Noname manuscript No.
(will be inserted by the editor)

Newtonian gravitation with radially varying velocity-dependent mass

Henrik Agerhäll

Received: date / Accepted: date

Abstract A new extension to Newtonian celestial mechanics is examined. We focus on the scenario of a point-like body with negligible mass orbiting a spherically symmetric massive body. We take the implicitly time-dependent mass of electrodynamics one step further. We let the mass of the orbiting body vary not only with the velocity, but also with the position within the gravitational field. We find a family of expressions for the gravitational acceleration that explains the anomalous precession of perihelion of the planets and in the strong field limit results in orbits in close agreement with the predictions of the Schwarzschild solution. Regarding the orbital velocity of a body in circular orbit and the acceleration of a body at rest, the new theory gives the same results as classically. This is not the case with the post-Newtonian expansion even if terms at the third post-Newtonian, 3PN, level are included. Arguably, the major benefit of the new theory is that it presents a method that is much less intricate and more practical to deal with than general relativity, while reproducing most of its results, at least in the spherically symmetric case. While the differences between the final expression and the corresponding expression from the post-Newtonian expansion are small and subtle, the new theory gives results that in several ways are closer to both the classical results and to what the Schwarzschild solution predicts.

Keywords Newton's laws · Schwarzschild solution · Perihelion precession

PACS 04.50.Kd · 04.25.Nx · 04.50.-h · 04.80.Cc · 95.10.Ce

Henrik Agerhäll
Strombergs väg 110, 907 28 Umeå, Sweden
Tel.: +46-7670-07670
E-mail: henrik.agerhell@gmail.com

1 Introduction

Although general relativity makes correct predictions, the sheer complexity of the theory has led people all throughout the last century to try to find alternative theories of gravity. We will show that most of the predictions of the Schwarzschild solution of general relativity can be reproduced by making a minor extension to the Newtonian theory. We will get an expression for gravitational acceleration that can be directly compared to the corresponding expression from the post-Newtonian expansion. We will see that the new theory has some benefits.

On scales and in field strengths typical of the solar system the predictions of general relativity is found to correspond very well with measurements (Will 2006). In the case of really strong fields, observed inspiraling rates of binary pulsars agree with the predictions of general relativity (Psaltis 2008) and soon detections of gravitational waves are expected to provide further evidence. On very short length scales loop quantum gravity (Rovelli 2011) and string theory (Mukhi 2011) try to combine general relativity with quantum mechanics. Other efforts to create alternative theories of gravity are motivated by their possible ability to better explain the observed expansion of the universe now and at earlier epochs (Elizalde et al. 2011). Alternative theories of gravity that give different results at large length scales are suggested to explain galaxy rotation curves (Moffat 2006). The possibility that the unexpected acceleration of the Pioneer spacecrafts could be explained with some new theory of gravity has also been investigated (Turyshev and Toth 2010). Within the framework of general relativity there are also research on new solutions to, for instance, account for cosmic jets (Chicone et al. 2011).

We will limit the discussion to the case of one body of negligible mass moving in a spherically symmetric gravitational field. The more general situation of a body moving under the gravitational influence of N point-like bodies is briefly examined in Appendix B. The new theory is based upon introducing an implicitly time-dependent mass into the Newtonian expression for gravitation. By letting the mass vary with the velocity and with the radial distance to the central mass, both of which vary with time, a new expression for the gravitational acceleration is found. The new expression explains the anomalous perihelion shifts of the planets and produces orbits as expected from the Schwarzschild solution even in the strong field limit. The characteristic shapes of strong field orbits expected from the Schwarzschild solution are reproduced all the way down to the Schwarzschild radius. Infalling objects, as seen from an external observer, will come to a halt at the Schwarzschild radius. The fact that the strong field behaviour of the new expression is virtually impossible to distinguish from what is expected from the Schwarzschild solution is the strongest result of the investigation.

Regarding the orbital velocity of a body in circular orbit the new theory reproduces the expression from the Schwarzschild solution and classical mechanics. The post-Newtonian expansion on the other hand, does not reproduce this velocity, not even if higher order terms are included. For the case of a static test-particle the new theory reproduces the classical expression for the acceleration. A natural extension of this work would be to see whether the new theory is consistent with effects of general relativity such as gravitational radiation and the geodetic effect. The assumption that the mass of the orbiting body can be considered as varying with the Lorentz factor when calculating how it will be accelerated by a gravitational field is not a very new one, as the mass can be interpreted as varying with the Lorentz factor also in, for instance, electrodynamics. The second assumption, that the mass of a body under the influence of a spherically symmetric gravitational field can be assumed as varying also with the distance to the center of the gravitational field, is new. There is no unique way to choose this dependence under the sole requirement that it must explain the anomalous perihelion shift. We will discuss the options. In section 2 we will start with classical mechanics and the classical Newtonian formulation of gravity. We then make two assumptions, extending the Newtonian theory. The end result is a new expression for gravitational acceleration that can be used directly in numerical celestial mechanics simulations and that causes bodies to move like the Schwarzschild solution predicts.

