Differential structures on path spaces.

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Abstract

We study general deformation structures on path spaces defined on manifolds and specialioze to those with memory. Furthermore, we highlight the construction of local operators on path space relatively compact to some position operator. The idea of this rather elementary work is to engage in the study of how perturbative string theory could be seen as a representation of a non commutative geometry. The latter, as studied in [1, 2] is acessible to direct physical interpretation.

1 Introduction

We engage in the study of differential operators on path space and are interested in particular in the construction of operators relatively compact to some position operator with a continuum spectrum so that quantum entanglement of space can survive at all scales as enunciated in [2].

2 Finite displacement structures.

Let $\mathcal{M}$ be a finite dimensional compact manifold and let path space $\mathcal{P}(\mathcal{M})$ be defined as the space of all continuous curves without self intersections defined on an interval $[a, b]$ equipped with the Vietoris topology[1]. As is well known, it can be given the structure of a locally convex manifold by using the exponential map associated to vectorfields. In general, let V be a vectorfield, then the free displacement over a parameter range λ is given by

$$\Delta_V^\lambda \gamma(s) = \exp_V(\lambda)\left[\gamma(s)\right]$$

which defines a curve with the same parameter domain. One can also consider extensionsof the curve either on the front or tail:

$$\Gamma_V^{\lambda,+}\gamma(s) = \exp_V(s-b)\left[\gamma(b)\right]$$

for $b+\lambda \geq s \geq b$ and $\gamma(s)$ otherwise. Therefore, the non-abelian semi group[2] of difference land is generated by $\Delta_V^\lambda, \Gamma_V^{\lambda,+}$ as well as the operations T and R^λ where $T\gamma(s) = \gamma(b+a-s)$ and $R^\lambda\gamma$ is the restriction of γ to $[a, a+\lambda]$ for[3]

[1]The case with self intersections can be treated by chopping the curve in suitable pieces or by considering multivalued fields.

[2]Not every generator has an inverse.

[3]One can also define the operation S^λ as the restriction of γ to $[b-\lambda, b]$. $S^\lambda = TR^\lambda T$ as an easy calculation reveals; also, T commutes with the displacements Δ_V^λ and $T\Gamma_V^{\lambda,+}T = \Gamma_V^{\lambda,-}$.

$\lambda \leq b - a$. We now study cases where V is dependent upon γ itself; in this context, two notions are particularly useful: (a) a curve γ and vectorfield V are locally alligned if and only if $\dot{\gamma}(t) = \alpha(t)V(\gamma(t))$ where $\alpha(t)$ is real valued and

$$\exp_V^{\lambda} \gamma(s) = \gamma \left(s + \int_s^{s+\lambda} \alpha(t)dt \right)$$

whenever the last expression between brackets remains smaller or equal to b and (b) V is called a velocity field if

$$\dot{\gamma}(t) = V(\gamma(t))$$

and the associated displacement

$$\exp_V^{\lambda} \gamma(s)$$

is given by the expression $\gamma(s + \lambda)$ for $a \leq s \leq b - \lambda$. We now resort to displacements with memory and depending upon a background vectorfield W as well.

2.1 Differences with memory and a background influence.

Basically, one can decide to displace a curve in a determinstic or stochastic manner and the latter only requires probability measures on the appropriate infinite dimensional spaces making the relevant objects into stochastic variables. For now, we will only present some deterministic options and require the presence of a non-degenerate Riemannian or Lorentzian metric h which canonically defines a Levi Civita connection ∇. That is, we consider vector valued functions $\omega(p, \gamma, h, t)$ defining a vector at p by means of the inverse exponential map $\exp_h^{-1}(p) : T\mathcal{M} \to T_p\mathcal{M}$ from a vector at $\gamma(t)$ constructed by means of local quantities $\dot{\gamma}(t), \ddot{\gamma}(t), \ldots$ where $\ddot{\gamma}(t)$ is defined as $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)$. Then,

