# Apsidal Motion Equations Questioned Through Application of Dimensional Analysis

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**Abstract:** By using the IS units and applying the dimensional analysis to the apsidal motion equations, we show more conclusively that their current formulation in astronomy and astrophysics is conceptually, mathematically and physically incorrect and highly questionable since it was erroneously used in several research articles. Consequently, we cannot use these apsidal motion equations, at least in their present form, as a decisive test of alternative gravity theories and theoretical models of internal stellar structure and evolution. Thus the main purpose of this paper is to draw theorists' attention to such a fallacy in order to establish new, correct, clear and concise equations.
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**Keywords:** Dimensional analysis, apsidal motion equations, eclipsing binary star systems

## 1. Introduction

Before entering into the heart of the subject of this paper, it is worthwhile to recall the meaning and role of the dimensional analysis (DA) in physical sciences.

### 1.1. Definition

In physics, DA is a robust tool to find qualitatively and quantitatively the relations among physical quantities by searching and checking the consistency and homogeneity of their physical dimensions and units. Further, DA is also the measure of physical quantities in consistent system of units, such as the SI system, in which the basic units are *meter*, *kilogram*, *second*, *ampere*, and *Kelvin* (m, kg, s, A, K)[1]. As logically it will turn out, the existence of consistent systems of measurement has nontrivial consequences.

### 1.2. Brief Overview

Historically, DA appeared early in the thoughts of physicists [1], and since then has been studied and used by some of the most famous of them (for example, Maxwell [2]; Einstein [3]; Rayleigh [4]).The first mathematical formulation of DA was published a century ago [5] but it is often associated with the first general exposition of the ideas published in *Nature* by Buckingham [6,7], see Pobeda and Georgievski [8] for more discussion of its origins. As such the fundamental theorem of DA is often referred to as the Buckingham π-theorem, which has since been further developed in many important textbooks [9-12]. This theorem describes how every physically meaningful equation involving n variables can be equivalently rewritten as an equation of n-m dimensionless parameters, where m is the numbers of fundamental dimensions used. Furthermore, and most importantly, it provides an economical method for com computing these dimensionless parameters from the given variables.

### 1.3. Basic Principle of DA

Originally, the basic principle of DA was known to Isaac Newton (1686), who referred to it as the "Great Principle of Similitude" [13]. Fundamentally, DA is based on the fact that: *The physical laws do not depend upon arbitrariness in the choice of the basic units of measurement.* A straightforward practical consequence is that any meaningful equation- and any inequality and inequation- must have the same dimensions in the left and right-hand sides.

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[1] Perhaps we should also include the mole (mol) and candela (cd).

Checking this is the basic way of performing DA. In science, DA is indeed routinely used to check the plausibility of derived equations and computations. It is also used to form reasonable hypotheses about complex physical situations that can be tested by experiment or by more developed theories of the phenomena and to categorize types of physical quantities and units based on their relations to or dependence on other units, or their dimensions if any. For example, the reader has probably encountered DA in his/her previous physics courses when he/she was admonished to' check his/her units' to ensure that the left and right hand sides of an equation had the same units- so that his/her calculation of a force had the units of $\mathrm{kgms}^{-2}$. In a sense, this is all there is to DA, although 'consistency checking of units' is certainly the most simple example of DA application, however, neglecting to do such operation may be led directly to the disastrous errors, particularly, in some research articles as we will see later.

## 2. Physical Dimensions

**Units of Measurement**

In this work we will adopt the SI system of units[2], which is described in some detail in the Physicist's Desk Reference [14] -which we will abbreviate as PDR from now on. As already mentioned, in the SI system the basic, or defined, units are {m, kg, s, A, K}. The definitions of these units in terms of fundamental physical processes are given in the PDR. All other units are derived. For example, the SI unit of energy, the joule (J), is equal to 1 kg $m^2$ $s^{-2}$. The derived units are also listed in PDR. The system is referred to as a L M T-class, since the defined units are length L, mass M, and time T (if we add thermal and electrical phenomena, then we have a L M T θ I-class in the SI system).