2 Theory

In the theory presented here, the mass of an object in orbit in the gravitational field can be perceived as varying with its relative velocity and radial distance to the central mass. In this paper, unless specifically stated otherwise, m always represents the "invariant" mass of the orbiting body, the mass as it would be interpreted if the body was at rest relative to the central mass and located infinitely far away.

We assume the case of a point-like body of mass m orbiting a spherically symmetric non-rotating body of mass M, $m << M$. According to the Newtonian formulation of gravity the acceleration of the orbiting body can be found from the expression:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\bar{v}) = -\frac{GMm}{r^2}\hat{r}. \tag{1}$$

The velocity v and radial distance r of the body of mass m are, throughout this paper, determined to be in relation to the center of the gravitational field. Classically the assumption is made that the time derivative of the mass m is zero in which case the expression for the gravitational acceleration of the body of mass m can be written as:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}\hat{r}. \tag{2}$$

We assume that the time derivative of the mass m is not zero and we substitute m on both sides of (1) according to $m = m\gamma$ where m is assumed invariant and

γ is a yet unknown expression for how the mass of the orbiting body varies with variables that vary with time. Instead of (2) we get

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} + \frac{v}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\hat{v} = -\frac{GM}{r^2}\hat{r}. \tag{3}$$

The primary objective of this paper is to find and motivate an expression for γ that when put into (3) results in an expression for the gravitational acceleration that can explain the anomalous precession of perihelion of the planets. We also want our model to be consistent with other theories of physics and experimental findings.

2.1 The Lorentz factor

In special relativity and electrodynamics as well as in preferred frame relativity theories the mass can be seen as varying with the velocity according to the *Lorentz factor*. We assume that the mass of the orbiting body can be perceived as varying with the Lorentz factor:

$$\gamma = \gamma(v(t)) = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{4}$$

Inserting (4) into (3) we get

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - \frac{v^2}{c^2}\right)\hat{v} + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v}. \tag{5}$$

In (5) the acceleration is split up into the part tangential to the velocity vector and the part perpendicular to the velocity vector using the scalar product and the cross product. The difference between (2) and (5) is that the latter has one extra term. By using (5) in a numerical simulation of the orbit of Mercury it is seen that it produces an anomalous perihelion shift that is, within measurement uncertainties, equal to one third of the observed anomalous perihelion shift.

2.2 The Lorentz factor and general relativity

In general relativity the velocity of light is always constant and equal to c in a gravitational field if measured by a local clock that measures *proper time*. At the same time it is a well known fact that the speed of light will appear to slow down in a gravitational field if a time-of-flight measurement is done with external clocks. This well-established experimental fact is known as *Shapiro delay*. The resolution to this apparent paradox is, according to the interpretation that we do in this paper, the *gravitational time dilation*. Gravitational time dilation causes clocks to slow down in a gravitational field by exactly the same factor that light slows down with. Thus, in proper time, the speed of light is invariant even if it will be interpreted as varying by an external clock, ideally at rest with respect to the gravitational field and positioned infinitely far away, measuring *coordinate time*. The speed of light in a gravitational field in coordinate time, $c(r)$, is related to the invariant speed of light as:

$$c(r) = c\sqrt{1 - \frac{2GM}{rc^2}}. \tag{6}$$

To make our model consistent with the concept of gravitational time dilation and the principle that the measured speed of light in a gravitational field locally always will be measured to be constant and equal to c we have to change (4) slightly.We replace the invariant c apparent in (4) with the speed of light in a gravitational field in coordinate time as shown in (6). We then get:

$$\gamma = \left(1 - \frac{v^2}{c^2[1 - 2GM/(rc^2)]}\right)^{-1/2}. \tag{7}$$

We still let the mass of the orbiting body vary with the Lorentz factor, but we compensate for the fact that the speed of light, as perceived with a non-local clock, varies within a gravitational field. We take it as a fact for the rest of this paper that the mass of the orbiting body must vary with the velocity as shown in (7). Letting the mass vary with the velocity in any other way would cause our model to have serious problems regarding consistency with electrodynamics and findings attributed to general relativity. Note that putting (7) into (3) results in

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r} \cdot \hat{v})\left(1 - \frac{v^2}{c^2[1 - 2GM/(rc^2)]} - \frac{v^4}{c^4[1 - 2GM/(rc^2)]^2}\right)\hat{v} \\ + \frac{GM}{r^2}(\hat{r} \times \hat{v}) \times \hat{v}, \tag{8}$$

which reduces to (5) in the weak field limit and thus do not help in explaining the anomalous perihelion shift. We still have two thirds of the anomalous perihelion shift left to explain.