$$V(p) = \int_a^b \omega(p, \gamma, h, t)dt$$

and we give now some examples where h is Lorentzian versus Riemannian. In case of a time oriented Lorentzian manifold and causal curves, one has to allow for an assymetric treatment of the past and future. More specifically $\omega(p, \gamma, h, t)$ vanishes for all t suh that $\gamma(t) \notin J^+(p) \cup J^-(p)$ and its functional form depends upon whether $\gamma(t) \in J^+(p)$ or $J^-(p)$ respectively. For an exclusively retarded prescription on flat Minkowski, one may choose

$$V(p) = \int_a^{\alpha_p} e^{-\mu(A(\gamma(t),p))}\dot{\gamma}(t)dt$$

where $\gamma(\alpha_p)$ is the intersection point of γ with $J^-(p)$ and $A(p, q)$ denotes the Alexandrov set between p and q. In case h is the standard flat Euclidean metric, it is natural to replace the volume of the Alexandrov set by the Euclidean distance between $\gamma(t)$ and p. When allowing for an external perturbation W, one could add to the above prescription terms of the kind

$$\alpha(p, \gamma)W(p) + \int_a^b \dot{\gamma}(s)h(\dot{\gamma}(s), W(\gamma(s)))\beta(\gamma(s), p)ds$$

as well as many other forms.

3 Construction of operators relatively compact to some position operator.

Let us first recall what it means for an operator A to be relatively compact to B. A is relatively compact with respect to B on a scale δ if and only if for any $\lambda \in \sigma(A)$ there exist $a \leq \lambda \leq b$ with $b - a < \delta$ and a hermitian projection operator $p^A_\lambda \prec p^A_{[a,b]}$(where by convention we choose the maximal a and minimal b)[4] such that the smallest $p^A_\lambda \prec p^B_\lambda$ obeying $p^B_\lambda B p^B_\lambda = B p^B_\lambda$ satisfies $p^B_\lambda \prec p^B_{[c,d]}$ where $c \leq d$ are optimally chosen. A direct consequence is the notion of spectral compression on a scale δ

$$C^\delta(A,B) = \sup_\lambda \frac{b-a}{d-c}.$$

Finally, we call the pair (A,B) relatively compact on a scale of δ when both are relatively compact on that scale with respect to one and another. Before we proceed, let us define differential operators on smooth functions $\Psi : \mathcal{P}(\mathcal{M}) \to \mathbb{R}$ by using the basic differences Δ and Γ. Let V be a smooth vectorfield, then

$$\partial_V \Psi(\gamma) = \lim_{\lambda \to 0} \frac{\Psi\left(\Delta^\lambda_V(\gamma)\right) - \Psi(\gamma)}{\lambda}$$

and using $\Gamma^{\lambda,\pm}_V$ one can define $\partial^\pm_V \Psi$. One notices that for elementary coordinate functions of endpoints, that is

$$x^\mu_f(\gamma) = x^\mu(\gamma(b))$$

the derivative

$$\partial^+_V x^\mu_f(\gamma) = V^\mu(\gamma(b)) = \partial_V x^\mu_f(\gamma).$$

3.1 Addition rule.

The derivative associated to a composition of ordinary displacements (not involving the T operator) obviously satisfies a chain rule depending upon the particular scaling limit which has been taken. Concretely, let $\alpha(\lambda,\delta)$ be a positive function of λ, δ which takes the value zero if and only if $\delta = \lambda = 0$. Then, the expression

$$\frac{\Psi\left(\Delta^\delta_V \Gamma^{\lambda,+}_W(\gamma)\right) - \Psi(\gamma)}{\alpha(\delta,\lambda)} = \frac{\delta}{\alpha(\delta,\lambda)} \frac{\Psi\left(\Delta^\delta_V \Gamma^{\lambda,+}_W(\gamma)\right) - \Psi\left(\Gamma^{\lambda,+}_W(\gamma)\right)}{\delta} + \frac{\lambda}{\alpha(\delta,\lambda)} \frac{\Psi\left(\Gamma^{\lambda,+}_W(\gamma)\right) - \Psi(\gamma)}{\lambda}$$

and the behaviour of this expression in the double limit depends upon the scaling $\lambda(\delta)$ as well as the function α. In case $\alpha(\lambda,\delta) = \frac{1}{2}(\lambda + \delta)$, the right hand side reduces to

$$\partial_V \Psi(\gamma) + \partial^+_W \Psi(\gamma)$$

in the scaling limit $\lambda = \delta$.

