Few recurrent series based on the difference between succesive primes

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Abstract. Despite the development of computer systems, the chains of succesive primes obtained through an iterative formula yet have short lenghts; for instance, the largest known chain of primes in arithmetic progression is an AP-26. We present here few formulas that might lead to interesting chains of primes.

I.

I.1.
I was studying twin primes when I noticed that the greater prime (q) from the pair can be obtained from the lesser one (p) through the formula q = p*(d/2) + d/2 + 1, where d is the difference between the two primes, namely 2 (the formula is obviously equivalent to q = p*1 + 1 + 1 = p + 2). I applied this formula to the following recurrence relation and I observed that it produces interesting results (*i.e.* many succesive primes):

Let be A(n+1) = A(n)*(d/2) + d/2 + 1, where A(0) is any chosen prime and d is the difference between A(m) and the next consecutive prime, if the process of iteration stops to the first composite term A(m) obtained. If the iteration is supposed to continue, the right definition is that d is the difference between A(m) and the smallest prime which is greater than A(m).

I list below few such series and I stop the iteration at the first composite term obtained.

We have for A(0) = 3:
A(1) = 3*1 + 1 + 1 = 5 (because d = (5 - 3)/2 = 1);
A(2) = 5*1 + 1 + 1 = 7 (because d = (7 - 5)/2 = 1);
A(3) = 7*2 + 2 + 1 = 17 (because d = (11 - 7)/2 = 2);
A(4) = 17*1 + 1 + 1 = 19 (because d = (19 - 17)/2 = 1);
A(5) = 19*2 + 2 + 1 = 41 (because d = (23 - 19)/2 = 2);
A(6) = 41*1 + 1 + 1 = 43 (because d = (43 - 41)/2 = 1);
A(7) = 43*2 + 2 + 1 = 89 (because d = (47 - 43)/2 = 2);
A(8) = 97*4 + 4 + 1 = 19^2 (because d = (97 - 89)/2 = 4).

We choose another prime as a starting term, of course not one resulted from the above iterative process (5, 7, 17, 19, 41, 43, 89), that would conduct to same result because the sequence depends only by the last term obtained.

We have for A(0) = 11:
A(1) = 11*1 + 1 + 1 = 13;
A(2) = 13*2 + 2 + 1 = 29;
A(3) = 29*1 + 1 + 1 = 31;
A(4) = 31*3 + 3 + 1 = 97;
A(5) = 97*2 + 2 + 1 = 197;
A(6) = 197*1 + 1 + 1 = 199;
A(7) = 199*6 + 6 + 1 = 1201;
A(8) = 1201*6 + 6 + 1 = 7213;
A(9) = 7213*3 + 3 + 1 = 23*941.

We have for A(0) = 23:
A(1) = 23*3 + 3 + 1 = 73;
A(2) = 73*3 + 3 + 1 = 223;
A(3) = 223*2 + 2 + 1 = 449;
A(4) = 449*4 + 4 + 1 = 1801;
A(5) = 1801*5 + 5 + 1 = 9011;
A(6) = 9011*1 + 1 + 1 = 9013;
A(7) = 9013*8 + 8 + 1 = 37*1949.

Note: We didn’t further obtained notable results (long chains of succesive primes) for starting terms 37, 47, 53, 59, 61 so it’s clear that the formula doesn’t conduct to such results if we choose randomly the starting prime.

I.2.
The starting term of the sequence is allowed to be a prime in absolute value; also the terms of the sequence.

We have for A(0) = -13:
A(1) = (-13)*1 + 1 + 1 = -11;
A(2) = (-11)*2 + 2 + 1 = -19;
A(3) = (-19)*1 + 1 + 1 = -17;
A(4) = (-17)*2 + 2 + 1 = -31;
A(5) = (-31)*1 + 1 + 1 = -29;
A(6) = (-29)*3 + 3 + 1 = -83;
A(7) = (-83)*2 + 2 + 1 = -163;
A(8) = (-163)*2 + 2 + 1 = -5*97.

We have for A(0) = -23:
A(1) = (-23)*2 + 2 + 1 = -43;
A(2) = (-43)*1 + 1 + 1 = -41;
A(3) = (-41)*2 + 2 + 1 = -79;
A(4) = (-79)*3 + 3 + 1 = -233;
A(5) = (-233)*2 + 2 + 1 = -463;

A(6) = (-463)*1 + 1 + 1 = -461;
A(7) = (-461)*2 + 2 + 1 = -919;
A(8) = (-919)*4 + 4 + 1 = -3671;
A(9) = (-3671)*2 + 2 + 1 = -97*227.

I.3.
The starting term of the sequence is a prime of the form 2*k*11 + 1.

We have for A(0) = 67:
A(1) = 67*2 + 2 + 1 = 137;
A(2) = 137*1 + 1 + 1 = 139;
A(3) = 139*5 + 5 + 1 = 701;
A(4) = 701*4 + 4 + 1 = 53^2.

We have for A(0) = 89: A(1) = 89*4 + 4 + 1 = 19^2.

We have for A(0) = 199: A(1) = 199*6 + 6 + 1 = 1201.

We have for A(0) = 331: A(1) = 331*3 + 3 + 1 = 997.

We have for A(0) = 353: A(1) = 353*3 + 3 + 1 = 1063.

We have for A(0) = 397: A(1) = 397*2 + 2 + 1 = 797.

We have for A(0) = 419: A(1) = 419*2 + 1 + 1 = 421.

We have for A(0) = 463: A(1) = 463*2 + 2 + 1 = 929.

We have for A(0) = 617: A(1) = 617*1 + 1 + 1 = 619.

Note: This choice of starting term seems to conduct to interesting but not outstanding results.