2.3 Mass varying with depth within a gravitational field

The problem at hand is now to derive some kind of secondary implicit time-dependence of the mass of the orbiting body that complements (7). The new γ shall, when put into (3), produce an expression for the gravitational acceleration of the orbiting body that in the weak field limit explains all of the anomalous perihelion shift. This can be achieved by letting the mass of the orbiting body vary with depth in the driving gravitational field. A general class of solutions to this problem is discussed in section 2.4. The method suggested throughout the rest of this paper is to let the mass of the orbiting body vary as

$$m(r) = m\left(1 - \frac{2GM}{rc^2}\right)^{-1}. \tag{9}$$

Combining (9) with (7) we get a new expression for γ,

$$\gamma = \left(1 - \frac{v^2}{c^2[1 - 2GM/(rc^2)]}\right)^{-1/2} \left(1 - \frac{2GM}{rc^2}\right)^{-1}. \tag{10}$$

By inserting (10) into (3) a new expression for the gravitational acceleration of the orbiting body is found as

$$\begin{aligned}\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - 3\frac{v^2}{c^2[1 - 2GM/(rc^2)]} + \frac{v^4}{c^4[1 - 2GM/(rc^2)]^2}\right)\hat{v} \\ + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v}. \end{aligned}\tag{11}$$

Expression (11) produces the correct value for the anomalous precession of perihelion of Mercury and the other planets as we will see in section 3. The reminder of this paper is dedicated to investigating the concequencies of (11) and comparing it to other methods of computing the gravitational acceleration including relativistic effects. What does it mean that the mass of a test-particle can be perceived as varying with depth in a gravitational field? For our purposes it is sufficient that (11) is an expression for the gravitational acceleration that very much is able to reproduce the expected orbits from the Schwarzschild solution. For the case of multiple bodies things get a bit more difficult. That case will be discussed in Appendix B.

2.4 Alternative assumptions

Under the sole requirement that an expression for how the mass of the orbiting body can be considered as varying within a gravitational field must be able to explain the remaining two thirds of the anomalous perihelion shift that (7) could not, (9) is not unique. From simulations it has been found that the expression:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - 3\frac{v^2}{c^2}\right)\hat{v} + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v} \tag{12}$$

is able to explain all of the anomalous perihelion shift. To get the anomalous precession right, we want to find an expression γ_r so that when we put (7) times γ_r into (3), the resulting expression for the acceleration will in the weak field limit reduce to (12). More specific we want an expression, γ_r, such that

$$\begin{aligned}\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - \frac{v^2}{c^2[1 - 2GM/(rc^2)]} - \frac{v^4}{c^4[1 - 2GM/(rc^2)]^2}\right)\hat{v} \\ -(\hat{r}\cdot\hat{v})\frac{v^2}{\gamma_r}\frac{\mathrm{d}\gamma_r}{\mathrm{d}r}\left(1 - \frac{v^2}{c^2[1 - 2GM/(rc^2)]}\right)\hat{v} \\ + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v} \end{aligned}\tag{13}$$

in the weak field limit reduces to (12). See Appendix A for details on the kind of calculations needed to get to (13). A general class of solutions to this problem is given by the expression

$$\gamma_r = \left(1 - \frac{aGM}{rc^2}\right)^{-2/a}, \tag{14}$$

in which a is an arbitrary constant. A γ_r on this form leads to

$$\begin{aligned}\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - \frac{v^2}{c^2[1-2GM/(rc^2)]} - \frac{v^4}{c^4[1-2GM/(rc^2)]^2}\right)\hat{v}\\ +\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(\frac{2v^2}{c^2[1-aGM/(rc^2)]} - \frac{2v^4}{c^4[1-2GM/(rc^2)][1-aGM/(rc^2)]}\right)\hat{v}\\ +\frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v},\end{aligned} \tag{15}$$

which reduces to (12) in the weak field limit. In (9), $a = 2$. Choosing $\gamma_r = e^{2GM/(rc^2)}$ would also lead to an expression for the gravitational acceleration that reduces to (12) in the weak field limit. If, for some reason, fine-tuning of the strong field behaviour is wanted these alternative assumptions could be investigated further. We do not follow up on any different way of choosing a in (14) or using exponential functions since (11) does a good job at reproducing the orbits expected from the Schwarzschild solution, even in the strong field limit.

2.5 A brief note on the energy

According to the Schwarzschild solution of general relativity the energy of a test-particle can be written as

$$E = mc^2\left(1 - \frac{v^2}{c^2[1-2GM/(rc^2)]}\right)^{-1/2}\left(1 - \frac{2GM}{rc^2}\right)^{1/2}. \tag{16}$$

Expression (16) reduces to the classical expressions for potential and kinetic energy in the slow velocity weak field limit. To check whether our model is consistent with (16) we can multiply (10) with an expression equal to unity according to:

$$\gamma = \left(1 - \frac{v^2}{c^2[1-2GM/(rc^2)]}\right)^{-1/2}\left(1 - \frac{2GM}{rc^2}\right)^{-1}\frac{\sqrt{1-2GM/(rc^2)}}{\sqrt{1-2GM/(rc^2)}}. \tag{17}$$

Expression (17) contains the expression for energy found in (16), except for a constant factor. If (16) were to be the correct expression for energy of the orbiting body in our theory it would have to remain constant during the orbit.

This means we should be able to treat the part of (17) containing the expression for the energy as a constant k and get

$$\gamma = k\left(1 - \frac{2GM}{rc^2}\right)^{-3/2}. \tag{18}$$

Inserting (18) into (3) results in

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - 3\frac{v^2}{c^2[1-2GM/(rc^2)]}\right)\hat{v} + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v}. \tag{19}$$

As is seen (19) is not identical to (11) so (16) can not be the correct expression for the energy according to our model. As (19) is not derived we are not going to investigate how it would affect orbits. It is a bit of a flaw of our model at its current state that we do not have an analytical solution of the equation of motion and no analytically derived expression for the energy either.

