Avoiding an imaginary connection in the Dirac Equation

Leonardo Pedro
Centro de Fisica Teorica de Particulas, Portugal
leonardo@cftp.ist.utl.pt

April 5, 2013

Abstract

In a Majorana basis, the Dirac equation for a free spin one-half particle is a 4x4 real matrix differential equation. When including the effects of the electromagnetic interaction, the Dirac equation is a complex equation due to the presence of an imaginary connection in the covariant derivative, related with the phase of the spinor.

In this paper we study the solutions of the Dirac equation with the null and Coulomb potentials and notice that there is a real matrix that squares to -1, relating the imaginary and real components of these solutions. We show that these solutions can be obtained from the solutions of two non-linear 4x4 real matrix differential equations with a real matrix as the connection of the covariant derivative.

Contents

1 Introduction

Ever since the publication of the Schrodinger equation in 1926, it is broadly accepted that the wave functions in non-relativistic Quantum Mechanics must be complex. The phase of the complex number is identified with the phase of the oscillation of the wave.

Ever since the publication of the Dirac equation in 1928 and the prediction of the positron in 1929, it is broadly accepted that the wave functions in relativistic Quantum Mechanics must also be complex. The phase of the complex number is also identified with the phase of the oscillation of the wave. Unlike in the non-relativistic case, there are both positive and negative energy solutions that can only be fully understood in the context of Quantum Field Theory.

In 1937, Majorana noticed that the Dirac equation for free particles is a real equation and it's solutions can be real wave functions, as long as, in the context of Quantum Field Theory, there is no difference between particles and anti-particles.

If a wave function is real, then the phase of the complex number is null and cannot be identified with the phase of the oscillation of the wave. This motivates us to study the phase of oscillation of the solutions of the Dirac equation.

2 Real Connection

The equations for the classical Majorana spinor fields ψ and χ and for the electromagnetic potential A_μ in Quantum Electrodynamics, can be written as:

$$(i\partial\!\!\!/ - m)\psi = ei A\!\!\!/ \chi \tag{2.1}$$
$$(i\partial\!\!\!/ - m)\chi = -ei A\!\!\!/ \psi \tag{2.2}$$
$$\partial^2 A_\mu - \partial_\mu \partial \cdot A = e\eta_{\mu\nu}(\psi^\dagger \gamma^0 \gamma^\nu \psi + \chi^\dagger \gamma^0 \gamma^\nu \chi) \tag{2.3}$$

These equations are invariant under the global Lorentz transformations $S \in Pin(1,3)$:

$$x \to \Lambda(S)x \tag{2.4}$$
$$\psi(x) \to S\psi(\Lambda(S)x) \tag{2.5}$$
$$\chi(x) \to \gamma^0 S^{-1\dagger}\gamma^0 \chi(\Lambda(S)x) \tag{2.6}$$
$$A_\mu(x) \to e\Lambda_\mu^{\ \nu}(S) A_\nu(x)(\gamma^0 S^{-1\dagger}\gamma^0 S^{-1}) \tag{2.7}$$

Usually the Dirac field $\Psi \equiv \psi + i\chi$ is defined and the equations are written as:

$$(i\partial\!\!\!/ - A\!\!\!/ - m)\Psi = 0 \tag{2.8}$$
$$\partial^2 A_\mu - \partial_\mu \partial \cdot A = e\eta_{\mu\nu}\Psi^\dagger \gamma^0 \gamma^\nu \Psi \tag{2.9}$$

Now we can easily see that these equations are also invariant under the local transformation:

$$\Psi \to e^{i\theta}\Psi \tag{2.10}$$
$$eA_\mu \to eA_\mu - \partial_\mu \theta \tag{2.11}$$

The electromagnetic potential is then identified with an imaginary connection, that is, the covariant derivative is written as:

$$\partial_\mu + iA_\mu \tag{2.12}$$

Now we make the question: is there another way of obtaining the same solutions but using a real (that is, real in a Majorana basis) connection? If we drop the linearity requirement, then the answer is yes. We need to assume that there is a real, space-time dependent, matrix J verifying $((i\gamma^0)J)^2 = -1$. Note that these conditions are invariant under the transform $J \to S^\dagger J S$ for $S \in Pin(1,3)$, that is:

$$((i\gamma^0)J)^2 \to (i\gamma^0 S^\dagger J S)^2 = (\pm S^{-1} i\gamma^0 J S)^2 = S^{-1}(\pm i\gamma^0 J)^2 S = -1 \tag{2.13}$$

Now we have the following equations:

$$(i\gamma^\mu(\partial_\mu - eA_\mu(x) i\gamma^0 J(x) - m)\psi(x) = 0 \tag{2.14}$$

$$(i\gamma^\mu(\partial_\mu - eA_\mu(x) i\gamma^0 J(x)) - m) i\gamma^0 J(x)\psi(x) = 0 \tag{2.15}$$

