An Elementary Proof of Oppermann's Conjecture

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ABSTRACT

We prove the Oppermann's conjecture: given an integer, $n > 1$, there is, at least, one prime between $n^2 - n$ and n^2, and, at least, another prime between n^2 and $n^2 + n$, using the prime-counting function and the Bertrand's Postulate.

1. INTRODUCTION

The Oppermann's conjecture, named after Ludvig Oppermann, in 1882, relates to distribution of the prime numbers. It states that, for any integer, $n > 1$, there is, at least, one prime between $n^2 - n$ and n^2, and, at least, another prime between n^2 and $n^2 + n$. We use the alternative statement:

Let $\pi(n)$ be the prime-counting function, that is, the number of prime numbers less than or equal to n. Then,

(1) $$\pi(n^2 - n) < \pi(n^2) < \pi(n^2 + n),$$

for $n > 1$. This means that between the square of a number n and the square of the same number plus (or minus) that number, there is a prime number. Or, equivalently,

(2) $$\pi(n^2) - \pi(n^2 - n) > 0$$

and

(3) $$\pi(n^2 + n) - \pi(n^2) > 0.$$

2. LEMMAS AND THEOREMS

LEMMA 1. (Bertrand's Postulate, actually a Theorem) *For any integer* $n > 3$, *there always exists, at least, one prime number,* p, *with* $n < p < 2n - 2$.

A weaker, but more elegant formulation is:

LEMMA 2. (Weak Bertrand's Postulate) *For every* $n > 1$ *there is always, at least, one prime number,* p, *such that* $n < p < 2n$.

THEOREM 1. *For* $n \geq 5$ *and* $n \in \mathbb{Z}_+$, *then*

(4) $$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1},$$

$$(5) \qquad \pi(n) = 2 + \sum_{k=5}^{n} \frac{1 - \cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] + \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]}{2 - 2\cos\left(\frac{2\pi}{z}\right)},$$

$$(6) \qquad \pi(n) = 2 - \sum_{k=5}^{n} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\},$$

$$(7) \qquad \pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi i\Gamma(z)}{z}}\right)\left(1 - e^{-\frac{2\pi i}{z}}\right)}{2 - 2\cos\left(\frac{2\pi}{z}\right)}.$$

Proof. Part 1. In [Dickson, pp. 427], H. Laurent noted that

$$(8) \qquad f(z) = \frac{e^{\frac{2\pi i\Gamma(z)}{z}} - 1}{e^{-\frac{2\pi i}{z}} - 1} = \begin{cases} 0, \textit{if } z \textit{ is composite} \\ 1, \textit{if } z \textit{ is prime} \end{cases},$$

for $z \geq 5$ and $z \in \mathbb{Z}_{+}$.

Observe that

$$f(z) = e^{\frac{2\pi i}{z}} \frac{\cos\left[\frac{2\pi\Gamma(z)}{z}\right] + i\sin\left[\frac{2\pi\Gamma(z)}{z}\right] - 1}{1 - e^{\frac{2\pi i}{z}}}$$

$$= e^{\frac{2\pi i}{z}} \frac{\cos\left[\frac{2\pi\Gamma(z)}{z}\right] + i\sin\left[\frac{2\pi\Gamma(z)}{z}\right] - 1}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$= -\left[\cos\left(\frac{2\pi}{z}\right) + i\sin\left(\frac{2\pi}{z}\right)\right] \frac{1 - \cos\left[\frac{2\pi\Gamma(z)}{z}\right] - i\sin\left[\frac{2\pi\Gamma(z)}{z}\right]}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right) - i\sin\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)} - \left\{\frac{i\sin\left(\frac{2\pi}{z}\right) - i\cos\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}\right\}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right) + \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right)}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)} - \frac{i\left\{\sin\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z}\right]\sin\left(\frac{2\pi}{z}\right) - \sin\left[\frac{2\pi\Gamma(z)}{z}\right]\cos\left(\frac{2\pi}{z}\right)\right\}}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}$$

$$= -\frac{\cos\left(\frac{2\pi}{z}\right) - \cos\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right] + i\left\{\sin\left(\frac{2\pi}{z}\right) - \sin\left[\frac{2\pi\Gamma(z)}{z} + \frac{2\pi}{z}\right]\right\}}{1 - \cos\left(\frac{2\pi}{z}\right) - i\sin\left(\frac{2\pi}{z}\right)}.$$

Using the identity

$$\frac{a+bi}{c-di}=\frac{ac-bd}{c^2+d^2}+i\frac{bc+ad}{c^2+d^2},$$

we find the following real part

(9) $$\Re[f(z)]=\frac{1-\cos\left(\frac{2\pi}{z}\right)-\cos\left[\frac{2\pi\Gamma(z)}{z}\right]+\cos\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

(10) $$=-\csc\left(\frac{\pi}{z}\right)\sin\left[\frac{\pi\Gamma(z)}{z}\right]\cos\left\{\frac{\pi[\Gamma(z)+1]}{z}\right\}$$

$$=\begin{cases}0, if\ z\ is\ composite\\ 1, if\ z\ is\ prime\end{cases},$$

for $z \geq 5$ and $z \in \mathbb{Z}_+$.

