A conjecture regarding the relation between Carmichael numbers and the sum of their digits

Marius Coman
Bucuresti, Romania
email: mariuscoman13@gmail.com

Abstract. Though they are a fascinating class of numbers, there are very many properties of Carmichael numbers still unstudied enough. I have always thought there is a connection between these numbers and the sum of their digits (few of them are also Harshad numbers). I try here to highlight such a possible connection.

Conjecture: For any Carmichael number C that has only prime factors of the form 6*k + 1 is true at least one of the following five relations:

(1) C is a Harshad number;
(2) If we note with s(m) the sum of the digits of the integer m then C is divisible by n*s(C) – n + 1, where n is integer;
(3) C is divisible by s((C + 1)/2);
(4) C is divisible by n*s((C + 1)/2) – n + 1, where n is integer;
(5) s(C) = s((C + 1)/2).

I verified below the conjecture for the first 23 Carmichael numbers of this type: 1729, 2821, 8911, 15841, 29341, 46657, 52633, 63973, 115921, 126217, 172081, 188461, 294409, 314821, 334153, 399001, 488881, 512461, 530881, 670033, 748657, 838201, 997633.

: 1729 is divisible by 19, where 19 = s(1729); so 1729 satisfies relation (1); also s((1729 + 1)/2) = s(865) = 19 so 1729 satisfies the relations (3) and (5) either;
: 2821 is divisible by 13, where 13 = s(2821); so 2821 satisfies relation (1); also s((2821 + 1)/2) = s(1411) = 7 and 2821 is divisible by 7 so 2821 satisfies the relation (3) either;
: 8911 is divisible by 19, where 19 = s(8911); so 8911 satisfies relation (1); also s((8911 + 1)/2) = s(4456) = 19 so 8911 satisfies the relations (3) and (5) either;
: s(15841) = s((15841 + 1)/2) = 19 and 15841 is divisible by 73 which is equal to 4*19 - 3; so 15841 satisfies relations (2), (4) and (5);

: s(29341) = s((29341 + 1)/2) = 19 and 29341 is divisible by 37 which is equal to 2*19 - 1; so 29341 satisfies relations (2), (4) and (5);
: s((46657 + 1)/2) = 19 and 46657 is divisible by 37, which is equal to 2*19 - 1; so 46657 satisfies relation (4);
: s(52633) = s((52633 + 1)/2) = 19 and 52633 is divisible by 73 which is equal to 4*19 - 3; so 52633 satisfies relations (2), (4) and (5);
: s(63973) = s((63973 + 1)/2) = s(31987) = 28; so 52633 satisfies relation (5);
: s(115921) = 19 and 115921 is divisible by 37 which is equal to 2*19 – 1;
: 126217 is divisible by 19, where 19 = s(126217); so 126217 satisfies relation (1); also s((126217 + 1)/2) = s(63109) = 19 so 126217 satisfies relations (3) and (5) either;
: s(172081) = s((172081 + 1)/2) = s(86041) = 19; so 172081 satisfies relation (5);
: s((188461 + 1)/2) = s(94231) = 19 and 188461 is divisible by 19; so 188461 satisfies relation (3); also s(188461) = 28 and 188461 is divisible by 109 which is equal to 4*28 – 3 so satisfies relation (2) either;
: s(294409) = 28 and 294409 is divisible by 109 which is equal to 4*28 - 3; so 294409 satisfies relation (2); s((294409 + 1)/2) = s(147205) = 19 and 294409 is divisible by 37, 73 and 109 which are equal to 19*2 – 1, 19*4 – 3 and 19*6 – 5 so 294409 satisfies relation (4) either;
: s(314821) = s((314821 + 1)/2) = s(157411) = 19; so 314821 satisfies relation(5);
: 334153 is divisible by 19, where 19 = s(334153); so 334153 satisfies relation (1);
: s(399001) = 22 and 399001 is divisible by 211 which is equal to 22*10 - 9; so 399001 satisfies relation(2);
: 488881 is divisible by 37, where 37 = s(488881); so 488881 satisfies relation (1);
: s(512461) = s((512461 + 1)/2) = s(256231) = 19; so 512461 satisfies relation(5);
: s(530881) = 22 and 530881 is divisible by 421 which is equal to 22*20 - 19; so 530881 satisfies relation(2);
: s(670033) = s((670033 + 1)/2) = s(335017) = 19; so 512461 satisfies relation(5);
: s(748657) = 37 and 748657 is divisible by 433 which is equal to 37*12 - 11; so 748657 satisfies relation(2);
: s((838201 + 1)/2) = s(419101) = 16 and 838201 is divisible by 61 and 151 which are equal to 16*4 – 3 and 16*10 - 5; so 748657 satisfies relation(4);
: s(997633) = s((997633 + 1)/2) = s(498817) = 37; so 997633 satisfies relation(5).