### 2.1. Dimension of a Physical Quantity

In order to understand more profoundly the dimension of a physical quantity, let us focus our attention on the following concrete illustration: consider the angular frequency, $\Omega$, of small oscillations of a simple pendulum of length $\ell$ and mass $m$ defined by

$$\Omega = \sqrt{g/\ell}\ , \tag{1}$$

where $g$ is the magnitude of local gravitational acceleration. To derive Eq.(1), we usually need to solve the differential equation that results from applying Newton's second law to the simple pendulum in question. Let us instead deduce Eq.(1) from dimensional considerations alone. What can $\Omega$ depend upon? It is reasonable to assume that the relevant variables are $m$, $\ell$ and $g$ -it seems hard to imagine others, at least for a simple pendulum. Now suppose that these symbols are replaced with M, L and T, referring to the dimensions of the physical quantities. With these replacements, the dimension of the physical quantity $\Omega$ is $\mathrm{T}^{-1}$. Similarly, the dimension of the physical quantity *velocity* is $\mathrm{LT}^{-1}$, and the dimension of the physical quantity *acceleration* is $\mathrm{LT}^{-2}$. Therefore, the dimension of the physical quantity $g/\ell$ should be $\mathrm{T}^{-2}$, and that of $\sqrt{g/\ell}$ should be $\mathrm{T}^{-1}$.

Consequently, the left hand side of Eq.(1) has the dimensional quantity $\mathrm{T}^{-1}$ and the right hand side of the same equation has also the dimensional quantity $\mathrm{T}^{-1}$ thus, Eq.(1) is qualitatively and quantitatively consistent, this double property is called: dimensional consistency and dimensional homogeneity. And the ratio

$$\sigma = \Omega \big/ \sqrt{g/\ell}\ , \tag{2}$$

[2] In some older texts this is referred to as the MKS system.

is invariant under a change of units; $\sigma$ is called a 'dimensionless number'. Since it does not depend upon the variables ( $m, \ell, g$ ), it is in fact a constant. In DA, any dimensionless quantity would conventionally have $\sigma = 1$. Following a convention suggested by Maxwell, we denote the dimensions of a physical quantity $\varphi$ by $[\varphi]$; thus $[\Omega] = \mathrm{T}^{-1}$ and $[\sigma] = 1$.

For example, what about more complicated physical quantities such as force? From Newton's second law expressed in scalar form, $F = m\,a$, so $[F] = [m][a] = \mathrm{MLT}^{-2}$. Proceeding in this way, we can easily construct the dimensions of any physical quantity.

## 3. First Application

In modern astronomy, the eclipsing binary star systems are a great stellar laboratory particularly for testing the alternative gravity theories *via* the study of apsidal motion which is generally explained as follows: it is well known that many eclipsing binary star systems have been found to display a gradual turning of the apsidal line of their elliptic orbit. This effect is supposed in most cases to result from distortions in one or both components, due to tidal interaction between the stars or to their own rotation. The derivation of the equation for classical apsidal motion (rate), $\dot{\omega}_{\mathrm{CL}}$, was first worked out by Cowling [15] and Sterne [16,17]. As applied to most of the known binary systems, the classical (tidal) term $\dot{\omega}_{\mathrm{CL}}$ is found to predominate in the apsidal motion; this term has been detected and investigated for about two dozen systems [18-21]. It is well recognized today that secular motion of the apsidal line will also be produced by general relativistic (GR) effects and was first studied by Livi-Cevita [22]; and Kopal [23].

Since the apsidal motion rate, $\dot{\omega}$, is by definition the derivation of the periastron longitude , $\omega$, with respect to time, that is to say

$$\dot{\omega} = \frac{d\omega}{dt}. \tag{3}$$

Thus according to Eq.(3), $\dot{\omega}$ has the physical dimensions of angular velocity or angular frequency, and by applying DA to Eq.(3), we get

$$[\dot{\omega}] = \mathrm{T}^{-1}. \tag{3.1}$$

If only the quadratic corrections are taken into account, the GR-contribution to the apsidal motion rate in the periastron longitude will be given, according to Fok [24], by the expression

$$\dot{\omega}_{\mathrm{GR}} = \frac{6\pi G(m_A + m_B)}{c^2 a(1-e^2)P}\ , \tag{4}$$

where $m_A$ and $m_B$ are, respectively, the masses of primary and secondary star expressed in terms of solar mass, $P$ is the orbital period, $e$ is the orbital eccentricity, $a$ is the semi major axis, $G$ and $c$ are, respectively, gravitational constant and light speed in vacuum. Furthermore, since the period, $P$, of any physical process has the dimensions of time, that is $[P] = \mathrm{T}$ and taking into account the fact that $[G] = \mathrm{L}^3\,\mathrm{M}^{-1}\,\mathrm{T}^{-2}$ and $[c] = \mathrm{L}\,\mathrm{T}^{-1}$; thus from all that and in view of Eq.(3.1), we find, after application DA to Eq.(4)

$$\mathrm{T}^{-1} = \mathrm{T}^{-1}. \tag{4.1}$$

As we can see, Eq.(4.1) satisfies the required dimensional consistency and homogeneity, hence, as consequence Eq.(4) is correct. Unfortunately, instead of Eq.(4), many authors [25-31] used the two following ones

$$\dot{\omega}_{\mathrm{GR}} = 5.45\times10^{-4}\,\frac{1}{1-e^2}\left(\frac{m_A+m_B}{P}\right)^{2/3}, \tag{5}$$

and/or

$$\dot{\omega}_{\mathrm{GR}} = 2.29\times10^{-3}\,\frac{(m_A+m_B)}{a(1-e^2)}\ . \tag{6}$$