The imaginary part is the following

(11) $$\Im[f(z)]=-\frac{\sin\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]-\sin\left[\frac{2\pi\Gamma(z)}{z}\right]-\sin\left(\frac{2\pi}{z}\right)}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{\sin\left(\frac{2\pi}{z}\right)+\sin\left[\frac{2\pi\Gamma(z)}{z}\right]-\sin\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}.$$

It follows from (9) and (11) that

(12) $$f(z)=\frac{1-\cos\left(\frac{2\pi}{z}\right)-\cos\left[\frac{2\pi\Gamma(z)}{z}\right]+\cos\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}+i\frac{\sin\left(\frac{2\pi}{z}\right)+\sin\left[\frac{2\pi\Gamma(z)}{z}\right]-\sin\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{1-\cos\left(\frac{2\pi}{z}\right)+i\sin\left(\frac{2\pi}{z}\right)-\cos\left[\frac{2\pi\Gamma(z)}{z}\right]+i\sin\left[\frac{2\pi\Gamma(z)}{z}\right]+\cos\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]-i\sin\left[\frac{2\pi\Gamma(z)}{z}+\frac{2\pi}{z}\right]}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{1-e^{-\frac{2\pi i}{z}}-e^{-\frac{2\pi i\Gamma(z)}{z}}+e^{-\left[\frac{2\pi i\Gamma(z)}{z}+\frac{2\pi i}{z}\right]}}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{-1\left(e^{-\frac{2\pi i}{z}}-1\right)+e^{-\frac{2\pi i\Gamma(z)}{z}}\left(e^{-\frac{2\pi i}{z}}-1\right)}{2-2\cos\left(\frac{2\pi}{z}\right)}$$

$$=\frac{\left(1-e^{-\frac{2\pi i\Gamma(z)}{z}}\right)\left(1-e^{-\frac{2\pi i}{z}}\right)}{2-2\cos\left(\frac{2\pi}{z}\right)}=\begin{cases}0, if\ z\ is\ composite\\ 1, if\ z\ is\ prime\end{cases},$$

for $z \geq 5$ and $z \in \mathbb{Z}_+$.

Part 2. The prime-counting function is the function counting the number of prime numbers less than or equal to some real number x. It is denoted by $\pi(x)$. From above definition, we have

$$\pi(x) = \sum_{p \leq x} 1.$$

With the restriction for the positive integers and greater than or equal to five, it follows that

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$

by (8),

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left[\frac{2\pi\Gamma(k)}{k}\right] + \cos\left[\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right]}{2 - 2\cos\left(\frac{2\pi}{k}\right)}$$

by (9),

$$\pi(n) = 2 - \sum_{k=5}^{n} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k) + 1]}{k}\right\}$$

by (10),

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi i \Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}$$

by (12). □

COROLLARY 1. *For* $x \in \mathbb{R}_{\geq 5}$, *then*

(13)
$$\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1},$$

(14)
$$\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left[\frac{2\pi\Gamma(k)}{k}\right] + \cos\left[\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right]}{2 - 2\cos\left(\frac{2\pi}{k}\right)},$$

(15)
$$\pi(x) = 2 - \sum_{k=5}^{\lfloor x \rfloor} \csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k) + 1]}{k}\right\},$$

(16)
$$\pi(x) = 2 + \sum_{k=5}^{\lfloor x \rfloor} \frac{\left(1 - e^{-\frac{2\pi i \Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}.$$

Proof. Is obvious by the definition of floor function: $\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \leq x\}$ and previous Theorem. □

THEOREM 2. (Oppermann's Theorem) *For any integer,* $n > 1$*, there is, at least, one prime number between* $n^2 - n$ *and* n^2*, and, at least, another prime number between* n^2 *and* $n^2 + n$*.*

Proof. Step 1. By use from (4), we find

$$\pi(n^2+n) = 2 + \sum_{k=5}^{n^2+n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}, \tag{17}$$

$$\pi(n^2) = 2 + \sum_{k=5}^{n^2} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \tag{18}$$

and

$$\pi(n^2-n) = 2 + \sum_{k=5}^{n^2-n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}. \tag{19}$$

