General Equation of Motion

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Abstract

In classical mechanics, this paper presents a general equation of motion, which can be applied in any reference frame (rotating or non-rotating) (inertial or non-inertial) without the necessity of introducing fictitious forces.

Introduction

The general equation of motion is a transformation equation between a reference frame S and a dynamic reference frame Š.

According to this paper, an observer S uses a reference frame S and a dynamic reference frame Š.

The dynamic position $\breve{\mathbf{r}}_a$, the dynamic velocity $\breve{\mathbf{v}}_a$, and the dynamic acceleration $\breve{\mathbf{a}}_a$ of a particle A of mass m_a relative to a dynamic reference frame Š, are given by:

$$\breve{\mathbf{r}}_a = \int \int (\mathbf{F}_a / m_a)\, dt\, dt$$

$$\breve{\mathbf{v}}_a = \int (\mathbf{F}_a / m_a)\, dt$$

$$\breve{\mathbf{a}}_a = (\mathbf{F}_a / m_a)$$

where $\mathbf{F}_a$ is the net force acting on particle A.

The dynamic angular velocity $\breve{\omega}_S$ and the dynamic angular acceleration $\breve{\alpha}_S$ of a reference frame S fixed to a particle S relative to a dynamic reference frame Š, are given by:

$$\breve{\omega}_S = \pm \left| (\mathbf{F}_1 / m_s - \mathbf{F}_0 / m_s) \cdot (\mathbf{r}_1 - \mathbf{r}_0) / (\mathbf{r}_1 - \mathbf{r}_0)^2 \right|^{1/2}$$

$$\breve{\alpha}_S = d(\breve{\omega}_S)/dt$$

where $\mathbf{F}_0$ and $\mathbf{F}_1$ are the net forces acting on the reference frame S in the points 0 and 1, $\mathbf{r}_0$ and $\mathbf{r}_1$ are the positions of the points 0 and 1 relative to the reference frame S, and m_s is the mass of particle S (the point 0 is the origin of the reference frame S and the center of mass of particle S) (the point 0 belongs to the axis of dynamic rotation, and the segment 01 is perpendicular to the axis of dynamic rotation) (the vector $\breve{\omega}_S$ is along the axis of dynamic rotation)

General Equation of Motion

The general equation of motion for two particles A and B relative to an observer S is:

$$m_a m_b(\mathbf{r}_a - \mathbf{r}_b) - m_a m_b(\breve{\mathbf{r}}_a - \breve{\mathbf{r}}_b) = 0$$

where m_a and m_b are the masses of particles A and B, $\mathbf{r}_a$ and $\mathbf{r}_b$ are the positions of particles A and B, $\breve{\mathbf{r}}_a$ and $\breve{\mathbf{r}}_b$ are the dynamic positions of particles A and B.

Differentiating the above equation with respect to time, we obtain:

$$m_a m_b\left[(\mathbf{v}_a - \mathbf{v}_b) + \breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_b)\right] - m_a m_b(\breve{\mathbf{v}}_a - \breve{\mathbf{v}}_b) = 0$$

Differentiating again with respect to time, we obtain:

$$m_a m_b\left[(\mathbf{a}_a - \mathbf{a}_b) + 2\breve{\omega}_S \times (\mathbf{v}_a - \mathbf{v}_b) + \breve{\omega}_S \times (\breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_b)) + \breve{\alpha}_S \times (\mathbf{r}_a - \mathbf{r}_b)\right] - m_a m_b(\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_b) = 0$$

Reference Frame

Applying the above equation to two particles A and S, we have:

$$m_a m_s\left[(\mathbf{a}_a - \mathbf{a}_s) + 2\breve{\omega}_S \times (\mathbf{v}_a - \mathbf{v}_s) + \breve{\omega}_S \times (\breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_s)) + \breve{\alpha}_S \times (\mathbf{r}_a - \mathbf{r}_s)\right] - m_a m_s(\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_s) = 0$$

If we divide by m_s and the reference frame S fixed to particle S $(\mathbf{r}_s = 0, \mathbf{v}_s = 0$, and $\mathbf{a}_s = 0)$ is rotating relative to the dynamic reference frame Š $(\breve{\omega}_S \neq 0)$, then we obtain:

$$m_a\left[\mathbf{a}_a + 2\breve{\omega}_S \times \mathbf{v}_a + \breve{\omega}_S \times (\breve{\omega}_S \times \mathbf{r}_a) + \breve{\alpha}_S \times \mathbf{r}_a\right] - m_a(\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_s) = 0$$

If the reference frame S is non-rotating relative to the dynamic reference frame Š $(\breve{\omega}_S = 0)$, then we obtain:

$$m_a \mathbf{a}_a - m_a(\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_s) = 0$$

If the reference frame S is inertial relative to the dynamic reference frame Š $(\breve{\omega}_S = 0$, and $\breve{\mathbf{a}}_s = 0)$, then we obtain:

$$m_a \mathbf{a}_a - m_a \breve{\mathbf{a}}_a = 0$$

that is:

$$m_a \mathbf{a}_a - \mathbf{F}_a = 0$$

where this equation is Newton’s second law.