The reasons why (5) and (6) are not satisfactory for astronomical purposes are that: these two equations are completely incorrect. To show their fallacy, let us of course apply DA to them, and we obtain

$$\mathrm{T}^{-1} = \mathrm{M}^{2/3}\,\mathrm{T}^{-2/3}, \tag{5.1}$$

and/or

$$\mathrm{T}^{-1} = \mathrm{ML}^{-1}. \tag{6.1}$$

That is to say contrary to Eq.(4.1), the two Eqs.(5.1) and (6.1) do not satisfy the required dimensional consistency and homogeneity; consequently Eqs.(5) and (6) are incorrect. The same mentioned authors [25-31] defined the relations between the sidereal period, $P_{\mathrm{s}}$, and the anomalistic period, $P_{\mathrm{a}}$, by the equation

$$P_{\mathrm{s}} = P_{\mathrm{a}}\left(1-\frac{\dot{\omega}}{360^\circ}\right), \tag{7}$$

and the relation between period of apsidal motion , $U$, and the anomalistic period by the equation

$$U = P_{\mathrm{a}}\times\frac{360^\circ}{\dot{\omega}}. \tag{8}$$

In view of the fact that the period of any physical process has the dimensions of time, therefore, by applying DA to Eqs.(7) and (8) and by taking into account (3.1), we find, respectively:

$$\mathrm{T} = \mathrm{T}\left(1-\mathrm{T}^{-1}\right), \tag{7.1}$$

$$\mathrm{T} = \mathrm{T}^2\ . \tag{8.1}$$

As the two Eqs.(7.1) and (8.1) do not satisfy the required dimensional consistency and homogeneity, that is why the two Eqs.(7) and (8) are incorrect. Bu the same authors [25-31] used correctly the following linear equation

$$\omega = \omega_0 + \dot{\omega}E, \tag{9}$$

that defines the periastron position $\omega$ at epoch $E$, where $\dot{\omega}$ is the rate of periastron advance, and $\omega_0$ is the position of periastron for the zero epoch $T_0$. It is clear, since $[\dot{\omega}] = \mathrm{T}^{-1}$ and $[E] = \mathrm{T}$ thus when we apply DA to Eq.(9), we obtain

$$1 = 1\ , \tag{9.1}$$

This means Eq.(9) is correct.

## 4. Second Application

The investigations of apsidal motion in eclipsing binary star systems is also very important as a stellar laboratory used observationally to verify the exactitude of modern theoretical models of internal stellar structure and evolution. In effect, these investigations are at present the main source of information on the inner structure of stars providing evidence for a high-density condensation toward the center of the star [32-37]. In this sense, the observed apsidal parameter $k_{2,\mathrm{obs}}$ plays a fundamental role to such models because it is intimately related to the variation in the density inside the star; and according to authors [25-31] this parameter may be derived with of the expression

$$k_{2,\mathrm{obs}} = \frac{1}{c_{21}+c_{22}}\frac{P_{\mathrm{a}}}{U} = \frac{1}{c_{21}+c_{22}}\frac{\dot{\omega}}{360^{\circ}}. \tag{10}$$

Where $c_{21}$ and $c_{22}$ are functions of the orbital eccentricity, fractional radii, the mass ratio of the components, and the ratio between rotational velocity of the stars and Keplerian velocity. Thus in accordance with this information, the apsidal parameter $k_{2,\mathrm{obs}}$ defined by Eq.(10) cannot be used as a test-parameter for the mentioned theoretical models because Eq.(10) is itself incorrect. Accordingly by applying DA to this equation, we find

$$1 = \mathrm{T}^{-1}. \tag{10.1}$$

As we can remark it, Eq.(10.1) does not satisfy the required dimensional consistency and homogeneity, therefore, as a direct consequence: Eq.(10) is incorrect.

## 5. Conclusion

In this paper, we have dimensionally analyzed the apsidal motion equations by applying DA, and we have shown that all the equations (5), (6), (7), (8) and (10) do not satisfy the required dimensional consistency and homogeneity. Therefore, all the mentioned equations are conceptually, mathematically and physically incorrect. Consequently, we cannot use these apsidal motion equations, at least in their present form, as a decisive test of alternative gravity theories and theoretical models of internal stellar structure and evolution.

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