,D_f)}}{T} + const \right]. \end{gathered} \tag{1.5}$$

From Eq.(1.5) one obtain

$$S_{D_f} = \frac{(\rho(T) + p(T)) V^{(D_t,D_f)}}{T} + const. \tag{1.6}$$

The law of energy conservation is

$$d\left[(\rho(T)+p(T))V^{(D_t,D_f)}\right] = V^{(D_t,D_f)}dp(T). \quad (1.7)$$

Substituting Eq.(1.4) into Eq.(1.7), it follows that

$$d\left[\frac{(\rho(T)+p(T))V^{(D_t,D_f)}}{T}\right] = 0. \quad (1.8)$$

Hence in thermal equilibrium, the entropy S per comoving fractal volume $V^{(D_t,D_f)}$ is conserved.
Let us define the fractal entropy density s^{D_f} :

$$s^{D_f} \triangleq \frac{S}{V^{(D_t,D_f)}} = \frac{\rho(T)+p(T)}{T}. \quad (1.9)$$

Remind that in $(3+1)$ spacetime the number density $n(T,\mu)$, energy density $\rho(T,\mu)$ and pressure $p(T,\mu)$ of a dilute, weakly-interacting gas of particles with g internal degrees of freedom is given in terms of its phase space distribution function $f(\vec{\mathbf{p}})$ is [30]:

$$n(T,\mu) = \frac{g}{(2\pi)^3}\int f(\vec{\mathbf{p}})\,d^3p,$$

$$\rho(T,\mu) = \frac{g}{(2\pi)^3}\int E(\vec{\mathbf{p}},m)f(\vec{\mathbf{p}})\,d^3p,$$

$$p(T,\mu) = \frac{g}{(2\pi)^3}\int \frac{|\vec{\mathbf{p}}|^2}{3E(\vec{\mathbf{p}},m)}f(\vec{\mathbf{p}})\,d^3p, \quad (1.10)$$

$$E^2(\vec{\mathbf{p}},m) = |\vec{\mathbf{p}}|^2 + m^2.$$

For a species in kinetic equilibrium phase space distribution function $f(\vec{\mathbf{p}})$ is

$$f(\vec{\mathbf{p}}) = \left[\exp\left(\frac{E(\vec{\mathbf{p}},m) - \mu}{T} \right) \pm 1 \right]^{-1} \tag{1.11}$$

where μ is the chemical potential of the species, and here and throughout $+1$ pertains to Fermi-Dirac species and -1 to Bose-Einstain species.
In $((D_{\mathbf{t}},D_{\mathbf{f}})+1)$ spacetime with $D_{\mathbf{t}} = 3$, the number density $n^{D_{\mathbf{f}}} \triangleq n(T,\mu,D_{\mathbf{f}})$, energy density $\rho^{D_{\mathbf{f}}} \triangleq \rho(T,\mu,D_{\mathbf{f}})$ and pressure $p^{D_{\mathbf{f}}} \triangleq p(T,\mu,D_{\mathbf{f}})$ of a dilute, weakly-interacting gas of particles with g internal degrees of freedom is given in terms of its phase space distribution function $f(\vec{\mathbf{p}})$ is

$$n(T,\mu,D_{\mathbf{f}}) = \frac{g}{(2\pi)^3}\int f(\vec{\mathbf{p}})\, d^{(3,D_{\mathbf{f}})}p = \frac{g}{(2\pi)^3}\int f(\vec{\mathbf{p}})\, |\vec{\mathbf{p}}|^{-\check{D}_{\mathbf{f}}} d^3p,$$

$$\rho(T,\mu,D_{\mathbf{f}}) = \frac{g}{(2\pi)^3}\int E(\vec{\mathbf{p}},m) f(\vec{\mathbf{p}})\, d^{(3,D_{\mathbf{f}})}p =$$

$$\frac{g}{(2\pi)^3}\int E(\vec{\mathbf{p}},m) f(\vec{\mathbf{p}})\, |\vec{\mathbf{p}}|^{-\check{D}_{\mathbf{f}}} d^3p,$$

$$p(T,\mu,D_{\mathbf{f}}) = \frac{g}{(2\pi)^3}\int \frac{|\vec{\mathbf{p}}|^2}{3E(\vec{\mathbf{p}},m)} f(\vec{\mathbf{p}})\, d^{(3,D_{\mathbf{f}})}p = \tag{1.12}$$

$$\frac{g}{(2\pi)^3}\int \frac{|\vec{\mathbf{p}}|^{2-\check{D}_{\mathbf{f}}}}{3E(\vec{\mathbf{p}},m)} f(\vec{\mathbf{p}})\, d^3p,$$

$$E^2(\vec{\mathbf{p}},m) = |\vec{\mathbf{p}}|^2 + m^2,$$

where $\check{D}_{\mathbf{f}} = D_{\mathbf{t}} - D_{\mathbf{f}}, D_{\mathbf{t}} = 3$.

