Divisor Function: Elementary Formulas and Asymptotic

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Abstract

We discovery some formulas for the divisor function, derived from a Vinogradov's formula and definitions these function, including the Ramanujan's sum. As well, we have developed a formula asymptotic, using the Euler-Maclaurin summation formula.

1. Introduction

Our main goal is the development of the following elementary formulas

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{r=0}^{j-1} e^{\frac{2\pi i n r}{j}},$$

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{r=0}^{j-1} \cos\left(\frac{2\pi n r}{j}\right),$$

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{r \bmod j} e^{\frac{2\pi i n r}{j}},$$

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{d|j} c_d(n),$$

and asymptotic formula

$$\sigma_k(n) \sim \frac{1}{2} + \frac{1}{2}\sum_{j=1}^{n} \cos\left(\frac{2\pi n(j-1)}{j}\right) j^{k-1} + \frac{1}{2\pi n}\sum_{j=1}^{n} \sin\left(\frac{2\pi n(j-1)}{j}\right) j^{k}$$

$$-\frac{\pi n}{6}\sum_{j=1}^{n} \sin\left(\frac{2\pi n(j-1)}{j}\right) j^{k-3} + \frac{n^2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{n}\frac{\sin(2\pi k(j-1))\cos\left(\frac{2\pi n(j-1)}{j}\right)}{(jk-n)(jk+n)} j^{k-1}$$

$$-\frac{n^3}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{j=1}^{n}\frac{\cos\big(2\pi k(j-1)\big)\sin\left(\frac{2\pi n(j-1)}{j}\right)}{(jk-n)(jk+n)} j^{k-2}.$$

2. DEFINITIONS, LEMMAS AND THEOREMS

DEFINITION 1 [1, page 310]:

$$d(n) = \sigma_0(n) := \sum_{d|n} 1.$$

DEFINITION 2 [1, page 310]:

$$\sigma(n) = \sigma_1(n) := \sum_{d|n} d.$$

DEFINITION 3 [1, page 310]:

$$\sigma_k(n) := \sum_{d|n} d^k.$$

LEMMA 1 [2, page 23]. *Let k be an integer, let n be an integer, and let*

$$E_j(n) = \sum_{r=0}^{j-1} e_j(nr) = \sum_{r=0}^{j-1} e^{\frac{2\pi inr}{j}}, \qquad \left(e_j(\alpha) = e^{\frac{2\pi i\alpha}{j}}\right).$$

Then

$$E_j(n) = \begin{cases} j \text{ if } j|n \\ 0 \text{ otherwise.} \end{cases}$$

THEOREM 1. *For $n \in \mathbb{Z}_{\geq 1}$,*

$$d(n) = \sum_{j=1}^{n} \frac{1}{j} \sum_{r=0}^{j-1} e^{\frac{2\pi inr}{j}}.$$

Proof. From Definition 1 and Lemma 1, it follows that

$$d(n) = \sigma_0(n) = \sum_{j=1}^{n} \frac{E_j(n)}{j} = \sum_{j=1}^{n} \frac{1}{j} \sum_{r=0}^{j-1} e^{\frac{2\pi inr}{j}}. \square$$

THEOREM 2. *For $n \in \mathbb{Z}_{\geq 1}$,*

$$\sigma(n) = \sum_{j=1}^{n} \sum_{r=0}^{j-1} e^{\frac{2\pi inr}{j}}.$$

Proof. From Definition 2 and Lemma 1, it follows that

$$\sigma(n)=\sigma_1(n)=\sum_{j=1}^{n}E_j(n)=\sum_{j=1}^{n}\sum_{r=0}^{j-1}e^{\frac{2\pi inr}{j}}. \square$$

