On the Representation of Integer Numbers by the sum of Cube roots

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ABSTRACT. We developed some formulas to represent integer numbers as the sum of cube roots.

Lemma 1. *If* $c = \frac{a}{b}$ *and* $b \neq 0$, *then*

$$ab^2 + ab^2c = b^3c + a^2b.$$

Proof. Suppose that

$$y = 1 + zy,$$

then, we can do

$$z = \frac{t-1}{t}$$

and

$$y = t.$$

Let $t = \frac{a}{b}$ in numerator and $t = c$ in denominator, such that $c = \frac{a}{b}$. Therefore,

$$\frac{a}{b} = 1 + \frac{\frac{a}{b} - 1}{c}\left(\frac{a}{b}\right),$$

$$\frac{a}{b} = 1 + \frac{(a-b)a}{b^2c},$$

$$\frac{a}{b} = \frac{b^2c + (a-b)a}{b^2c},$$

$$ab^2c = b^3c + a^2b - ab^2,$$

$$ab^2 + ab^2c = b^3c + a^2b. \square$$

Corollary 1. *If* $b = \frac{a}{c}$ *or* $b = a$ *or* $b = 0$ *and* $c \neq 0$, *then* b *satisfies the equation*

$$cx^3 - a(c+1)x^2 + a^2x = 0.$$

Proof. We substitute $x = b$ in Lemma1 and use a bit of algebraic manipulation. $\square$

Lemma 2. *If* $c = \frac{a}{b}$ *and* $b \neq 0$, *then*

$$a^2b + ab^2c^2 = a^3 + b^3c^2.$$

Proof. Suppose that

$$y = 1 + zy^2,$$

then, we can do

$$z = \frac{t-1}{t^2}$$

and

$$y = t.$$

Let $t = \frac{a}{b}$ and $t^2 = c^2$, such that $c = \frac{a}{b}$. Therefore,

$$\frac{a}{b} = 1 + \frac{\frac{a}{b} - 1}{c^2}\left(\frac{a}{b}\right)^2,$$

$$\frac{a}{b} = 1 + \frac{(a-b)a^2}{b^3c^2},$$

$$\frac{a}{b} = \frac{b^3c^2 + (a-b)a^2}{b^3c^2},$$

$$a = \frac{b^3c^2 + (a-b)a^2}{b^2c^2},$$

$$ab^2c^2 = b^3c^2 + a^3 - a^2b,$$

$$a^2b + ab^2c^2 = a^3 + b^3c^2. \square$$

Corollary 2. *If* $b = \frac{a}{c}$ *or* $b = -\frac{a}{c}$ *or* $b = a$ *and* $c \neq 0$, *then* b *satisfies the equation*

$$c^2x^3 - ac^2x^2 - a^2x + a^3 = 0.$$

Proof. Substitute $b = x$ in Lemma 2 and use a bit of algebraic manipulation. □

Theorem 1. *For* $c \in \mathbb{Z}_{>1}$, *then any integer* a *has the following representation of the two cube roots*

$$a = \sqrt[3]{\frac{1}{2}\left(-\frac{a^3(c+1)}{c^2} + \sqrt{\frac{a^6(c+1)^2}{c^4} - \frac{4a^6(c^2+c+1)^3}{27c^6}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{a^3(c+1)}{c^2} - \sqrt{\frac{a^6(c+1)^2}{c^4} - \frac{4a^6(c^2+c+1)^3}{27c^6}}\right)}.$$

Proof. By Cardano's formula, a root of $x^3 + px + q = 0$ is given by

(1) $$x_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} + \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}.$$

Suppose that $x = a$, and

(2) $$x^3 = -px - q.$$

By Corollary 1, we obtain

$$c(-px - q) - a(c+1)x^2 + a^2x = 0,$$

$$-a(c+1)x^2 + (a^2 - cp)x - cq = 0,$$

$$a(c+1)x^2 - (a^2 - cp)x + cq = 0.$$

By Bhaskara's formula, we find

(3) $$x = \frac{(a^2 - cp) \pm \sqrt{(a^2 - cp)^2 - 4acq(c+1)}}{2a(c+1)}.$$

On the other hand, by Corollary 2 and (2), we obtain

$$c^2(-px - q) - ac^2x^2 - a^2x + a^3 = 0,$$

$$-\,ac^2x^2 - (a^2 + c^2p)x + a^3 - c^2q = 0.$$

By Bhaskara's formula, we put

(4) $$x = -\frac{a^2 + c^2p \pm \sqrt{(a^2 + c^2p)^2 + 4ac^2(a^3 - c^2q)}}{2ac^2}.$$