3 Results

We are going to test analytically what our model says regarding the orbital velocity of a body in circular orbit and the acceleration of a static test-particle. By computer simulations we will investigate what the model predicts regarding weak field perihelion shifts and strong field orbits. Our results will be compared to the post-Newtonian expansion and the Schwarzschild solution. The standard method to incorporate relativity into celestial mechanics computations in weak fields such as those in our solar system is by applying the post-Newtonian expansion (Brumberg 2007; Brumberg 2010). We are going to focus on the version of the post-Newtonian expansion that is the most standard and used by for instance the Jet Propulsion Laboratory. This version is derived from the *isotropic, parametrized post-Newtonian (PPN) n-body point mass metric* (Moyer 2000; Seidelmann 2006) which is an approximate solution to the Einstein field equations. Most of the terms of this expression vanish in the case of only one point-like body of negligible mass moving in a spherically symmetric gravitational field. Carried out to the first post-Newtonian, 1PN, level, the only non-vanishing terms are, here written together with the classical Newtonian acceleration:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}\left(1 - \frac{4GM}{rc^2} + \frac{v^2}{c^2}\right)\hat{r} + \frac{4GM}{r^2}(\hat{r}\cdot\hat{v})\frac{v^2}{c^2}\hat{v}. \tag{20}$$

Expression (20) can also be derived from the *Schwarzschild isotropic one-body point mass metric* (Moyer 2000). We write the expression, (11), that we derived to account for the gravitational acceleration in this new model once again:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - 3\frac{v^2}{c^2[1-2GM/(rc^2)]} + \frac{v^4}{c^4[1-2GM/(rc^2)]^2}\right)\hat{v} + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v}. \tag{21}$$

3.1 Orbital velocity in circular motion

For thel case of a circular orbit all the non-classical terms of (21) vanish while only one of the non-classical terms of (20) vanishes. By setting the right part of (20) equal to the centrifugal term, that appears if we switch to polar coordinates, the orbital velocity of a body in circular orbit using (20) becomes

$$v = \sqrt{\frac{GM}{r}\frac{[1 - 4GM/(rc^2)]}{[1 - GM/(rc^2)]}}. \tag{22}$$

The expression for the gravitational acceleration from the post-Newtonian expansion carried out to the third, 3PN, post-Newtonian order is more complex than (20) (Blanchet and Iyer 2003). For the case of a circular orbit the 3PN expression gives a slightly more complex expression for the orbital velocity:

$$v = \sqrt{\frac{GM}{r}\left[1 - \frac{4GM}{rc^2} + 9\left(\frac{GM}{rc^2}\right)^2 - 16\left(\frac{GM}{rc^2}\right)^3\right]\left(1 - \frac{GM}{rc^2}\right)^{-1/2}}. \tag{23}$$

The classical expression for the orbital velocity of a body in circular orbit, $v = \sqrt{GM/r}$, is correct also according to the Schwarzschild solution. The only way an expression that is supposed to account for relativistic effects can give the classical expression for the orbital velocity of a body in circular motion is if all other terms than the classical Newtonian term vanish in circular motion. As an example, using the values $c = 3\cdot 10^8 m/s$ and $GM/r = 9 \cdot 10^8 m^2/s^2$ (22) gives the orbital velocity 29 999.999 55 m/s. Including the higher order terms of (23) still gives an orbital velocity of 29 999.999 550 00 m/s rounded off to 13 siginificant digits. Classically and according to our model the orbital velocity will be 30 000 m/s. As the Earth has roughly a circular orbit around the Sun with an orbital velocity of 30 000 m/s the difference in distance travelled in a year using (20) instead of (21) would be approximately $(30\,000 - 29\,999.999\,55)\cdot 365 \cdot 24 \cdot 60 \cdot 60 = 14\,191$ metres.

3.2 Acceleration of a static test-particle

For the case of no motion at all (20) reduces to

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}\left(1 - \frac{4GM}{rc^2}\right)\hat{r}. \tag{24}$$

Expression (21) on the other hand reduces to the classical result of $\bar{a} = -(GM/r^2)\hat{r}$. Note that (24) is only the predicted acceleration of a static test particle according to the post-Newtonian expansion as carried out to the first post-Newtonian, 1PN, level. From higher order post-Newtonian expansions we can find the acceleration of a static test-particle according to the post-Newtonian expansion carried out to the third, 3PN, level [2] as:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}\left[1 - \frac{4GM}{rc^2} + 9\left(\frac{GM}{rc^2}\right)^2 - 16\left(\frac{GM}{rc^2}\right)^3\right]\hat{r}. \tag{25}$$

3.3 Anomalous precession of perihelion

Table 1 shows simulated perihelion shifts of four planets using (21) and (20). Initial conditions for the simulations are taken from tabulated values of the

Table 1 Anomalous perihelion shift of four planets

Planet	Precession per revolution in radians using (20)	Precession per revolution in radians using (21)
Mercury	$5.023\,063 \cdot 10^{-7}$	$5.023\,166 \cdot 10^{-7}$
Venus	$2.572\,525 \cdot 10^{-7}$	$2.570\,865 \cdot 10^{-7}$
Earth	$1.863\,169 \cdot 10^{-7}$	$1.864\,092 \cdot 10^{-7}$
Mars	$1.233\,150 \cdot 10^{-7}$	$1.233\,634 \cdot 10^{-7}$

orbital velocites at aphelion and the aphelion distances of the planets. We see that under solar system conditions (20) and (21) result in very much the same perihelion precession. For the case of Mercury, a precession of $5.0232 \cdot 10^{-7}$ radians per revolution corresponds to 42.99 arcseconds per century if the number of revolutions per century for Mercury is taken to be 414.9378. We see that in the solar system our model gives almost identically the same result as the post-Newtonian expansion does.