THEOREM 3. *For $n\in\mathbb{Z}_{\geq 1}$ and $k\in\mathbb{Z}_{\geq 0}$, then*

$$\sigma_k(n)=\sum_{j=1}^{n}j^{k-1}\sum_{r=0}^{j-1}e^{\frac{2\pi inr}{j}}.$$

Proof. From Definition 3 and Lemma 1, it follows that

$$\sigma_k(n)=\sum_{j=1}^{n}j^{k-1}E_j(n)=\sum_{j=1}^{n}j^{k-1}\sum_{r=0}^{j-1}e^{\frac{2\pi inr}{j}}. \square$$

LEMMA 2. *Let k be an integer, let n be an integer, and let*

$$C_j(n)=\sum_{r=0}^{j-1}\cos\left(\frac{2\pi nr}{j}\right).$$

Then

$$C_j(n)=\begin{cases} j \text{ if } j|n \\ 0 \text{ else.}\end{cases}$$

Proof. In the Lemma 1, observe that the $\Im\left(S_j(n)\right)=\sum_{r=0}^{j-1}\sin\left(\frac{2\pi nr}{j}\right)=\left[\cos\left(\frac{\pi n}{j}\right)-\cos\left(\frac{(2j-1)\pi n}{j}\right)\right]\csc\left(\frac{\pi n}{j}\right)=0$, since $j=1,2,3,\ldots$ and $n=1,2,3,\ldots$; on the other hand, $\Re\left(S_j(n)\right)=\sum_{r=0}^{j-1}\cos\left(\frac{2\pi nr}{j}\right)$. So, it follows that

$$E_j(n)=\sum_{r=0}^{j-1}e_j(nr)=\sum_{r=0}^{j-1}e^{\frac{2\pi inr}{j}}=\sum_{r=0}^{j-1}\cos\left(\frac{2\pi nr}{j}\right)+i\sum_{r=0}^{j-1}\sin\left(\frac{2\pi nr}{j}\right)=\sum_{r=0}^{j-1}\cos\left(\frac{2\pi nr}{j}\right)+i\cdot 0$$

$$=\sum_{r=0}^{j-1}\cos\left(\frac{2\pi nr}{j}\right)=C_j(n)=\begin{cases} j \text{ if } j|n \\ 0 \text{ else,}\end{cases}$$

and we complete the proof. □

THEOREM 4. *For $n\in\mathbb{Z}_{\geq 1}$, then*

$$d(n)=\sum_{j=1}^{n}\frac{1}{j}\sum_{r=0}^{j-1}\cos\left(\frac{2\pi nr}{j}\right).$$

Proof. From Definition 1 and Lemma 2, it follows that

$$d(n)=\sigma_0(n)=\sum_{j=1}^{n}\frac{C_j(n)}{j}=\sum_{j=1}^{n}\frac{1}{j}\sum_{r=0}^{j-1}\cos\left(\frac{2\pi nr}{j}\right). \square$$

THEOREM 5. *For $n \in \mathbb{Z}_{\geq 1}$, then*

$$\sigma(n) = \sum_{j=1}^{n} \sum_{r=0}^{j-1} \cos\left(\frac{2\pi nr}{j}\right).$$

Proof. From Definition 2 and Lemma 2, it follows that

$$\sigma(n) = \sigma_1(n) = \sum_{j=1}^{n} C_j(n) = \sum_{j=1}^{n} \sum_{r=0}^{j-1} \cos\left(\frac{2\pi nr}{j}\right). \square$$

THEOREM 6. *For $n \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 0}$, then*

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{r=0}^{j-1} \cos\left(\frac{2\pi nr}{j}\right).$$

Proof. From Definition 3 and Lemma 2, it follows that

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} C_j(n) = \sum_{j=1}^{n} j^{k-1} \sum_{r=0}^{j-1} \cos\left(\frac{2\pi nr}{j}\right). \square$$

THEOREM 7. *For $n \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 0}$, then*

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{r \bmod j} e^{\frac{2\pi i n r}{j}}.$$