Equation of Motion

From the general equation of motion it follows that the acceleration $\mathbf{a}_a$ of a particle A of mass m_a relative to a reference frame S fixed to a particle S of mass m_s, is given by:

$$\mathbf{a}_a = \frac{\mathbf{F}_a}{m_a} - 2\,\breve{\omega}_S \times \mathbf{v}_a - \frac{\mathbf{F}_S^a}{m_s}$$

where $\mathbf{F}_S^a$ is the net force acting on the reference frame S in the point A $(\mathbf{r}_a)$

This paper considers that Newton's first and second laws are false. Therefore, in this paper there is no need to introduce fictitious forces.

Universal Position

Applying the general equation of motion to a particle A of mass m_a and to the center of mass of the universe of mass m_{cm}, we have:

$$m_a m_{cm}(\mathbf{r}_a - \mathbf{r}_{cm}) - m_a m_{cm}(\breve{\mathbf{r}}_a - \breve{\mathbf{r}}_{cm}) = 0$$

Dividing by m_{cm} and considering that $\breve{\mathbf{r}}_{cm}$ is always zero, then we obtain:

$$m_a(\mathbf{r}_a - \mathbf{r}_{cm}) - m_a\breve{\mathbf{r}}_a = 0$$

that is:

$$m_a\mathbf{r}_a^{cm} - \int\int \mathbf{F}_a \; dt \; dt = 0$$

where $\mathbf{r}_a^{cm}$ is the position of particle A relative to the center of mass of the universe.

General Principle

From the general equation of motion it follows that the total position $\tilde{\mathbf{R}}_{ij}$ of a system of biparticles of mass M_{ij} $(M_{ij} = \sum_i \sum_{j>i} \; m_i m_j)$, is given by:

$$\tilde{\mathbf{R}}_{ij} = \sum_i \sum_{j>i} \frac{m_i m_j}{M_{ij}} \left[(\mathbf{r}_i - \mathbf{r}_j) - (\breve{\mathbf{r}}_i - \breve{\mathbf{r}}_j)\right] = 0$$

From the general equation of motion it follows that the total position $\tilde{\mathbf{R}}_i$ of a system of particles of mass M_i $(M_i = \sum_i \; m_i)$ relative to an observer S fixed to a particle S, is given by:

$$\tilde{\mathbf{R}}_i = \sum_i \frac{m_i}{M_i} \left[(\mathbf{r}_i - \mathbf{r}_s) - (\breve{\mathbf{r}}_i - \breve{\mathbf{r}}_s)\right] = 0$$

Therefore, the total position $\tilde{\mathbf{R}}_{ij}$ of a system of biparticles and the total position $\tilde{\mathbf{R}}_i$ of a system of particles are always in equilibrium.

Kinetic Force

The kinetic force $\mathbf{F}\mathrm{K}_{a|b}$ exerted on a particle A of mass m_a by another particle B of mass m_b relative to an observer S, is given by:

$$\mathbf{F}\mathrm{K}_{a|b} = \frac{m_a m_b}{m_{cm}} \left[(\mathbf{a}_a - \mathbf{a}_b) + 2\,\breve{\omega}_S \times (\mathbf{v}_a - \mathbf{v}_b) + \breve{\omega}_S \times (\breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_b)) + \breve{\alpha}_S \times (\mathbf{r}_a - \mathbf{r}_b)\right]$$

where m_{cm} is the mass of the center of mass of the universe.

From the previous equation it follows that the net kinetic force $\mathbf{F}\mathrm{K}_a$ acting on a particle A of mass m_a, is given by:

$$\mathbf{F}\mathrm{K}_a = m_a \left[(\mathbf{a}_a - \mathbf{a}_{cm}) + 2\,\breve{\omega}_S \times (\mathbf{v}_a - \mathbf{v}_{cm}) + \breve{\omega}_S \times (\breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_{cm})) + \breve{\alpha}_S \times (\mathbf{r}_a - \mathbf{r}_{cm})\right]$$

where $\mathbf{r}_{cm}$, $\mathbf{v}_{cm}$, and $\mathbf{a}_{cm}$ are the position, the velocity, and the acceleration of the center of mass of the universe.