[4]One can require that p^A_λ satisfies $p^A_\lambda A p^A_\lambda = A p^A_\lambda$ but this is not mandatory.

3.2 Coordinatization.

Let us first construct suitable coordinate systems so that explicit calculations become possible. That is, let $B(\mathcal{M})$ be a countable basis of vectorfields on $\mathcal{M}$ and likewise consider $B(\mathcal{O})$ to be a filter of local subbases meaning $B(\mathcal{O}) \subset B(\mathcal{V})$ for $\mathcal{O} \subset \mathcal{V}$. If $\mathcal{O}$ is a set of s points and $\mathcal{M}$ is n-dimensional, then the dimension of $B(\mathcal{O})$ is given by sn. Obviously, it is sufficient to construct charts based around a curve γ an open set $\mathcal{O}$ around it and some open interval $(-\delta, \delta)$; more specifically, consider the subspace of $T\mathcal{O}$ consisting of vectorfields V of sup-norm one on $\mathcal{O}$ (with respect to some Riemannian metric h) and study the actions

$$T[\lambda, \delta, V](\gamma) = \Delta^{\lambda}_{V} \Gamma^{\delta,+}_{W_\gamma}(\gamma)$$

where W_γ is some non-vanishing background field with γ as an integral curve. Now, while the action above is globally uniquely defined regarding V in the sense that in case

$$T[\lambda, \delta, V_1](\gamma) = T[\lambda, \delta, V_2](\gamma)$$

for all λ, δ and γ, then $V_1 = V_2$ and ordinary rescalings have been excluded already by using the sup norm. However, the Gribov problem that for *some* δ_i, V_i the above expressions might be equal remains and appropriate identifications have to be made. Obviously, the solution to this problem is to further restrict to vectorfields defined from the maximal W_γ future extension $\gamma_{\max}$ of γ by using the exponential map $\exp^h_\lambda$. One knows that $\exp^h_\lambda[V(\gamma_{\max}(s))]$ is a bijection on $(\lambda, s) \in (-d, d) \times \gamma_{\max}$ for some suitable $d > 0$ given the unit sup-norm vectorfield V. However, we cannot disgard focal points, otherwise curves were not allowed to curl and have to stay transversal with respect to nonintersecting geodesic bundles. Therefore, we define multivalued vectorfields V to be consistent if and only if differentiable integral curves are well defined (in the sense that they are well defined whenever V is). Hence, we work with vectorfields of unit sup-norm on $\gamma_{\max}$ which may be geodesically mapped to a multivalued field in case the geodesics cross. In particular, let $\gamma_{\max} : [a, c] \to \mathcal{M}$ be a (possibly self-intersecting) curve in $\overline{\mathcal{O}}$ where $a < b < c$ and consider as basis (in L^2 norm) of the function space of continuous functions on $[a, c]$ the family $\sin\left(n\pi \frac{s-a}{c-a}\right), \cos\left(n\pi \frac{s-a}{c-a}\right)$ where $n \in \mathbb{N}$. Moreover, let $E_i(s)$ be some parallel transported basis with respect to h, then every (possibly multivalued) orthonormal vectorfield V on $\gamma_{\max}$ can be uniquely written as

$$V(\gamma_{\max}(s)) = \sum_{i=1}^{n} \left(a_{i,0} + \sum_{n=1}^{\infty} \left(a_{i,n} \sin\left(n\pi \frac{s-a}{c-a}\right) + b_{i,n} \cos\left(n\pi \frac{s-a}{c-a}\right)\right)\right) E_i(s)$$