Proof. In [3, p.7], we encounter

(1)
$$\sum_{r \bmod j} e^{\frac{2\pi i n r}{j}} = \begin{cases} j \text{ if } j|n \\ 0 \text{ contrariwise.} \end{cases}$$

From (1) and definition 3, we find

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{\substack{r \bmod j \\ 0 \leq r \leq j-1}} e^{\frac{2\pi i n r}{j}} = \sum_{j=1}^{n} j^{k-1} \sum_{r \bmod j} e^{\frac{2\pi i n r}{j}}. \square$$

THEOREM 8. *For $n \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 0}$, then*

$$\sigma_k(n) = \sum_{j=1}^{n} j^{k-1} \sum_{d|j} c_d(n),$$

where $c_d(n)$ is the Ramanujan's sum.

Proof. In [4, p. 180], Ramanujan define

$$\eta_j(n) := \sum_{r=0}^{j-1} \cos\left(\frac{2\pi nr}{j}\right)$$

and he relates to the Ramanujan's sum

$$\eta_j(n) = \sum_{d|j} c_d(n), \tag{2}$$

which is defined by

$$c_q(n) := \sum_{\substack{a=1 \\ \gcd(a,q)=1}}^{q} e^{2\pi i \frac{a}{q} n}.$$

Substituting (2) in Theorem 6, we complete the proof. □

3. ASYMPTOTIC FORMULAE

LEMMA 9. *For $n, r \in \mathbb{N}$ and $f \in C^{(2r)}[0, n]$, we have*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{1}{2}[f(0) + f(n)] + \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right]$$

$$+(-1)^r \sum_{k=1}^{\infty} \int_0^n \frac{e^{2\pi ikt} + e^{-2\pi ikt}}{(2\pi k)^{2r}} f^{(2r)}(t)dt,$$

where B_{2k} are the Bernoulli numbers.

Proof. See [5, p. 1]. □

THEOREM 10. *For $n \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 0}$, then*

$$\sigma_k(n) \sim \frac{1}{2} + \frac{1}{2}\sum_{j=1}^{n} \cos\left(\frac{2\pi n(j-1)}{j}\right) j^{k-1} + \frac{1}{2\pi n}\sum_{j=1}^{n} \sin\left(\frac{2\pi n(j-1)}{j}\right) j^{k}$$

$$-\frac{\pi n}{6}\sum_{j=1}^{n} \sin\left(\frac{2\pi n(j-1)}{j}\right) j^{k-3} + \frac{n^2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{n} \frac{\sin(2\pi k(j-1))\cos\left(\frac{2\pi n(j-1)}{j}\right)}{(jk-n)(jk+n)} j^{k-1}$$

$$-\frac{n^3}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{j=1}^{n} \frac{\cos\big(2\pi k(j-1)\big)\sin\left(\frac{2\pi n(j-1)}{j}\right)}{(jk-n)(jk+n)} j^{k-2}.$$

Proof. We evaluate the sum $\sum_{r=0}^{j-1} \cos\left(\frac{2\pi nr}{j}\right)$, using the Lemma 9 and $r = 1$, clear that

$$\sum_{r=0}^{j-1} \cos\left(\frac{2\pi nr}{j}\right) \sim \int_0^{j-1} \cos\left(\frac{2\pi nx}{j}\right) dx + \frac{1}{2}\left[1 + \cos\left(\frac{2\pi n(j-1)}{j}\right)\right] \tag{3}$$

$$+\frac{B_2}{2}\left[\left(\frac{2\pi(j-1)n}{j^2}-\frac{2\pi n}{j}\right)\sin\left(\frac{2\pi(j-1)n}{j}\right)\right]$$

$$+\frac{4\pi^2n^2}{j^2}\sum_{k=1}^{\infty}\int_0^{j-1}\frac{e^{2\pi ikt}+e^{-2\pi ikt}}{(2\pi k)^2}\cos\left(\frac{2\pi nt}{j}\right)dt$$