We compare (3) with (4), and find

$$-\frac{a^2 + c^2p}{2ac^2} = \frac{a^2 - cp}{2a(c+1)}$$

and

$$\frac{(a^2 + c^2p)^2 + 4ac^2(a^3 - c^2q)}{4a^2c^4} = \frac{(a^2 - cp)^2 - 4acq(c+1)}{4a^2(c+1)^2}.$$

Solving the system of equations above, we get

(5) $$p = -\frac{a^2(c^2 + c + 1)}{c^2}$$

and

(6) $$q = \frac{a^3(c+1)}{c^2}.$$

Therefrom, we substitute (5) and (6) in (1) and let $x_1 = a$, so we complete the proof. □

1. Examples

For $a = 3$ and $c = 2$, then

$$3 = \sqrt[3]{\frac{1}{2}\left(-\frac{81}{4} + \frac{15i\sqrt{3}}{2}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{81}{4} - \frac{15i\sqrt{3}}{2}\right)};$$

for $a = 3$ and $c = 3$, then

$$3 = \sqrt[3]{\frac{1}{2}\left(-12 + \frac{70i}{3\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-12 - \frac{70i}{3\sqrt{3}}\right)};$$

for $a = 4$ and $c = 2$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(-48 + \frac{160i}{3\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-48 - \frac{160i}{3\sqrt{3}}\right)};$$

for $a = 4$ and $c = 3$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(-\frac{256}{9} + \frac{4480i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{256}{9} - \frac{4480i}{81\sqrt{3}}\right)};$$

for $a = 5$ and $c = 2$, then

$$5 = \sqrt[3]{\frac{1}{2}\left(-\frac{375}{4} + \frac{625i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{375}{4} - \frac{625i}{6\sqrt{3}}\right)};$$

for $a = 5$ and $c = 3$, then

$$5 = \sqrt[3]{\frac{1}{2}\left(-\frac{500}{9} + \frac{8750i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{500}{9} - \frac{8750i}{81\sqrt{3}}\right)}.$$

Corollary 3. *For $c \in \mathbb{Z}_{>1}$, then any rational number $\frac{a}{b}$ and $b \neq 0$, has the following representation by a sum of the two cube roots*

$$\frac{a}{b} = \sqrt[3]{-\frac{1}{2}\left[\sqrt{\frac{a^6(c+1)^2}{b^6c^4} - \frac{4a^6(c^2+c+1)^3}{27b^6c^6}} + \frac{a^3(c+1)}{b^3c^2}\right]} + \sqrt[3]{\frac{1}{2}\left[\sqrt{\frac{a^6(c+1)^2}{b^6c^4} - \frac{4a^6(c^2+c+1)^3}{27b^6c^6}} - \frac{a^3(c+1)}{b^3c^2}\right]}.$$

Proof. Let $a \to \frac{a}{b}$ in Theorem 1, this completes the proof. □

2. Examples

For $a = 2, b = 3$ and $c = 2$, then

$$\frac{2}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{2}{9} + \frac{20i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{2}{9} - \frac{20i}{81\sqrt{3}}\right)};$$

for $a = 2, b = 3$ and $c = 3$, then

$$\frac{2}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{32}{243} + \frac{560i}{2187\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{32}{243} - \frac{560i}{2187\sqrt{3}}\right)};$$

for $a = 1, b = 3$ and $c = 2$, then

$$\frac{1}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{1}{36} + \frac{5i}{162\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{1}{36} - \frac{5i}{162\sqrt{3}}\right)};$$

for $a = 1, b = 3$ and $c = 3$, then

$$\frac{1}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{4}{243} + \frac{70i}{2187\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{243} - \frac{70i}{2187\sqrt{3}}\right)}.$$

Theorem 2. *For $c \in \mathbb{Z}_{\geq 1}$, then any integer a has the following representation by the sum of the two cube roots*

$$a = \sqrt[3]{\frac{1}{2}\left(\frac{a^3}{c} + \sqrt{\frac{a^6}{c^2} - \frac{4a^6(c-1)^3}{27c^3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{a^3}{c} - \sqrt{\frac{a^6}{c^2} - \frac{4a^6(c-1)^3}{27c^3}}\right)}.$$

Proof. By Cardano's formula, a root of $x^3 + px + q = 0$ is given by

(7) $$x_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} + \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}.$$