I.4.
The starting term of the sequence is allowed to be a square of a prime. The starting value of d is now the difference between the starting term and the smaller prime bigger than this (*i.e.* d = 29 – 25 = 4 for starting term 25) and after that d is the difference between A(m) and the next consecutive prime.

We have for A(0) = 3^2 = 9:
A(1) = 9*1 + 1 + 1 = 11 which is prime.

We have for A(0) = 5^2 = 25:
A(1) = 25*2 + 2 + 1 = 53 which is prime.

We have for A(0) = 7^2 = 49:
A(1) = 49*2 + 2 + 1 = 101 which is prime.

We have for A(0) = 11^2 = 121:
A(1) = 121*3 + 3 + 1 = 367 which is prime;
A(2) = 367*3 + 3 + 1 = 1105 which is Fermat pseudoprime to base 2.

We have for A(0) = 13^2 = 169:
A(1) = 169*2 + 2 + 1 = 341 which is Fermat pseudoprime to base 2.

Note: We didn't observed further a notable pattern.

I.5.
The starting term of the series is a prime but the value of d is now the difference between A(m) and the second consecutive prime (*e.g.* d = 7 – 3 = 4; d = 11 - 5 = 6; d = 13 – 7 = 6 and so on).

We have for A(0) = 3:
A(1) = 3*2 + 2 + 1 = 9 = 3^2.

We have for A(0) = 5:
A(1) = 5*3 + 3 + 1 = 19;
A(2) = 19*5 + 5 + 1 = 121 = 11^2.

We have for A(0) = 7:
A(1) = 7*3 + 3 + 1 = 25 = 5^2.

We have for A(0) = 11:
A(1) = 11*3 + 3 + 1 = 37.

We have for A(0) = 13:
A(1) = 13*3 + 3 + 1 = 43.

Note: We didn't observed further a notable pattern.

I.6.
The starting term of the series is a prime but the value of d is now the difference between A(m) and the third consecutive prime (*e.g.* d = 11 – 3 = 8; d = 13 - 5 = 8; d = 17 – 7 = 10 and so on).

We have for A(0) = 3: A(1) = 3*4 + 4 + 1 = 17.

We have for A(0) = 5: A(1) = 5*4 + 4 + 1 = 5^2.

We have for A(0) = 7: A(1) = 7*5 + 5 + 1 = 41.

We have for A(0) = 11: A(1) = 11*4 + 4 + 1 = 7^2.

We have for A(0) = 13: A(1) = 13*5 + 5 + 1 = 71.

We have for A(0) = 17: A(1) = 17*6 + 6 + 1 = 109.

We have for A(0) = 19: A(1) = 19*6 + 6 + 1 = 11^2.

We have for A(0) = 23: A(1) = 23*7 + 7 + 1 = 13^2.

We have for A(0) = 29: A(1) = 29*6 + 6 + 1 = 181.

We have for A(0) = 31: A(1) = 31*6 + 6 + 1 = 193.

We have for A(0) = 37: A(1) = 37*5 + 5 + 1 = 191.

We have for A(0) = 41: A(1) = 41*6 + 6 + 1 = 11*23.

We have for A(0) = 43: A(1) = 43*8 + 8 + 1 = 353.

We have for A(0) = 47: A(1) = 43*7 + 7 + 1 = 337.

We have for A(0) = 53: A(1) = 53*7 + 7 + 1 = 379.

We have for A(0) = 59: A(1) = 59*6 + 6 + 1 = 19^2.

We have for A(0) = 61: A(1) = 61*6 + 6 + 1 = 373.

We have for A(0) = 67: A(1) = 67*6 + 6 + 1 = 409.

We have for A(0) = 71: A(1) = 71*6 + 6 + 1 = 433.

We have for A(0) = 73:
A(1) = 73*7 + 7 + 1 = 593;
A(2) = 593*7 + 7 + 1 = 4159;
A(3) = 4159*26 + 27 + 1 = 108161;
A(4) = 108161*15 + 15 + 1 = 1622431;
A(5) = 1622431*20 + 20 + 1 = 32448641;
A(6) = 32448641*19 + 19 + 1 = 21247*29017.

Note: This formula seems also to conduct to interesting results.

II.

We define now another resembling series, based again on the differences between consecutive primes:

Let be A(n+1) = (A(n)+ 1))*(d_1/2) + d_2/2) + 1, where A(0) is any chosen prime, d_1 is the difference between A(m) and the next consecutive prime and d_2 is the difference between A(m) and the second consecutive prime (the definition is made under the presumtion that process of iteration stops to the first composite term obtained).

I list below few such series:

We have for A(0) = 3:
A(1) = 4*(1 + 2) + 1 = 13;
A(2) = 14*(2 + 3) + 1 = 71;
A(3) = 72*(1 + 4) + 1 = 19^2.

We have for A(0) = 5:
A(1) = 6*(1 + 3) + 1 = 5^2.

We have for A(0) = 7:
A(1) = 8*(2 + 3) + 1 = 41;
A(2) = 42*(1 + 3) + 1 = 13^2.

We have for A(0) = 11:
A(1) = 12*(1 + 3) + 1 = 7^2.

We have for A(0) = 23:
A(1) = 24*(3 + 4) + 1 = 13^2.

Note: This formula could also lead to interesting results, but these results still seem to depend on the choice of the starting term.

Conclusion: I believe that especially the first formula is appealing because it is such easy to compute, though the longest chains of primes that I obtained so far using it are just 9 primes long: 11, 13, 29, 31, 97, 197, 199, 1201, 7213 (for starting term 11) and -23, -43, -41, -79, -233, -463, -461, -919, -3671 (for starting term -23). But, on the other side, neither Cunningham chains or AP chains were much more longer before being largely computed.