$$=\frac{j}{2\pi n}\sin\left(\frac{2\pi n(j-1)}{j}\right)+\frac{1}{2}\left[1+\cos\left(\frac{2\pi n(j-1)}{j}\right)\right]-\frac{\pi n}{6j^2}\sin\left(\frac{2\pi(j-1)n}{j}\right)$$

$$+\frac{4\pi^2n^2}{j^2}\sum_{k=1}^{\infty}\frac{1}{(2\pi k)^2}\int_0^{j-1}\left(e^{2\pi ikt}+e^{-2\pi ikt}\right)\cos\left(\frac{2\pi nt}{j}\right)dt$$

$$=\frac{j}{2\pi n}\sin\left(\frac{2\pi n(j-1)}{j}\right)+\frac{1}{2}\left[1+\cos\left(\frac{2\pi n(j-1)}{j}\right)\right]-\frac{\pi n}{6j^2}\sin\left(\frac{2\pi(j-1)n}{j}\right)$$

$$+\frac{n^2}{j^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{j\left[jk\sin(2\pi k(j-1))\cos\left(\frac{2\pi n(j-1)}{j}\right)-n\cos(2\pi k(j-1))\sin\left(\frac{2\pi n(j-1)}{j}\right)\right]}{\pi(jk-n)(jk+n)}$$

$$=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi n(j-1)}{j}\right)+\frac{j}{2\pi n}\sin\left(\frac{2\pi n(j-1)}{j}\right)-\frac{\pi n}{6j^2}\sin\left(\frac{2\pi n(j-1)}{j}\right)$$

$$+\frac{n^2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi k(j-1))\cos\left(\frac{2\pi n(j-1)}{j}\right)}{k(jk-n)(jk+n)}-\frac{n^3}{j\pi}\sum_{k=1}^{\infty}\frac{\cos\big(2\pi k(j-1)\big)\sin\left(\frac{2\pi n(j-1)}{j}\right)}{k^2(jk-n)(jk+n)}.$$

Substituting (3) in Theorem 6, we find

$$\sigma_k(n)\sim\sum_{j=1}^{n}j^{k-1}\left[\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi n(j-1)}{j}\right)+\frac{j}{2\pi n}\sin\left(\frac{2\pi n(j-1)}{j}\right)-\frac{\pi n}{6j^2}\sin\left(\frac{2\pi n(j-1)}{j}\right)\right]$$

$$+\frac{n^2}{\pi}\sum_{j=1}^{n}j^{k-1}\left[\sum_{k=1}^{\infty}\frac{\sin(2\pi k(j-1))\cos\left(\frac{2\pi n(j-1)}{j}\right)}{k(jk-n)(jk+n)}\right]$$

$$-\sum_{j=1}^{n}j^{k-1}\left[\frac{n^3}{j\pi}\sum_{k=1}^{\infty}\frac{\cos\big(2\pi k(j-1)\big)\sin\left(\frac{2\pi n(j-1)}{j}\right)}{k^2(jk-n)(jk+n)}\right]$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{j=1}^{n}\cos\left(\frac{2\pi n(j-1)}{j}\right)j^{k-1}+\frac{1}{2\pi n}\sum_{j=1}^{n}\sin\left(\frac{2\pi n(j-1)}{j}\right)j^{k}$$

$$-\frac{\pi n}{6}\sum_{j=1}^{n}\sin\left(\frac{2\pi n(j-1)}{j}\right)j^{k-3}+\frac{n^2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{n}\frac{\sin(2\pi k(j-1))\cos\left(\frac{2\pi n(j-1)}{j}\right)}{(jk-n)(jk+n)}j^{k-1}$$

$$-\frac{n^3}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{j=1}^{n}\frac{\cos\big(2\pi k(j-1)\big)\sin\left(\frac{2\pi n(j-1)}{j}\right)}{(jk-n)(jk+n)}j^{k-2}.\ \square$$

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