Suppose that $x = \frac{a}{c}$, thus,

(8) $$x^3 = -px - q.$$

By Corollary 2, we obtain

$$-ac^2x^2 - (a^2 + c^2p)x + a^3 - c^2q = 0,$$

by Baskara's formula

(9) $$x = -\frac{a^2 + c^2 p \pm \sqrt{(a^2 + c^2 p)^2 + 4ac^2(a^3 - c^2 q)}}{2ac^2}.$$

By Corollary 3, we encounter

$$c^3 x(-px - q) - ac^3(-px - q) - a^3 x + a^4 = 0,$$

$$-c^3 p x^2 + (ac^3 p - c^3 q - a^3)x + a^4 + ac^3 q = 0.$$

Again, by Baskara's formula

(10) $$x = -\frac{a^3 + c^3 q - ac^3 p \pm \sqrt{(ac^3 p - c^3 q - a^3)^2 + 4c^3 p(a^4 + ac^3 q)}}{2c^3 p}.$$

Compare (7) with (8), and we find

$$\frac{a^2 + c^2 p}{2ac^2} = \frac{a^3 + c^3 q - ac^3 p}{2c^3 p}$$

and

$$\frac{\sqrt{(a^2 + c^2 p)^2 + 4ac^2(a^3 - c^2 q)}}{2ac^2} = \frac{\sqrt{(ac^3 p - c^3 q - a^3)^2 + 4c^3 p(a^4 + ac^3 q)}}{2c^3 p} \therefore$$

$$\frac{(a^2 + c^2 p)^2 + 4ac^2(a^3 - c^2 q)}{4a^2 c^4} = \frac{(ac^3 p - c^3 q - a^3)^2 + 4c^3 p(a^4 + ac^3 q)}{4c^6 p^2}.$$

Solving the system of equations above, we get

(10) $$p = -\frac{a^2(c - 1)}{c}$$

and

(11) $$q = -\frac{a^3}{c}$$

or

(12) $$p = -\frac{a^2(c^2 + c + 1)}{c^2}$$

and

(13) $$q = \frac{a^3(c + 1)}{c^2}.$$

The solutions (12) and (13) are equals to solutions (5) and (6); thereof, we replace (10) and (11) in (7), and let $x_1 = a$, this completes the proof. □

3. Examples

For $a = 3$ and $c = 2$, then

$$3=\sqrt[3]{\frac{1}{2}\left(\frac{27}{2}+\frac{15\sqrt{3}}{2}\right)}+\sqrt[3]{\frac{1}{2}\left(\frac{27}{2}-\frac{15\sqrt{3}}{2}\right)};$$

for $a=3$ and $c=5$, then

$$3=\sqrt[3]{\frac{1}{2}\left(\frac{27}{5}+\frac{33}{5}i\sqrt{\frac{3}{5}}\right)}+\sqrt[3]{\frac{1}{2}\left(\frac{27}{5}-\frac{33}{5}i\sqrt{\frac{3}{5}}\right)};$$

for $a=4$ and $c=2$, then

$$4=\sqrt[3]{\frac{1}{2}\left(32+\frac{160}{3\sqrt{3}}\right)}+\sqrt[3]{\frac{1}{2}\left(32-\frac{160}{3\sqrt{3}}\right)};$$

for $a=5$ and $c=2$, then

$$5=\sqrt[3]{\frac{1}{2}\left(\frac{125}{2}+\frac{625}{6\sqrt{3}}\right)}+\sqrt[3]{\frac{1}{2}\left(\frac{125}{2}-\frac{625}{6\sqrt{3}}\right)}$$

for $a=4$ and $c=5$, then

$$4=\sqrt[3]{\frac{1}{2}\left(\frac{64}{5}+\frac{704i}{15\sqrt{15}}\right)}+\sqrt[3]{\frac{1}{2}\left(\frac{64}{5}-\frac{704i}{15\sqrt{15}}\right)};$$

for $a=4$ and $c=3$, then

$$4=\sqrt[3]{\frac{1}{2}\left(-\frac{256}{9}+\frac{4480i}{81\sqrt{3}}\right)}+\sqrt[3]{\frac{1}{2}\left(-\frac{256}{9}-\frac{4480i}{81\sqrt{3}}\right)};$$

for $a=4$ and $c=7$, then

$$4=\sqrt[3]{\frac{1}{2}\left(\frac{64}{7}+\frac{320i}{7\sqrt{7}}\right)}+\sqrt[3]{\frac{1}{2}\left(\frac{64}{7}-\frac{320i}{7\sqrt{7}}\right)};$$

for $a=5$ and $c=8$, then

$$5=\sqrt[3]{\frac{1}{2}\left(\frac{125}{8}+\frac{2125i}{24\sqrt{6}}\right)}+\sqrt[3]{\frac{1}{2}\left(\frac{125}{8}-\frac{2125i}{24\sqrt{6}}\right)}.$$