3.4 Strong field simulations

We numerically integrate orbits using classical Runge-Kutta based on our expression for gravitational acceleration, (21), to investigate what is expected from the orbits in the strong field regime. The simulations are based on tabulated values for the planet Mercury at aphelion. The mass of the Sun times the gravitational constant is taken to be $GM = 1.3278 \cdot 10^{20} Nm^2/kg^2$. The initial velocity is taken to be $v = 38\,860 m/s$ and the initial radial distance from the Sun is $6.982 \cdot 10^{10} m$. In the simulations we then increase the initial orbital velocity with a factor that we call the *scale factor* and we decrease the initial radial distance by the square of the *scale factor*. Using the classical Newtonian expression for gravitational acceleration this scheme would only reproduce orbits of the same shape but at different distances from the Sun. When we use (21) to calculate the acceleration this is no longer true.

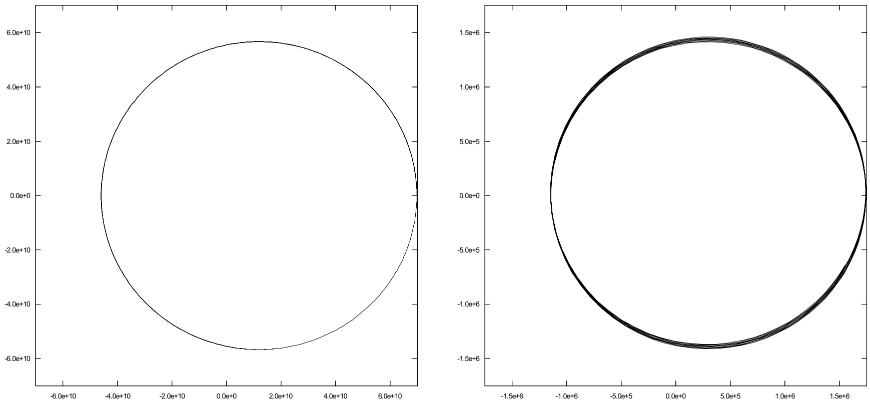

Fig. 1 Orbits using scale factors 1 (left) and 200 (right)

In Fig. 1 we see that at a scale factor of 200, corresponding to a planet orbiting 40 000 times closer to a central mass of the same mass as the Sun than the planet Mercury, the perihelion shift is easy to detect visually. In Fig. 2 we see that using a scale factor of 1357 the perihelion shift between successive perihelions is very close to $\pi/2$ radians. The scales on the axes of all of the simulations show distance in metres.

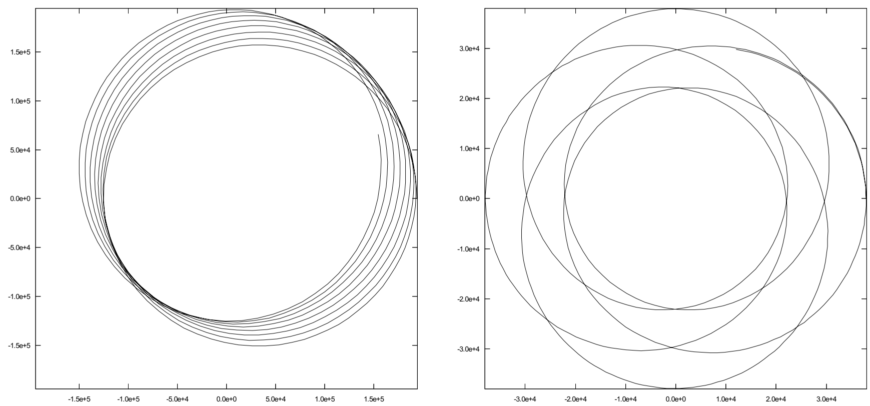

Fig. 2 Orbits using scale factors 600 (left) and 1357 (right)

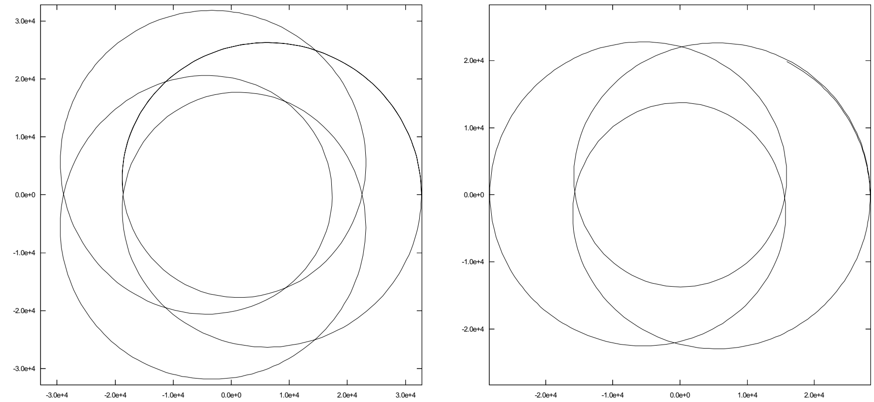

Fig. 3 Orbits using scale factors 1460 (left) and 1571 (right)

