Relations between Distorted and Original Angles in STR

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Using the Oblique-Length Contraction Factor, which is a generalization of Lorentz Contraction Factor, one shows several trigonometric relations between distorted and original angles of a moving object lengths in the Special Theory of Relativity.

1 Introduction

The lengths at oblique angle to the motion are contracted with the Oblique-Length Contraction Factor $OC(v, \theta)$, defined as [1-2]:

$$OC(v,\theta) = \sqrt{C(v)^2 \cos^2\theta + \sin^2\theta} \tag{1}$$

where $C(v)$ is just Lorentz Factor:

$$C(v) = \sqrt{1 - \frac{v^2}{c^2}} \in [0,1] \text{ for } v \in [0, \mathrm{c}]. \tag{2}$$

Of course

$$0 \le OC(v,\theta) \le 1. \tag{3}$$

The Oblique-Length Contraction Factor is a generalization of Lorentz Contractor $C(v)$, because: when $\theta = 0$, or the length is moving along the motion direction, then $OC(v, 0) = C(v)$. Similarly

$$OC(v,\pi) = OC(v, 2\pi) = C(v). \tag{4}$$

Also, if $\theta = \pi/2$, or the length is perpendicular on the motion direction, then $OC(v, \pi/2) = 1$, i.e. no contraction occurs. Similarly $OC(v, \frac{3\pi}{2}) = 1$.

2 Tangential relations between distorted acute angles vs. original acute angles of a right triangle

Let's consider a right triangle with one of its legs along the motion direction (Fig. 1).

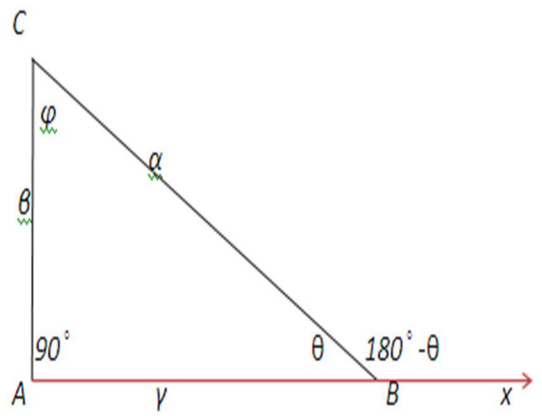

Fig. 1:

$$\tan\theta = \frac{\beta}{\gamma} \tag{5}$$

$$\tan(180^\circ - \theta) = -\tan\theta = \frac{\beta}{\gamma} \tag{6}$$

After contraction of the side AB (and consequently contraction of the oblique side BC) one gets (Fig. 2):

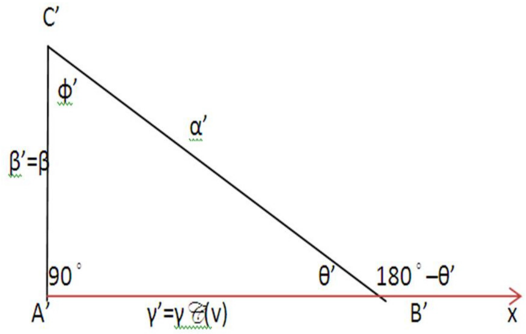

Fig. 2:

$$\tan(180^\circ - \theta') = -\tan\theta' = -\frac{\beta'}{\gamma'} = -\frac{\beta}{\gamma C(v)}. \tag{7}$$

Then:

$$\frac{\tan(180^\circ - \theta')}{\tan(180^\circ - \theta)} = \frac{-\frac{\beta}{\gamma C(v)}}{-\frac{\beta}{\gamma}} = \frac{1}{C(v)}. \tag{8}$$

Therefore

$$\tan(\pi - \theta') = -\frac{\tan(\pi - \theta)}{C(v)} \tag{9}$$

and consequently

$$\tan(\theta') = \frac{\tan(\theta)}{C(v)} \tag{10}$$

or

$$\tan(B') = \frac{\tan(B)}{C(v)} \tag{11}$$

which is the Angle Distortion Equation, where θ is the angle formed by a side travelling along the motion direction and another side which is oblique on the motion direction.

The angle θ is increased (i.e. $\theta' > \theta$).

$$\tan\varphi = \frac{\gamma}{\beta} \quad \text{and} \quad \tan\varphi' = \frac{\gamma'}{\beta'} = \frac{\gamma C(v)}{\beta} \tag{12}$$

whence:

$$\frac{\tan\varphi'}{\tan\varphi} = \frac{\frac{\gamma C(v)}{\beta}}{\frac{\gamma}{\beta}} = C(v). \tag{13}$$

So we get the following Angle Distortion Equation:

$$\tan\varphi' = \tan\varphi \cdot C(v) \tag{14}$$

or

$$\tan C' = \tan C \cdot C(v) \tag{15}$$

where φ is the angle formed by one side which is perpendicular on the motion direction and the other one is oblique to the motion direction.