Note: We observed a subset a Carmichael numbers: the numbers 399001 = 31*61*211 and 530881 = 13*97*421 have both the sum of their digits s(C) = 22 and s((C + 1)/2) = 25; also, C is divisible by n*s(C) – n + 1, where n is their greatest prime factor.

Note: Many other Carmichael numbers have resembling properties, the ones that have only prime factors of the form 6*k – 1 for instance, but I didn’t find yet another category of Carmichel numbers that could be set in such a closed form.

Note: For many Carmichael number C that are also Harshad number is true that s(C) = s((C + 1)/2).

Note: For the odd Harshad numbers H that I checked, the first one that satisfy the relation s(H) = s((H + 1)/2) is the number 1387, the fifth Poulet number, which yet again connect this property with Fermat pseudoprimes.

Observation: I also noticed few relations based on the sum of the digits that are satisfied by a Poulet number P that has only two prime factors, both of the form 6*k + 1:

(1) s(P) = s((P + 1)/2);
(2) Both prime factors of P can be written as n*s((P + 1)/2) + 1, where n is integer;
(3) Both prime factors of P can be written as n*s((P + 1)/2) + n + 1, where n is integer;
(4) Both prime factors of P can be written as n*s((P + 1)/2) - n + 1, where n is integer;
(5) Both prime factors of P can be written as n*s(P) - n + 1, where n is integer.

I considered the first 15 Poulet numbers of this type: 1387, 2071, 4033, 4681, 5461, 7957, 10261, 14491, 18721, 23377, 31609, 31621, 42799, 49141, 49981 (for a list of Poulet numbers with two prime factors see the sequence A214305 in OEIS).

: s(1387) = s((1387 + 1)/2) = s(694) = 19, so 1387 satisfies relation (1);
: s(2071) = s((2071 + 1)/2) = s(1351) = 10, so 2071 satisfies relation (1);
: s(4033) = s((4033 + 1)/2) = s(2017) = 10, so 4033 satisfies relation (1);
: s(4681) = 19 and s((4681 + 1)/2) = s(2341) = 10 and 4681 is divisible with 31 which is equal to 3*10 + 1 also with 151 which is equal to 15*10 + 1, so 4681 satisfies relation (2);

: s(5461) = 16 and s((5461 + 1)/2) = s(2731) = 13 and 4681 is divisible with 43 which is equal to 3*13 + 4 also with 127 which is equal to 9*13 + 10, so 1387 satisfies relation (3);
: s(7957) = s((7957 + 1)/2) = s(3979) = 28, so 7957 satisfies relation (1);
: s(10261) = 10 and s((10261 + 1)/2) = s(5131) = 10 and 10261 is divisible with 31 which is equal to 3*10 + 1 also with 331 which is equal to 33*10 + 1, so 10261 satisfies relation (2);
: s(14491) = s((14491 + 1)/2) = s(7246) = 19, so 14491 satisfies relation (1);
: s(18721) = s((18721 + 1)/2) = s(9361) = 19, so 18721 satisfies relation (1);
: s(23377) = 22 and s((23377 + 1)/2) = s(11689) = 25 and 23377 is divisible with 97 which is equal to 4*25 - 3 also with 241 which is equal to 10*25 - 9, so 23377 satisfies relation (4);
: s(31609) = s((31609 + 1)/2) = s(15805) = 19, so 31609 satisfies relation (1);
: s(31621) = 13 and s((31621 + 1)/2) = s(15811) = 16 and 31621 is divisible with 103 which is equal to 6*16 + 7 also with 307 which is equal to 18*16 + 19, so 31621 satisfies relation (3);
: s(42799) = 31 and s((42799 + 1)/2) = s(21400) = 7 and 42799 is divisible with 127 which is equal to 18*7 + 1 also with 337 which is equal to 48*7 + 1, so 42799 satisfies relation (2);
: s(49141) = s((49141 + 1)/2) = s(24571) = 19, so 49141 satisfies relation (1);
: s(49981) = 31 and 49981 is divisible with 151 which is equal to 31*5 - 4 also with 331 which is equal to 31*11 – 10, so 49981 satisfies relation (1).

Conclusion: The relation between the Fermat pseudoprimes and the sum of their digits seems to be obvious even that there are probably better ways to express this relation (I actually only wanted to highlight few such possible ways). The property of a composite odd integer n to be divisible with s((n + 1)/2) deserves further study, also the property of a Harshad odd number n to have s(n) = s((n + 1)/2): we saw that the smallest such number with this property is a Fermat pseudoprime to base 2, the number 1387. It would also be interesting to see what numbers that are products of more than three prime factors of the form 6*k + 1 and are not Carmichael numbers satisfy the relations from the conjecture.