Corollary 4. *For* $c\in\mathbb{Z}_{\geq 1}$, *then any rational number* $\frac{a}{b}$ *and* $b\neq 0$, *has the following representation by the sum of the two cube roots*

$$\frac{a}{b}=\sqrt[3]{\frac{1}{2}\left[\frac{a^3}{b^3c}-\sqrt{\frac{a^6}{b^6c^2}-\frac{4a^6(c-1)^3}{27b^6c^3}}\right]}+\sqrt[3]{\frac{1}{2}\left[\frac{a^3}{b^3c}+\sqrt{\frac{a^6}{b^6c^2}-\frac{4a^6(c-1)^3}{27b^6c^3}}\right]}$$

Proof. Let $a \to \frac{a}{b}$ in Theorem 2, this completes the proof. □

4. Examples

For $a = 1, b = 3$ and $c = 2$, then

$$\frac{1}{3} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{54} + \frac{5}{162\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{54} - \frac{5}{162\sqrt{3}}\right)};$$

for $a = 1, b = 4$ and $c = 2$, then

$$\frac{1}{4} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{128} + \frac{5}{384\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{128} - \frac{5}{384\sqrt{3}}\right)};$$

for $a = 1, b = 5$ and $c = 2$, then

$$\frac{1}{5} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{250} + \frac{1}{150\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{250} - \frac{1}{150\sqrt{3}}\right)};$$

for $a = 1, b = 6$ and $c = 2$, then

$$\frac{1}{6} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{432} + \frac{5}{1296\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{432} - \frac{5}{1296\sqrt{3}}\right)}.$$

Corollary 5. *For* $c \in \mathbb{Z}_{\geq 1}$, *then*

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{c} + \sqrt{\frac{1}{c^2} - \frac{4(c-1)^3}{27c^3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{c} - \sqrt{\frac{1}{c^2} - \frac{4(c-1)^3}{27c^3}}\right)}.$$

Proof. Simplifying the right-hand side of Theorem 2 and dividing both members for a. □

5. Examples

For $c = 2$,

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{2} + \frac{5}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{2} - \frac{5}{6\sqrt{3}}\right)};$$

for $c = 5$,

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{5} + \frac{11i}{15\sqrt{15}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{5} - \frac{11i}{15\sqrt{15}}\right)};$$

for $c = 7$,

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{7} + \frac{5i}{7\sqrt{7}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{7} - \frac{5i}{7\sqrt{7}}\right)}.$$

Lemma 3. *For* $c \in \mathbb{Z}_{>1}$, *then*

$$1 = \sqrt[3]{\frac{1}{2}\left(-\frac{c+1}{c^2} + \sqrt{\frac{(c+1)^2}{c^4} - \frac{4(c^2+c+1)^3}{27c^6}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{c+1}{c^2} - \sqrt{\frac{(c+1)^2}{c^4} - \frac{4(c^2+c+1)^3}{27c^6}}\right)}.$$

Proof. Simplifying the right-hand side of Theorem 1 and dividing both members for a. □

Theorem 4. *For* $n \in \mathbb{Z}_{>1}$, *then* n *has the following representation by the sum of cube roots*

$$n = \sum_{k=2}^{n+1}\left[\sqrt[3]{\frac{1}{2}\left(-\frac{k+1}{k^2} + \sqrt{\frac{(k+1)^2}{k^4} - \frac{4(k^2+k+1)^3}{27k^6}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{k+1}{k^2} - \sqrt{\frac{(k+1)^2}{k^4} - \frac{4(k^2+k+1)^3}{27k^6}}\right)}\right].$$

Proof. Using the Lemma 3 and finite induction, this completes the proof. □

6. Examples

$$1 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)};$$

$$2 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} + \frac{70i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} - \frac{70i}{81\sqrt{3}}\right)};$$

$$3 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} + \frac{70i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} - \frac{70i}{81\sqrt{3}}\right)}$$

$$+ \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} + \frac{9i\sqrt{3}}{32}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} - \frac{9i\sqrt{3}}{32}\right)};$$

$$4 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} + \frac{70i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} - \frac{70i}{81\sqrt{3}}\right)}$$

$$+ \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} + \frac{9i\sqrt{3}}{32}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} - \frac{9i\sqrt{3}}{32}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{6}{25} + \frac{308i}{375\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{6}{25} - \frac{308i}{375\sqrt{3}}\right)}.$$