In Fig. 3 we see that using scale factors 1460 and 1571 we get anomalous perihelion shifts of $2\pi/3$ and π radians between successive perihelions. A scale factor of 1571 means that we start the simulation with the orbiting planet at a distance of $69\,820\,000\,000/1571^2 \approx 28\,290$ metres from the center of the gravitational field and with an initial velocity of $38\,860 \cdot 1571 \approx 6.1 \cdot 10^7$ metres per second in a direction perpendicular to the radial vector from the central mass. In Fig. 4 we see that at a scale factor of 1668 the orbiting planet circulates two times around the central mass before returning to its aphelion distance. We also see that at a scale factor of 1682.167 the orbiting planet circulates three times around the central mass between aphelions.

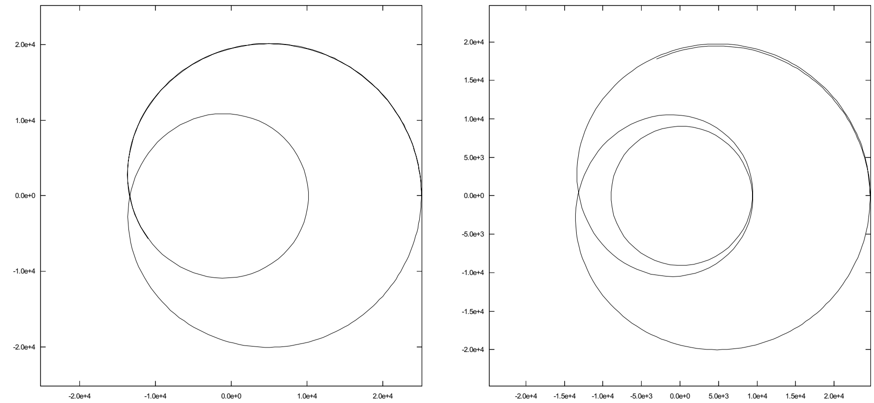

Fig. 4 Orbits using scale factors 1668 (left) and 1682.167 (right)

The strange shapes of the orbits in the strong field regime are expected from the Schwarzschild solution. In Fig. 5 we see that at a scale factor of 1682.452 the orbiting planet circulates four times around the central mass between aphelions. At a scale factor of 1682.457 54 the orbiting planet is very close to being captured by the central mass.

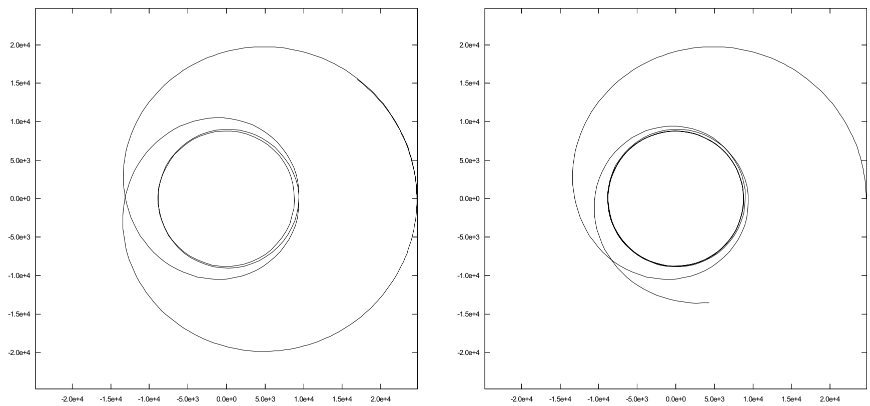

Fig. 5 Orbits using scale factors 1682.452 (left) and 1682.457 54 (right)