Step 2.Subtracting (18) from (17), we have

$$\pi(n^2+n) - \pi(n^2) = 2 + \sum_{k=5}^{n^2+n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \left[2 + \sum_{k=5}^{n^2} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}\right] \tag{20}$$

$$= 2 + \sum_{k=5}^{2n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} + \sum_{k=2n+1}^{n^2+n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \left[2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} + \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}\right]$$

$$= \pi(2n) + \sum_{k=2n+1}^{n^2+n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \pi(n) - \sum_{k=n+1}^{n^2} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=2n+1}^{n^2} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} + \sum_{k=n^2+1}^{n^2+n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \sum_{k=2n+1}^{n^2} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}$$

$$= \pi(2n) - \pi(n) + \sum_{k=n^2+1}^{n^2+n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} - \sum_{k=n+1}^{2n} \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1}.$$

By (8) we have the inequality

$$0 = \min_{k \in \mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right) \leq \frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \leq \max_{k \in \mathbb{Z}_{\geq 5}} \left(\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right) = 1. \tag{21}$$

From (20) and (21), it follows that

$$\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}\min_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)-\sum_{k=n+1}^{2n}\min_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)\leq \pi(n^2+n)-\pi(n^2)$$
$$\leq \pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}\max_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)-\sum_{k=n+1}^{2n}\max_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right),$$

$$\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}0-\sum_{k=n+1}^{2n}0\leq \pi(n^2+n)-\pi(n^2)\leq \pi(2n)-\pi(n)+\sum_{k=n^2+1}^{n^2+n}1-\sum_{k=n+1}^{2n}1,$$

$$\pi(2n)-\pi(n)\leq \pi(n^2+n)-\pi(n^2)\leq \pi(2n)-\pi(n)+n-n,$$

$$\pi(2n)-\pi(n)\leq \pi(n^2+n)-\pi(n^2)\leq \pi(2n)-\pi(n),$$

for $n\in\mathbb{Z}_{\geq 5}$.

Step 3. For $n=2$, $\pi(2^2+2)-\pi(2^2)=\pi(6)-\pi(4)=3-2=1>0$; for $=3$, $\pi(3^2+3)-\pi(3^2)=\pi(12)-\pi(9)=5-4=1>0$, for $n=4$, $\pi(4^2+4)-\pi(4^2)=\pi(20)-\pi(16)=8-6=2>0$.

Step 4. Subtracting (19) from (18), we have

$$(22)\quad \pi(n^2)-\pi(n^2-n)=2+\sum_{k=5}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\left[2+\sum_{k=5}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]$$

$$=2+\sum_{k=5}^{2n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}+\sum_{k=2n+1}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\left[2+\sum_{k=5}^{n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}+\sum_{k=n+1}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]$$

$$=\pi(2n)+\sum_{k=2n+1}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\pi(n)-\sum_{k=n+1}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$=\pi(2n)-\pi(n)+\sum_{k=2n+1}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\sum_{k=n+1}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$=\pi(2n)-\pi(n)+\sum_{k=2n+1}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}+\sum_{k=n^2-n+1}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\sum_{k=n+1}^{2n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\sum_{k=2n+1}^{n^2-n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}$$

$$=\pi(2n)-\pi(n)+\sum_{k=n^2-n+1}^{n^2}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}-\sum_{k=n+1}^{2n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}.$$

From (21) and (22), it follows that

$$\pi(2n)-\pi(n)+\sum_{k=n^2-n+1}^{n^2}\min_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)-\sum_{k=n+1}^{2n}\min_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)\leq \pi(n^2)-\pi(n^2-n)$$
$$\leq \pi(2n)-\pi(n)+\sum_{k=n^2-n+1}^{n^2}\max_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right)-\sum_{k=n+1}^{2n}\max_{k\in\mathbb{Z}_{\geq 5}}\left(\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right),$$

$$\pi(2n)-\pi(n)+\sum_{k=n^2-n+1}^{n^2}0-\sum_{k=n+1}^{2n}0 \leq \pi(n^2)-\pi(n^2-n) \leq \pi(2n)-\pi(n)+\sum_{k=n^2-n+1}^{n^2}1-\sum_{k=n+1}^{2n}1,$$

$$\pi(2n)-\pi(n) \leq \pi(n^2+n)-\pi(n^2) \leq \pi(2n)-\pi(n)+ n-n,$$

$$\pi(2n)-\pi(n) \leq \pi(n^2+n)-\pi(n^2) \leq \pi(2n)-\pi(n),$$

for $n \in \mathbb{Z}_{\geq 5}$.

Step 5. For $n=2$, $\pi(2^2)-\pi(2^2-2)=\pi(4)-\pi(2)=2-1=1>0$; for $=3$, $\pi(3^2)-\pi(3^2-3)=\pi(9)-\pi(6)=4-3=1>0$, for $n=4$, $\pi(4^2)-\pi(4^2-4)=\pi(16)-\pi(12)=6-5=1>0$. This is completes the proof. □

References

[Dickson] Dickson, Leonard Eugene, *History of the Theory of Numbers, Volume I: Divisibility and Primality*, Dover, 2005.