On A Property of Pascal's Triangle

Germán Andrés Paz
(2000) Rosario, Santa Fe, Argentina
germanpaz_ar@hotmail.com

Abstract

In this simple Math exercise we show a property of Pascal's Triangle. More precisely, we show that if a is any positive odd integer, then $\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 1$. Moreover, we prove that if b is any positive even integer, then $\binom{b}{1} - \binom{b}{2} + \binom{b}{3} - \binom{b}{4} + \cdots + \binom{b}{b-1} - \binom{b}{b} = 1$.

1 Preliminaries

Throughout this paper the numbers a, b, i, j, n, r and x are always positive integers.

2 Lemmas

Lemma 1. *It is correct to say that* $\binom{n}{r} = \binom{n}{n-r}$.

Proof. According to the binomial coefficient formula, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}. \tag{1}$$

Now, according to (1) we have

$$\begin{aligned}\binom{n}{n-r} &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= \frac{n!}{(n-r)!(n-n+r)!}\end{aligned}$$

$$= \frac{n!}{(n-r)!r!}$$
$$= \frac{n!}{r!(n-r)!}.$$

Since $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{n}{n-r} = \frac{n!}{r!(n-r)!}$, then $\binom{n}{r} = \binom{n}{n-r}$. □

Lemma 2. *If n is any positive integer, then $\binom{n}{n} = 1$.*

Proof. As we already know, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

This means that

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1,$$

which proves the lemma. □

Lemma 3. *If a is any positive odd integer, then $\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 1$.*

Proof. In order to prove that $\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 1$, we need to prove that $\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots - \binom{a}{a-1} = 0$, since according to Lemma 2 we have $\binom{a}{a} = 1$.

In the expression

$$\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots - \binom{a}{a-1} \tag{2}$$

we have an even number of binomial coefficients. In (2) we can see that $\binom{a}{x}$ has a positive sign when x is odd, whereas $\binom{a}{x}$ has a negative sign when x is even. According to Lemma 1 we have

$$\binom{a}{1} = \binom{a}{a-1}$$
$$\binom{a}{2} = \binom{a}{a-2}$$
$$\binom{a}{3} = \binom{a}{a-3}$$

$$\binom{a}{4} = \binom{a}{a-4}$$

$$\ldots$$

When x is even, then $a-x$ is odd. On the other hand, when x is odd then $a-x$ is even. This means that x and $a-x$ never have the same parity. According to what we said before, this implies that in (2) two terms $\binom{a}{x}$ and $\binom{a}{a-x}$ always have different signs. In other words, if one of these terms is positive then the other is negative, which means that they cannot be both positive or negative at the same time. Besides, we have $\binom{a}{x} = \binom{a}{a-x}$ (see Lemma 1).

All this means that in (2) for every binomial coefficient $\binom{a}{x}$ we have the binomial coefficient $-\binom{a}{x}$ (in (2) each positive $\binom{a}{x}$ has its 'negative counterpart'). We can also say that in (2) for every $-\binom{a}{x}$ we have the binomial coefficient $\binom{a}{x}$. This means that

$$\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots - \binom{a}{a-1} = 0.$$

Since

$$\binom{a}{a} = 1,$$

then

$$\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots - \binom{a}{a-1} + \binom{a}{a} = 1. \qquad \square$$

Lemma 4. *It is correct to say that* $\binom{n}{1} - \binom{n-1}{1} = 1$.

Proof. We have

$$\begin{aligned}\binom{n}{1} - \binom{n-1}{1} &= \frac{n!}{1!(n-1)!} - \frac{(n-1)!}{1!(n-1-1)!} \\ &= \frac{n!}{(n-1)!} - \frac{(n-1)!}{(n-2)!} \\ &= n-(n-1) \\ &= n-n+1 \\ &= 1,\end{aligned}$$

which proves the lemma. $\square$

Lemma 5. *It is correct to say that* $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

Proof. We know that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Now, we have

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!((n-1)-(r-1))!} + \frac{(n-1)!}{r!((n-1)-r)!} \\
&= \frac{(n-1)!}{(r-1)!(n-1-r+1)!} + \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\
&= \frac{r(n-1)! + (n-r)(n-1)!}{r!(n-r)!} \\
&= \frac{(n-1)!(r+(n-r))}{r!(n-r)!} \\
&= \frac{(n-1)!n}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!}.
\end{aligned}$$

Since $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{n!}{r!(n-r)!}$, then $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. $\square$

Lemma 6. *If b is any positive even integer, then* $\binom{b}{1} - \binom{b}{2} + \binom{b}{3} - \binom{b}{4} + \cdots + \binom{b}{b-1} - \binom{b}{b} = 1$.