in the L^2 sense. Since normalization with respect to the sup-norm is technically akward, it is much easier to consider the map $\exp^h_1(V(\gamma_{\max}(s)))$ wherever the image of the latter is a connected curve[5] since $\exp^h_1(V(\gamma_{\max}(s))) = \exp^h_\lambda\left(\frac{1}{\lambda} V(\gamma_{\max}(s))\right)$. Therefore, the infinite dimensional coordinate charts look like $(\lambda, a_{i,0}, a_{i,n}, b_{i,n})$ with as chart mapping

$$\exp^h_1\left(V^\lambda(s)\right) R^{b-a+\lambda}(\gamma_{\max})$$

[5]Taking into account that some pieces may drop off the manifold.

where

$$V^{\lambda}(s) = \sum_{i=1}^{n} \left(a_{i,0} + \sum_{n=1}^{\infty} \left(a_{i,n} \sin \left(n\pi \frac{s-a}{b+\lambda-a} \right) + b_{i,n} \cos \left(n\pi \frac{s-a}{b+\lambda-a} \right) \right) \right) E_i(s)$$

and the reader may verify that everything is well defined. Notice that not every vectorfield defined in this way has a continuous representant and that therefore holes are to be pinched in this chart; however, the set of vectors with a continuous representant is open in the compact-open topology on $\mathbb{R}^{\infty}$ and everything below has to be understood in this way.

3.3 Explicit expressions of operators.

We now calculate leading terms for the standard differential operators depending upon nonlocal functions of local geometric tensors derived from the metric h and give exact expressions in the Euclidean case. Let us start with the difference operator Δ^{μ}_{W}; that is, we calculate

$$\Delta^{\mu}_{W}(p)^{co}(s) = \left(\exp^{h}_{1}\left(\gamma_{\max}(s)\right)\right)^{-1} \left(\Delta^{\mu}_{W}\left(\exp^{h}_{1}\left(V^{\lambda}(s)\right)\right)\left(R^{b-a+\lambda}\left(\gamma_{\max}(s)\right)\right)\right)$$

where $p = (\lambda, a_{i,0}, a_{i,n}, b_{i,n})$ and rescale to the entire interval $[a, c]$. More precisely, we calculate the first and second derivatives with respect to μ and W as far as we can for a general Riemannian metric, the Lorentzian case being somewhat more subtle. Concerning the μ-derivative, one needs to remember the notion of Fermi transport and calculate the geodesic difference equation. Let V be a vectorfield, then the Fermi derivative of W along V is defined as

$$D^{F}_{V}(W) = \nabla_V W - \frac{h(\nabla_V W, V)}{h(V,V)} V$$

and coincides with the standard Levi-Civita derivative in case V has geodesic integral curves. Therefore, let $\gamma(s,t)$ be a one parameter family of geodesic curves in the sense that

$$s \to \gamma(s,t)$$

defines a geodesic in affine parametrization for all t. We wish to write down an evolution equation for the orthogonal part $Z_{+} = Z - \frac{h(Z,V)}{h(V,V)}V$ of the geodesic deviation vector $Z = \partial_t \gamma(s,t)$ and note that $\mathcal{L}_V Z = [V, Z] = 0$ by construction. Taking into account that $D^{F}_{V} Z_{+} = \nabla_V Z_{+}$, we arrive at

$$\frac{D^F}{ds} Z_{+} = \nabla_V Z_{+}$$

and

$$\frac{D^F}{ds}\frac{D^F}{ds} Z_{+} = \nabla_V \nabla_V Z_{+} = \nabla_V \nabla_Z V = -R(Z,V)V = -R(Z_{+},V)V$$

due to the geodesic equation. Let us study more in particular the geodesic congruence

$$\exp^{s}_{tZ+V}(p)$$

where $s \geq 0$, $t \in (-\delta, \delta)$ and $h(Z, V) = 0$ at $t = 0$. Consider an $n-1$ bein $E_i(s)$ perpendicular to $V(s) \equiv \frac{d}{ds}\exp_V^s(p)$ satisfying $\nabla_{V(s)}E_i(s) = 0$; then with $Z(s) = Z^i(s)E_i(s)$ the above equation reduces to