The angle φ is decreased (i.e. $\varphi' < \varphi$). If the traveling right triangle is oriented the opposite way (Fig. 3)

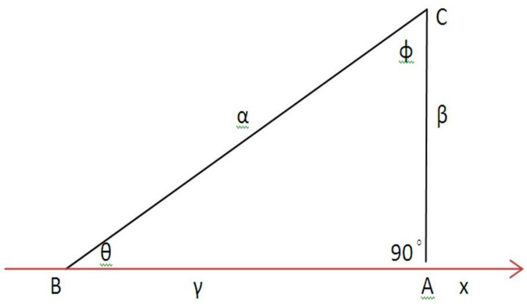

Fig. 3:

$$\tan\theta = \frac{\beta}{\gamma} \quad \text{and} \quad \tan\varphi = \frac{\gamma}{\beta}. \tag{16}$$

Similarly, after contraction of side AB (and consequently contraction of the oblique side BC) one gets (Fig. 4)

$$\tan\theta' = \frac{\beta'}{\gamma'} = \frac{\beta}{\gamma C(v)} \tag{17}$$

and

$$\tan\varphi' = \frac{\gamma'}{\beta'} = \frac{\gamma C(v)}{\beta} \tag{18}$$

$$\frac{\tan\theta'}{\tan\theta} = \frac{\frac{\beta}{\gamma C(v)}}{\frac{\beta}{\gamma}} = \frac{1}{C(v)} \tag{19}$$

or

$$\tan\theta' = \frac{\tan\theta}{C(v)} \tag{20}$$

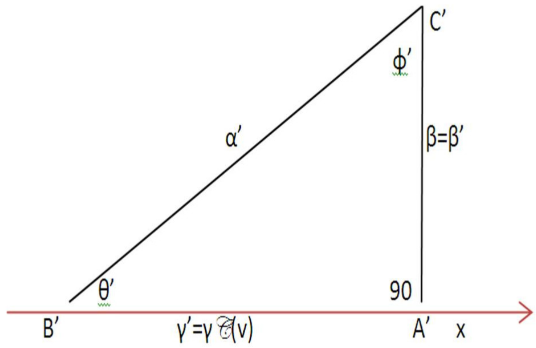

Fig. 4:

and similarly

$$\frac{\tan\varphi'}{\tan\varphi} = \frac{\frac{\gamma C(v)}{\beta}}{\frac{\gamma}{\beta}} = C(v) \tag{21}$$

or

$$\tan\varphi' = \tan\varphi \cdot C(v). \tag{22}$$

Therefore one got the same Angle Distortion Equations for a right triangle traveling with one of its legs along the motion direction.

3 Tangential relations between distorted angles vs. original angles of a general triangle

Let's suppose a general triangle ΔABC is travelling at speed v along the side BC as in Fig. 5.

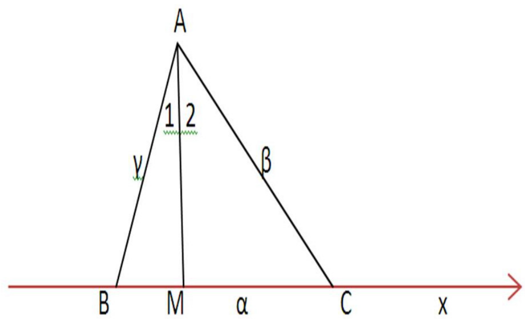

Fig. 5:

The height remains not contracted: $AM \equiv A'M'$. We can split this figure into two traveling right sub-triangles as in Fig. 6.

In the right triangles $\Delta A'M'B'$ and respectively $\Delta A'M'C'$ one has

$$\tan B' = \frac{\tan B}{C(v)} \quad \text{and} \quad \tan \mathrm{C}' = \frac{\tan \mathrm{C}}{\mathrm{C}(v)}. \tag{23}$$

Also

$$\tan A_1' = \tan A_1 C(v) \quad \text{and} \quad \tan A_2' = \tan A_2 C(v). \tag{24}$$

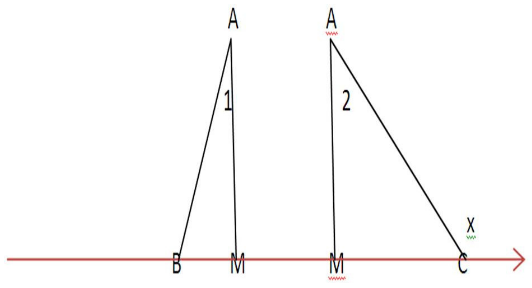

Fig. 6:

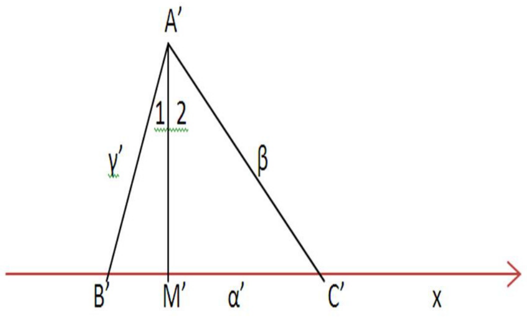

Fig. 7:

But

$$\begin{aligned}\tan A' &= \tan(A'_1 + A'_2) = \frac{\tan A'_1 + \tan A'_2}{1 - \tan A'_1 \tan A'_2}\\ &= \frac{\tan A_1 C(v) + \tan A_2 C(v)}{1 - \tan A_1 C(v) \tan A_2 C(v)}\\ &= C(v) \cdot \frac{\tan A_1 + \tan A_2}{1 - \tan A_1 \tan A_2 C(v)^2}\\ &= C(v) \cdot \frac{\frac{\tan A_1 + \tan A_2}{1 - \tan A_1 \tan A_2} \cdot (1 - \tan A_1 \tan A_2)}{1 - \tan A_1 \tan A_2 C(v)^2}\\ &= C(v) \cdot \frac{\tan(A_1 + A_2)}{1} \cdot \frac{1 - \tan A_1 \tan A_2}{1 - \tan A_1 \tan A_2 C(v)^2}.\end{aligned}$$

$$\tan A' = C(v) \cdot \tan(A) \cdot \frac{1 - \tan A_1 \tan A_2}{1 - \tan A_1 \tan A_2 C(v)^2}. \tag{25}$$

We got

$$\tan A' = \tan(A) \cdot C(v) \cdot \frac{1 - \tan A_1 \tan A_2}{1 - \tan A_1 \tan A_2 C(v)^2} \tag{26}$$

Similarly we can split this Fig. 7 into two traveling right sub-triangles as in Fig. 8.

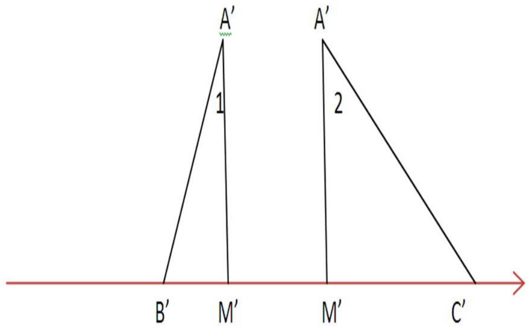

Fig. 8:

4 Other relations between the distorted angles and the original angles

1. Another relation uses the Law of Sine in the triangles ΔABC and respectively $\Delta A'B'C'$:

$$\frac{\alpha}{\sin A} = \frac{\beta}{\sin B} = \frac{\gamma}{\sin C} \tag{27}$$

$$\frac{\alpha'}{\sin A'} = \frac{\beta'}{\sin B'} = \frac{\gamma'}{\sin C'}. \tag{28}$$

After substituting

$$\alpha' = \alpha C(v) \tag{29}$$

$$\beta' = \beta \mathcal{O}C(v, C) \tag{30}$$

$$\gamma' = \gamma \mathcal{O}C(v, B) \tag{31}$$

into the second relation one gets:

$$\frac{\alpha C(v)}{\sin A'} = \frac{\beta \mathcal{O}C(v, C)}{\sin B'} = \frac{\gamma \mathcal{O}C(v, B)}{\sin C'}. \tag{32}$$

Then we divide term by term the previous equalities:

$$\frac{\frac{\alpha}{\sin A}}{\frac{\alpha C(v)}{\sin A'}} = \frac{\frac{\beta}{\sin B}}{\frac{\beta \mathcal{O}C(v, C)}{\sin B'}} = \frac{\frac{\gamma}{\sin C}}{\frac{\gamma \mathcal{O}C(v, B)}{\sin C'}} \tag{33}$$

whence one has:

$$\frac{\sin A'}{\sin A \cdot C(v)} = \frac{\sin B'}{\sin B \cdot \mathcal{O}C(v, C)} = \frac{\sin C'}{\sin C \cdot \mathcal{O}C(v, B)}. \tag{34}$$

2. Another way:

$$A' = 180° - (B' + C') \quad \text{and} \quad \mathrm{A} = 180° - (\mathrm{B} + \mathrm{C}) \tag{35}$$

$$\begin{aligned}\tan A' &= \tan[180° - (B' + C')] = -\tan(B' + C')\\ &= -\frac{\tan B' + \tan C'}{1 - \tan B' \cdot \tan C'}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\dfrac{\tan B}{C(v)} + \dfrac{\tan C}{C(v)}}{1 - \tan B \cdot \tan C / C(v)^2} \\
&= -\frac{1}{C(v)} \cdot \frac{\tan B + \tan C}{1 - \tan B \cdot \tan C / C(v)^2} \\
&= -\frac{\tan(B+C)}{C(v)} \cdot \frac{1 - \tan B \tan C}{1 - \tan B \cdot \tan C / C(v)^2} \\
&= -\frac{-\tan[180^\circ - (B+C)]}{C(v)} \cdot \frac{1 - \tan B \cdot \tan C}{1 - \tan B \cdot \tan C / C(v)^2} \\
&= \frac{\tan A}{C(v)} \cdot \frac{1 - \tan B \cdot \tan C}{1 - \tan B \cdot \tan C / C(v)^2}.
\end{aligned}
$$

We got

$$
\tan A' = \frac{\tan A}{C(v)} \cdot \frac{1 - \tan B \cdot \tan C}{1 - \tan B \cdot \tan C / C(v)^2}. \qquad (36)
$$

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