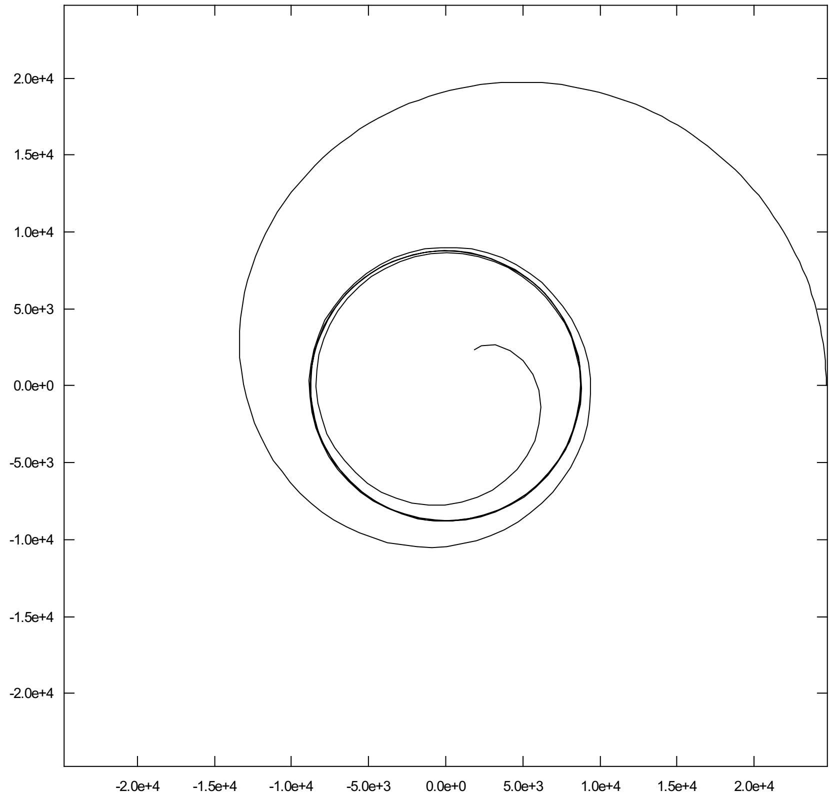

Fig. 6 Orbit using scale factor 1682.457 55

In Fig. 6 we see the orbiting body spinning around in a circle before falling down towards the Schwarzschild radius, which is at $r = 2GM/c^2 \approx 2954.75$ metres from the origin. Do note that the body does not cross the Schwarzschild radius and fall in to $r = 0$, since it actually halts, or very nearly halts, at the Schwarzschild radius. This is the expected behaviour from the Schwarzschild solution of general relativity, if Schwarzschild coordinates are used. The fact that our theory reproduces this behaviour at the Schwarzschild radius is the strongest result of the entire investigation.

4 Discussion

We set out to find a way of letting the mass of a test-particle subject to a Newtonian gravitational force from a spherically symmetric gravitational field vary implicitly with time. Our goal was to explain the anomalous perihelion shift. We assumed that the mass of the test-particle varies with velocity as in electrodynamics, but compensating for the Shapiro effect. We decided that we could not let the mass vary with velocity in any other way without violating other laws of physics. As this velocity-dependence could only explain one third of the perihelion shift we set out to find how to let the mass of the test-particle vary with position within a gravitational field. We found a family of solutions. We chose one of the options although the weak field behaviour of all solutions are virtually identical. If a different strong field behaviour is wanted the other options could be investigated.

The new model explains all of the anomalous perihelion shift in the weak field limit. Regarding the acceleration of a static test-particle our model reproduces the classical expression. Regarding the orbital velocity of a body in circular motion the new model reproduces the classical expression just as the Schwarzschild solution does. The post-Newtonian expansion, even at the third post-Newtonian level, predicts a slightly different orbital velocity. Our model produces orbits of shapes that are expected from the Schwarzschild solution also in the strong field limit. Even the fact that, according to the Schwarzschild solution, a test-particle will come to a halt at the Schwarzschild radius is reproduced.

We have found the orbits based on numerical integration and the new expression for gravitational acceleration. The fact that we have not found an analytical solution to the equations of motion from our definition of the gravitational acceleration can be seen as the weakest point of the investigation. A natural extension of this work would be to find such equations of motion analytically and to extend the comparison with the Schwarzschild solution. An idea on how to extend the theory to the case of a body under the gravitational influence of N point-like bodies is discussed earlier (Agerhäll 2007) and in Appendix B. However, extending the theory to the situation of N bodies in a more rigorous manner would also be a good extension. Another reinterpretation of the Schwarzschild solution where the the problem of adding the contribution from several bodies by simple superposition is claimed to have been found (Montanus 2005). Other possible future work include to see whether the new theory is or can be made compatible with phenomena attributed to general relativity such as gravitational radiation and the geodetic effect.

Summarizing, we have found a new expression for gravitational acceleration that very much produces the orbits expected from the Schwarzschild solution of general relativity right down to the Schwarzschild radius. We have not tested the post-Newtonian expansion far into the strong field regime but as the post-Newtonian expansion is a weak field expansion the new theory should work better in the strong field Schwarzschild regime. As far as the perihelion shift goes the expressions are equally good. The new theory reproduces the classical acceleration for the case of no relative velocity. The new theory also reproduces the classical expression, as does the Schwarzschild solution, for the orbital velocity of a body in circular orbit. This is not the case for the post-Newtonian expansion. While the two last results are nice, there is a coordinate freedom inherent in general relativity and what we have interpreted as a radial vector is under the general relativity formalism considered to be a gauge dependent object, so one must be careful not to draw too strong conclusions from them.