The equation for A_μ can be written as:

$$\partial^2 A^\mu - \partial^\mu \partial_\nu A^\nu = e\psi^\dagger \gamma^0 \gamma^\mu \psi + e\psi^\dagger J^\dagger \gamma^\mu \gamma^0 J\psi \tag{2.16}$$

We can see that for a global $S \in Pin(1,3)$ we have:

$$x \to \Lambda(S)x \tag{2.17}$$

$$\psi(x) \to S\psi(\Lambda(S)x) \tag{2.18}$$

$$J(x) \to S^{-1\dagger} J(\Lambda x) S^{-1} \tag{2.19}$$

$$eA_\mu(x) i\gamma^0 J(x) \to e\Lambda_\mu{}^\nu(S) A_\nu(x) S i\gamma^0 J(\Lambda x) S^{-1} \tag{2.20}$$

We can write the previous two real equations as one complex equation as:

$$(i\gamma^\mu(\partial_\mu - eA_\mu(x) i - m)(1 + \gamma^0 J(x))\psi(x) = 0 \tag{2.21}$$

Now we can see that there is another transform that leaves the equations invariant:

$$\psi \to e^{i\gamma^0 J\theta}\psi \tag{2.22}$$

$$(1 + \gamma^0 J)\psi \to e^{i\theta}(1 + \gamma^0 J)\psi \tag{2.23}$$

$$eA_\mu \to eA_\mu + \partial_\mu \theta \tag{2.24}$$

Where θ is a real function of the space-time. Although we get a very similar equation with QED, there is a fundamental difference: the connection is real, the equations are non-linear and as a consequence we get, from the start a projector in the complex equation. In QED, this projector appears only in the final solutions, not in the equations.

3 Free particle

When the electromagnetic potential is null, we have:

$$\psi_p(x) = e^{-i\frac{\not p}{m} p\cdot x}\psi_p(0) \tag{3.1}$$

$$J_p(x) = \frac{\not p \gamma^0}{m} \tag{3.2}$$

We can check that $J_p(x)$ is hermitian and that $\psi_p(x) \to S\psi_p(\Lambda(S)x)$, $J_p(x) \to S^{-1\dagger} J_p(\Lambda(S)x) S^{-1\dagger}$

4 Hydrogen Atom

The Dirac equation for the Hydrogen atom is:

$$i\gamma^0(i\not{\partial} - e\not{A} - m)\Psi = 0 \tag{4.1}$$

With $A_i = 0$, $A_0 = -\frac{e}{r}$. The term with the potential is imaginary, therefore, the equation is complex.

We define the matrix:

$$\Lambda_{nlm\epsilon} = \Big(\frac{f_{nl\epsilon}(r)}{r} + \frac{g_{nl\epsilon}(r)}{r} i\gamma^r\Big)\Omega_{lm}\frac{1+\epsilon\sigma^3}{2} \tag{4.2}$$

Where $\epsilon = \pm 1$. If f and g are such that the following equations hold:

$$(E_{nl} + \frac{e^2}{r} - m)\frac{f_{nl\epsilon}(r)}{r} + (\partial_r + \frac{1-\epsilon l}{r})\frac{g_{nl\epsilon}(r)}{r} = 0 \tag{4.3}$$

$$(-E_{nl} - \frac{e^2}{r} - m)\frac{g_{nl\epsilon}(r)}{r} + (\partial_r + \frac{1+\epsilon l}{r})\frac{f_{nl\epsilon}(r)}{r} = 0 \tag{4.4}$$

We will not solve these equations here, the solution can be seen in [1].

Then Λ verifies:

$$i\gamma^0(i\vec{\not{\partial}} - m)\Lambda_{nlm\epsilon}\frac{1+\gamma^0}{2} = i(E_{nl} + \frac{e^2}{r})\Lambda_{nlm\epsilon}\frac{1+\gamma^0}{2} \tag{4.5}$$

The solution to Dirac equation is:

$$\Psi = \Lambda_{nlm\epsilon}e^{-i\gamma^0 E_{nl}x^0}\frac{1+\gamma^0}{2}\psi \tag{4.6}$$

Where ψ is a fixed Majorana spinor. We can now check that

$$\Psi = \frac{1+\gamma^0 J(x)}{2}\Lambda_{nlm\epsilon}e^{-i\gamma^0 E_{nl}x^0}\psi \tag{4.7}$$

Where

$$J(x) = \frac{\Big(f_{nl\epsilon}(r) - g_{nl\epsilon}(r) i\gamma^r\Big)^2}{f^2_{nl\epsilon}(r) - g^2_{nl\epsilon}(r)} \tag{4.8}$$

And we can check that $(i\gamma^0 J)^2 = -1$.

References

[1] P. S. Riseborough, "Advanced Quantum Mechanics," May, 2011. http://www.math.temple.edu/~prisebor/Advanced.pdf.