$$\frac{d^2}{ds^2}Z^i(s) = -R^i{}_{njn}(\exp_V^s(p))Z^j(s)h(V,V)$$

with $Z(0) = 0$ and $\frac{D}{ds}Z(0) = Z$. Last, but not least, we need a method to find excellent approximations to solutions of the geodesic equation and the geodesic deviation in particular; first, choose a coordinate system such that $|h_{\alpha\beta} - \delta_{\alpha\beta}| < \epsilon$ and the Christoffel symbols $\Gamma^{\alpha}_{\beta\delta}$ are all smaller in absolute value than $\epsilon > 0$ as well and we proceed with a Newton-Rhapson iteration scheme. That is, linearize the geodesic equation for a bundle of geodesics given by integral curves of some vectorfield V

$$V^{\mu}\partial_{\mu}V^{\nu} + \Gamma^{\nu}_{\alpha\beta}V^{\alpha}V^{\beta} = 0$$

by means of the equation

$$V_i^{\mu}\partial_{\mu}V_{i+1}^{\nu} + \Gamma^{\nu}_{\alpha\beta}V_i^{\alpha}V_{i+1}^{\beta} = 0$$

where $i \geq 1$ and at each iteration step fixed initial data are held for some maximal $n-1$ surface Σ. For example, in such coordinate system, one can pick Σ as $x^n = 0$ and consider

$$V_1(x^1 + W^1(x^1, \ldots, x^{n-1})x^n, \ldots, x^{n-1} + W^{n-1}(x^1, \ldots, x^{n-1})x^n, x^nW^n(x^1, \ldots, x^{n-1})) = W(x^1, \ldots, x^{n-1})$$

where W, transversal to Σ, is a vectorfield on Σ kept fixed at any iteration step. We should check that the above procedure converges to a fix point (an obvious fact in one dimension where $V_3 = V_2$). In that regard, it is easier to use the notation $A_{\alpha} = \partial_{\alpha} + \Gamma_{\alpha}$ and frame the convergence of the Newton-Rhapson procedure within the context of that family of operators; defining $V_{i+1} = V_i + \delta V_i$ where $\delta V_i = 0$ on Σ, one obtains that

$$V_1^{\alpha}\Gamma^{\mu}_{\alpha\beta}V_1^{\beta} + V_1^{\alpha}\partial_{\alpha}\delta V_1^{\mu} + V_1^{\alpha}\Gamma^{\mu}_{\alpha\beta}\delta V_1^{\beta} = 0$$

and it is a matter of uniformly controlling the behaviour of δV_i in a neighborhood of Σ given that the equation $\nabla_{V_i}\delta V_i = 0$ is equivalent to $\delta V_i = 0$. It is natural to consider the Hilbert space $\mathcal{H}$ of vectorfields vanishing on Σ and study spectral properties of the operators $A_i = V_i^{\alpha}A_{\alpha}$; hence, we construct the Green kernels $G^{\alpha}_{i\,\beta}(x,y)$ satisfying

$$A^{\alpha}_{i\,\gamma}G^{\gamma}_{i\,\beta}(x,y) = \sqrt{h}(y)\delta^n(x,y)\delta^{\alpha}_{\beta}$$

and with boundary conditions $G_i(x,y) = 0$ whenever $x \in \Sigma$. It is most convenient to go to a V adapted coordinate system $(x^1, \ldots, x^{n-1}, x^n)$ where the first $n-1$ digits constitute the standard coordinates of the intersection point of the unique integral curve of V with Σ and x^n is the affine parameter vanishing when $x \in \Sigma$. Covariance of the above equation then implies

$$\frac{\partial}{\partial x^n}G^{\alpha}_{\beta}(x,y) + \Gamma^{\alpha}_{n\gamma}(x)G^{\gamma}_{\beta}(x,y) = \sqrt{h}(y)\delta^n(x,y)\delta^{\alpha}_{\beta}$$