Proof. In order to better understand the proof of this lemma, we will start with an example: we will prove that

$$\binom{6}{1} - \binom{6}{2} + \binom{6}{3} - \binom{6}{4} + \binom{6}{5} - \binom{6}{6} \tag{3}$$

equals 1.

According to Lemma 5, the expression (3) can be written as

$$\begin{gathered}
\binom{6}{1} - \left(\binom{5}{1} + \binom{5}{2}\right) + \left(\binom{5}{2} + \binom{5}{3}\right) - \left(\binom{5}{3} + \binom{5}{4}\right) + \\
+ \left(\binom{5}{4} + \binom{5}{5}\right) - 1.
\end{gathered} \tag{4}$$

Let us look at the binomial coefficients of the form $\binom{5}{x}$ where $2 \le x \le 4$. When $2 \le x \le 4$, the binomial coefficient $\binom{5}{x}$ appears twice in (4): once with a positive sign and another time with a negative sign. On the other hand, we have $\binom{6}{1} - \binom{5}{1} = 1$ and $\binom{5}{5} - 1 = 1 - 1 = 0$. This means that (4) can be written as

$$\binom{6}{1} - \binom{5}{1} - \binom{5}{2} + \binom{5}{2} + \binom{5}{3} - \binom{5}{3} - \binom{5}{4} + \binom{5}{4} + \binom{5}{5} - 1. \quad (5)$$

It is easy to see that the result of (5) equals 1, since that expression equals

$$\binom{6}{1} - \binom{5}{1} + \binom{5}{5} - 1,$$

and

$$\binom{6}{1} - \binom{5}{1} + \binom{5}{5} - 1 = 1 + 1 - 1 = 1.$$

In general, if b is any positive even integer then the expression

$$\binom{b}{1} - \binom{b}{2} + \binom{b}{3} - \binom{b}{4} + \cdots + \binom{b}{b-1} - \binom{b}{b}$$

can be written as

$$\binom{b}{1} - \binom{b-1}{1} - \binom{b-1}{2} + \binom{b-1}{2} + \binom{b-1}{3} - \binom{b-1}{3} - \binom{b-1}{4} +$$

$$+ \binom{b-1}{4} + \cdots + \binom{b-1}{b-3} - \binom{b-1}{b-3} - \binom{b-1}{b-2} + \binom{b-1}{b-2} + \binom{b-1}{b-1} - 1.$$

Within this expression, for every $\binom{b-1}{x}$ where $2 \le x \le b-2$ there is a binomial coefficient $-\binom{b-1}{x}$ (we can also say that for every $-\binom{b-1}{x}$ there is a binomial coefficient $\binom{b-1}{x}$). This means that the above expression equals

$$\binom{b}{1} - \binom{b-1}{1} + \binom{b-1}{b-1} - 1,$$

which clearly equals 1, since $\binom{b}{1} - \binom{b-1}{1} = 1$ and $\binom{b-1}{b-1} - 1 = 0$. Therefore, we have proved that

$$\binom{b}{1} - \binom{b}{2} + \binom{b}{3} - \binom{b}{4} + \cdots + \binom{b}{b-1} - \binom{b}{b} = 1. \qquad \square$$

3 Conclusion

According to Lemma 3, if a is any positive odd integer then

$$\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 1. \tag{6}$$

According to Lemma 6, if b is any positive even integer then

$$\binom{b}{1} - \binom{b}{2} + \binom{b}{3} - \binom{b}{4} + \cdots + \binom{b}{b-1} - \binom{b}{b} = 1. \tag{7}$$

We can also say that

$$-\binom{a}{0} + \binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 0$$

and

$$-\binom{b}{0} + \binom{b}{1} - \binom{b}{2} + \binom{b}{3} - \binom{b}{4} + \cdots + \binom{b}{b-1} - \binom{b}{b} = 0.$$

The equality (6) can be expressed as

$$\sum_{i=1}^{\frac{a+1}{2}} \binom{a}{2i-1} - \sum_{j=1}^{\frac{a-1}{2}} \binom{a}{2j} = 1,$$

whereas the equality (7) can be expressed as

$$\sum_{i=1}^{\frac{b}{2}} \binom{b}{2i-1} - \sum_{j=1}^{\frac{b}{2}} \binom{b}{2j} = 1.$$

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