The net kinetic force $\mathbf{F}\mathrm{K}_{ab}$ and the net dynamic force $\mathbf{F}\mathrm{D}_{ab}$, both acting on a biparticle AB of mass $m_a m_b$, are given by:

$$\mathbf{F}\mathrm{K}_{ab} = m_a m_b (\mathbf{F}\mathrm{K}_a / m_a - \mathbf{F}\mathrm{K}_b / m_b)$$

$$\mathbf{F}\mathrm{D}_{ab} = m_a m_b (\mathbf{F}\mathrm{D}_a / m_a - \mathbf{F}\mathrm{D}_b / m_b)$$

$$\longrightarrow$$

$$\mathbf{F}\mathrm{K}_{ab} = m_a m_b \left[(\mathbf{a}_a - \mathbf{a}_b) + 2\,\breve{\omega}_S \times (\mathbf{v}_a - \mathbf{v}_b) + \breve{\omega}_S \times (\breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_b)) + \breve{\alpha}_S \times (\mathbf{r}_a - \mathbf{r}_b)\right]$$

$$\mathbf{F}\mathrm{D}_{ab} = m_a m_b (\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_b)$$

$$\longrightarrow$$

$$\mathbf{F}\mathrm{K}_{ab} - \mathbf{F}\mathrm{D}_{ab} = 0$$

$$\longrightarrow$$

$$\mathbf{F}\mathrm{T}_{ab} = 0$$

Therefore:

The kinetic acceleration $\left[d^2(\mathbf{r}_a - \mathbf{r}_b)/dt^2\right]_{\check{S}}$ of a biparticle AB is related to the kinetic force.

The dynamic acceleration $\left[d^2(\breve{\mathbf{r}}_a - \breve{\mathbf{r}}_b)/dt^2\right]_{\check{S}}$ of a biparticle AB is related to the dynamic forces (gravitational force, electromagnetic force, etc.)

The total force $\mathbf{F}\mathrm{T}_{ab}$ acting on a biparticle AB is always in equilibrium.

Appendix

From the general principle the following equations are obtained:

12 equations for a biparticle AB relative to an observer S:

$$\frac{1}{x}\left[(\mathbf{r}_a-\mathbf{r}_b)^y\times\left[\frac{d^z(\mathbf{r}_a-\mathbf{r}_b)}{dt^z}\right]_{\check{\text{S}}}\right]^x-\frac{1}{x}\left[(\check{\mathbf{r}}_a-\check{\mathbf{r}}_b)^y\times\left[\frac{d^z(\check{\mathbf{r}}_a-\check{\mathbf{r}}_b)}{dt^z}\right]_{\check{\text{S}}}\right]^x=0$$

12 equations for a particle A relative to an observer S fixed to a particle S:

$$\frac{1}{x}\left[(\mathbf{r}_a-\mathbf{r}_s)^y\times\left[\frac{d^z(\mathbf{r}_a-\mathbf{r}_s)}{dt^z}\right]_{\check{\text{S}}}\right]^x-\frac{1}{x}\left[(\check{\mathbf{r}}_a-\check{\mathbf{r}}_s)^y\times\left[\frac{d^z(\check{\mathbf{r}}_a-\check{\mathbf{r}}_s)}{dt^z}\right]_{\check{\text{S}}}\right]^x=0$$

Where:

x takes the value 1 or 2 (1 vector equation, and 2 scalar equation)

y takes the value 0 or 1 (0 linear equation, and 1 angular equation)

z takes the value 0 or 1 or 2 (0 position equation, 1 velocity equation, and 2 acceleration equation)

Observations:

$\mathbf{r}_s = 0$, $\mathbf{v}_s = 0$, and $\mathbf{a}_s = 0$ relative to the reference frame S.

If y takes the value 0 then the symbol $\times$ should be removed from the equation.

$\left[d^z(...)/dt^z\right]_{\check{\text{S}}}$ means z-th time derivative relative to the dynamic reference frame Š.

On the other hand, these 24 equations would be valid even if Newton's third law were false.

Bibliography

A. Einstein, Relativity: The Special and General Theory.

E. Mach, The Science of Mechanics.

R. Resnick and D. Halliday, Physics.

J. Kane and M. Sternheim, Physics.

H. Goldstein, Classical Mechanics.

L. Landau and E. Lifshitz, Mechanics.