From Eq.(1.11)-Eq.(1.12) by using polar coordinate with $\varpi = |\vec{\mathbf{p}}|$, one obtain

$$
\begin{aligned}
n(T,\mu,D_{\mathbf{f}}) &= \frac{g}{\pi^2}\int_0^\infty \frac{\varpi^{2-\check{D}_{\mathbf{f}}}}{\exp[(E(\varpi,m)-\mu)/T]\pm 1}d\varpi,\\
\rho(T,\mu,D_{\mathbf{f}}) &= \frac{g}{\pi^2}\int_0^\infty \frac{E(\varpi,m)\varpi^{2-\check{D}_{\mathbf{f}}}}{\exp[(E(\varpi,m)-\mu)/T]\pm 1}d\varpi,\\
p(T,\mu,D_{\mathbf{f}}) &= \frac{g}{3\pi^2}\int_0^\infty \frac{\varpi^{4-\check{D}_{\mathbf{f}}}}{E(\varpi,m)[\exp[(E(\varpi,m)-\mu)/T]\pm 1]}d\varpi,\\
E(\varpi,m) &= \sqrt{\varpi^2+m^2}\,,\check{D}_{\mathbf{f}} = D_{\mathbf{t}}-D_{\mathbf{f}},D_{\mathbf{t}} = 3.
\end{aligned}
\tag{1.13}
$$

In the relativistic limit ($T \gg m$), for $T \gg \mu$, from Eqs.(1.13) we obtain

$$
\begin{aligned}
n(T,\mu,D_{\mathbf{f}}) &\propto \frac{g}{\pi^2}\int_0^\infty \frac{\varpi^{2-\check{D}_{\mathbf{f}}}}{\exp[\varpi/T]\pm 1}d\varpi = gT^{3-\check{D}_{\mathbf{f}}}\left(\frac{1}{\pi^2}\int_0^\infty \frac{u^{2-\check{D}_{\mathbf{f}}}}{\exp[u]\pm 1}du\right),\\
\rho(T,\mu,D_{\mathbf{f}}) &\propto \frac{g}{\pi^2}\int_0^\infty \frac{\varpi^{3-\check{D}_{\mathbf{f}}}}{\exp[\varpi/T]\pm 1}d\varpi = gT^{4-\check{D}_{\mathbf{f}}}\left(\frac{1}{\pi^2}\int_0^\infty \frac{u^{3-\check{D}_{\mathbf{f}}}}{\exp[u]\pm 1}du\right),\\
p(T,\mu,D_{\mathbf{f}}) &\propto \frac{g}{3\pi^2}\int_0^\infty \frac{\varpi^{3-\check{D}_{\mathbf{f}}}}{\exp[\varpi/T]\pm 1}d\varpi = \frac{1}{3}gT^{4-\check{D}_{\mathbf{f}}}\left(\frac{1}{\pi^2}\int_0^\infty \frac{u^{3-\check{D}_{\mathbf{f}}}}{\exp[u]\pm 1}du\right),\\
\check{D}_{\mathbf{f}} &= 3-D_{\mathbf{f}}.
\end{aligned}
\tag{1.14}
$$

From Eqs.(1.14) finally we obtain

$$\rho(T,\mu,D_{\mathbf{f}}) \propto \rho_B(D_{\mathbf{f}})gT^{4-\check{D}_{\mathbf{f}}} = c_B g T^{1+D_{\mathbf{f}}} \text{ (BOSE)}$$

$$\rho(T,\mu,D_{\mathbf{f}}) \propto \rho_F(D_{\mathbf{f}})gT^{4-\check{D}_{\mathbf{f}}} = c_F g T^{1+D_{\mathbf{f}}} \text{ (FERMI)}$$

$$n(T,\mu,D_{\mathbf{f}}) \propto n_B(D_{\mathbf{f}})gT^{3-\check{D}_{\mathbf{f}}} = n_B g T^{D_{\mathbf{f}}} \quad \text{(BOSE)} \tag{1.15}$$

$$n(T,\mu,D_{\mathbf{f}}) \propto n_F(D_{\mathbf{f}})gT^{3-\check{D}_{\mathbf{f}}} = n_F g T^{D_{\mathbf{f}}} \quad \text{(FERMI)}$$

$$p(T,\mu,D_{\mathbf{f}}) = \frac{1}{3}\rho(T,\mu,D_{\mathbf{f}}).$$

In $((D_{\mathbf{t}},D_{\mathbf{f}})+1)$ spacetime with $D_{\mathbf{t}} = 3$, the total energy density $\rho_{\mathbf{tot}}^{D_{\mathbf{f}}} \triangleq \rho_{\mathbf{tot}}(T,D_{\mathbf{f}})$ and pressure $p_{\mathbf{tot}}^{D_{\mathbf{f}}} \triangleq p_{\mathbf{tot}}(T,D_{\mathbf{f}})$ of all species in equilibrium can be expressed in terms of the photon temperature T :