and the latter is most easily solved by noticing that for $x^n > 0$

$$\frac{\partial}{\partial x^n}\left(\mathcal{T}G(x,y)\,\exp\left(\int_0^{x^n}\Gamma_n(x^1, \ldots, x^{n-1}, s)ds\right)\right) = \mathcal{T}\exp\left(\int_0^{x^n}\Gamma_n(x^1, \ldots, x^{n-1}, s)ds\right)\sqrt{h}(y)\delta^n(x,y)$$

where the operation $\mathcal{T}$ orders the expression in decreasing values of time. Consequently, $G(x,y)$ is given by

$$\sqrt{h}(y)\mathcal{T}\int_0^{x^n} dt\left(\exp\left(\int_0^t \Gamma_n(x^1,\ldots,x^{n-1},s)ds\right)\delta^n(x^1,\ldots,x^{n-1},t,y)\right)\exp\left(-\int_0^{x^n}\Gamma_n(x^1,\ldots,x^{n-1},s)ds\right).$$

The above formula has a nice geometric interpretation in the sense that $\int G(x,y)f(y)d^ny$ only depends on the values of f on the integral curve of V through x on the past of x and obviously, this statement is coordinate independent. It is clear that, although formal proofs of convergence can be set up, this method is not going to be of much help since calculating integral curves in closed form is usually not possible and therefore one is working with approximations of approximations.

Another, much more geometrical and direct method, consists in constructing geodesics as an ordered integral from some initial values; that is, choose δ small enough such that variations of Γ become small on scales of $W\delta$ and construct a piecewise linear curve starting on Σ at x_0 with initial direction $W(x_0)$ and length of the first linear piece $\delta\sqrt{h(W(x_0),W(x_0))}$ ending at x_1. At x_1, twist the vector $W(x_0)$ by an amount of $-\delta W(x_0)\Gamma(x_0)W(x_0)$ and repeat this procedure n times such that $n\delta$ remains constant in the limiting procedure. It is very easy to track the evolution of the geodesic deviation in this way by constructing two geodesics with nearly identical initial conditions in phase space and comparing the endpoints at identical parameter lengths.

Finally, exact computations are only possible in highly symmetric spaces such as Euclidean space, higher dimensional tori or some n-dimensional sphere; we shall treat here to some detail the first case. Since general exact computations of non-local operators are out of reach, we merely compute the derivative $\frac{d}{d\lambda}\Delta^{\lambda}_{W\,|\lambda=0}$; in general, one has to obtain the Fourier coefficients of

$$V^{\kappa}(s)+\lambda W\left(R^{b-a+\kappa}(\gamma_{\max})(s)+V^{\kappa}(s)\right)$$

with respect to a parallel transported vielbein $E_i(s)$. As an example, let

$$\gamma:\left[0,\frac{1}{2}\right]\to\mathbb{R}^2: s\to(0,s)$$

and $\gamma_{\max}$ be the unique maximal extension to $[0,1]$. With $V^0(s)$ as before and $W(x,y)=(1,y^2)$, we are left with Fourier decomposing

$$\left(\sum_{m=1}^{\infty}\left(a_{2,m}\sin(2\pi ms)+b_{2,m}\cos(2\pi ms)\right)+a_{2,0}+s\right)^2$$

in order to determine the corresponding vectorfield on $\mathbb{R}^{\infty}$. Obviously, the reader can immediately construct a few differential operators forming a relatively compact pair on any scale to the endpoint operator

$$x^i_f(\gamma)=a_{i,0}+\sum_{m=1}^{\infty}(-1)^m b_{i,m}$$

on the space of square integrable functions on path space[6] for the same reason that ∂_y and x are. More interesting examples are to be constructed later on.

References

[1] Johan Noldus, Foundations of a theory of quantum gravity, Vixra:1106.0029

[2] Johan Noldus, Topological quantum manifolds, preprint Vixra:1106.0028

[6] We will come back to the accurate description later on.