$$\rho_{\mathbf{tot}}(T,D_{\mathbf{f}}) = T^{4-\check{D}_{\mathbf{f}}} \sum_{i=1}^{N} \left(\frac{T_i}{T}\right)^{4-\check{D}_{\mathbf{f}}} \times$$

$$\frac{g}{\pi^2}\int_0^\infty \frac{E(\varpi,m_i/T)\varpi^{2-\check{D}_{\mathbf{f}}}}{\exp[(E(\varpi,m_i/T)-\mu_i/T)]\pm 1}d\varpi, \tag{1.16}$$

$$p_{\mathbf{tot}}(T,D_{\mathbf{f}}) = T^{4-\check{D}_{\mathbf{f}}} \sum_{i=1}^{N} \left(\frac{T_i}{T}\right)^{4-\check{D}_{\mathbf{f}}} \times$$

$$\frac{g}{3\pi^2}\int_0^\infty \frac{\varpi^{4-\check{D}_{\mathbf{f}}}}{E(\varpi,m_i)[\exp[(E(\varpi,m_i/T)-\mu_i/T)]\pm 1]}d\varpi,$$

where N is a total number of all species, and we have taken into account the possibility that the species i may have a thermal distribution,but with a different temperature than that of the photons.

Since the energy density and pressure of a non-relativistic species i.e.,one with mass $m_i \gg T$, is exponentially smaller than that a relativistic species i.e.,one with mass $m_i \ll T$, it is a very good approximation to include only the relativistic species in the sums for $\rho_{\mathbf{tot}}(T,D_{\mathbf{f}})$ and $p_{\mathbf{tot}}(T,D_{\mathbf{f}})$. In this case the expressions given by

Eq.(1.16) greatly simplify:

$$\rho_{\mathbf{tot}}(T,D_{\mathbf{f}}) = c_\rho(D_{\mathbf{f}})g_* g_*(T,D_{\mathbf{f}})T^{4-\check{D}_{\mathbf{f}}} = c_\rho(D_{\mathbf{f}})g_*(T,D_{\mathbf{f}})T^{1+D_{\mathbf{f}}},$$
$$p_{\mathbf{tot}}(T,D_{\mathbf{f}}) = \frac{1}{3}c_p(D_{\mathbf{f}})g_* g_*(T,D_{\mathbf{f}})T^{4-\check{D}_{\mathbf{f}}}\frac{1}{3}c_p(D_{\mathbf{f}})g_*(T,D_{\mathbf{f}})T^{1+D_{\mathbf{f}}}, \qquad (1.17)$$

where $g_*(T)$ counts the total number of effectively massless degrees of freedom (those species with mass $m_i \ll T$), and

$$g_*(T) = \sum_{i=1}^{N_B} g_i\left(\frac{T_i}{T}\right)^{4-\check{D}_{\mathbf{f}}} + \frac{7}{8}\sum_{i=1}^{N_F} g_i\left(\frac{T_i}{T}\right)^{4-\check{D}_{\mathbf{f}}} =$$
$$\sum_{i=1}^{N_B} g_i\left(\frac{T_i}{T}\right)^{1+D_{\mathbf{f}}} + \frac{7}{8}\sum_{i=1}^{N_F} g_i\left(\frac{T_i}{T}\right)^{1+D_{\mathbf{f}}}, \qquad (1.18)$$

where N_B is the total number of bosons and N_F is the total number of fermions. Hence the entropy density $s(T,D_{\mathbf{f}})$ is dominated by the contribution of the relativistic particles and a very good approximation is

$$s(T,D_{\mathbf{f}}) = c(D_{\mathbf{f}})g_{*S}(T,D_{\mathbf{f}})T^{D_{\mathbf{f}}}, \qquad (1.19)$$

where

$$g_{*S}(T,D_{\mathbf{f}}) = \sum_{i=1}^{N_B} g_i\left(\frac{T_i}{T}\right)^{1+D_{\mathbf{f}}} + \frac{7}{8}\sum_{i=1}^{N_F} g_i\left(\frac{T_i}{T}\right)^{1+D_{\mathbf{f}}}. \qquad (1.20)$$

Conservation of $S_{D_{\mathbf{f}}}$ implies that $s(T,D_{\mathbf{f}}) \propto R^{-D_{\mathbf{f}}}(t)$, and therefore the quantity

$$g_{*S}(T,D_{\mathbf{f}})T^{D_{\mathbf{f}}}R^{D_{\mathbf{f}}} \qquad (1.21)$$

remains constant as the Universe expands.

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