Appendix A: Derivation of the acceleration in detail

We show the derivation of (11) in detail. Expression (10) can be written:

$$\gamma = \gamma_{\mathrm{rv}}\gamma_{\mathrm{r}} = \frac{1}{\sqrt{1 - v^2/c^2 - 2GM/(rc^2)}} \frac{1}{\sqrt{1 - 2GM/(rc^2)}}. \tag{A.1}$$

We define $\gamma_{\mathrm{r}} = 1/\sqrt{1 - 2GM/(rc^2)}$ and $\gamma_{\mathrm{rv}} = 1/\sqrt{1 - v^2/c^2 - 2GM/(rc^2)}$. We insert (A.1) into (3), $\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} + \frac{v}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\hat{v} = -\frac{GM}{r^2}\hat{r}$, and get

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} + \frac{v}{\gamma_{\mathrm{rv}}}\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}\hat{v} + \frac{v}{\gamma_{\mathrm{rv}}}\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}\hat{v} + \frac{v}{\gamma_{\mathrm{r}}}\frac{\mathrm{d}\gamma_{\mathrm{r}}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}\hat{v} = -\frac{GM}{r^2}\hat{r}. \tag{A.2}$$

To get further with the derivation (A.2) is split into the part tangential to the velocity vector and the part perpendicular to the velocity vector. To get the tangential part we take the scalar product of (A.2) and the unit vector in the direction of the velocity:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t}\cdot\hat{v} + \frac{v}{\gamma_{\mathrm{rv}}}\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}\hat{v}\cdot\hat{v} + \frac{v}{\gamma_{\mathrm{rv}}}\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}\hat{v}\cdot\hat{v} + \frac{v}{\gamma_{\mathrm{r}}}\frac{\mathrm{d}\gamma_{\mathrm{r}}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}\hat{v}\cdot\hat{v} = -\frac{GM}{r^2}\hat{r}\cdot\hat{v}. \tag{A.3}$$

Identifying

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t}\cdot\hat{v} = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{A.4}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v(\hat{r}\cdot\hat{v}) \tag{A.5}$$

we can write

$$\left(1 + \frac{v}{\gamma_{\mathrm{rv}}}\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}v}\right)\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v^2}{\gamma_{\mathrm{rv}}}\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}r}\hat{r}\cdot\hat{v} + \frac{v^2}{\gamma_{\mathrm{r}}}\frac{\mathrm{d}\gamma_{\mathrm{r}}}{\mathrm{d}r}\hat{r}\cdot\hat{v} = -\frac{GM}{r^2}\hat{r}\cdot\hat{v}. \tag{A.6}$$

Now we make the identifications

$$\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}v} = \frac{v}{c^2}\gamma_{\mathrm{rv}}^3 \tag{A.7}$$

$$\frac{\mathrm{d}\gamma_{\mathrm{rv}}}{\mathrm{d}r} = -\frac{GM}{r^2c^2}\gamma_{\mathrm{rv}}^3 \tag{A.8}$$

$$\frac{\mathrm{d}\gamma_{\mathrm{r}}}{\mathrm{d}r} = -\frac{GM}{r^2c^2}\gamma_{\mathrm{r}}^3 \tag{A.9}$$

and write

$$\left(1 + \frac{v^2\gamma_{\mathrm{rv}}^2}{c^2}\right)\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v^2}{c^2}\frac{GM}{r^2}\gamma_{\mathrm{rv}}^2(\hat{r}\cdot\hat{v}) + \frac{v^2}{c^2}\frac{GM}{r^2}\gamma_{\mathrm{r}}^2(\hat{r}\cdot\hat{v}) = -\frac{GM}{r^2}\hat{r}\cdot\hat{v}. \tag{A.10}$$

Realizing that $1 + v^2\gamma_{\mathrm{rv}}^2/c^2 = \gamma_{\mathrm{rv}}^2/\gamma_{\mathrm{r}}^2$ and rewriting we get:

$$\frac{\gamma_{\mathrm{rv}}^2}{\gamma_{\mathrm{r}}^2}\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - \frac{v^2}{c^2}\gamma_{\mathrm{rv}}^2 - \frac{v^2}{c^2}\gamma_{\mathrm{r}}^2\right). \tag{A.11}$$

Realizing that $\gamma_{\rm r}^2/\gamma_{\rm rv}^2 = 1 - v^2/\{c^2[1-2GM/(rc^2)]\}$ and rewriting we get:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - 3\frac{v^2}{c^2[1-2GM/(rc^2)]} + \frac{v^4}{c^4[1-2GM/(rc^2)]^2}\right). \tag{A.12}$$

To get to the full expression for the gravitational acceleration in vector form we must take also the part of (A.2) that is perpendicular to the velocity vector. We note that the three non-classical terms of (A.2) vanish if we take the vector component of them in the direction perpendicular to the velocity. For that reason, in the direction perpendicular to the velocity we end up with the same acceleration as classically. We write the full expression as:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - 3\frac{v^2}{c^2[1-2GM/(rc^2)]} + \frac{v^4}{c^4[1-2GM/(rc^2)]^2}\right)\hat{v} + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v}. \tag{A.13}$$

We have to use the cross product twice to get the direction of the perpendicular acceleration right. Expression (A.13) is identical to (11).

Appendix B: Outline to a generalization to the case of N point-like bodies

For the case of a test-particle in a spherically symmetric field the questions of in reference to what the distance and the velocity measures apparent in our model for gravitational acceleration,

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = -\frac{GM}{r^2}(\hat{r}\cdot\hat{v})\left(1 - 3\frac{v^2}{c^2[1-2GM/(rc^2)]} + \frac{v^4}{c^4[1-2GM/(rc^2)]^2}\right)\hat{v} + \frac{GM}{r^2}(\hat{r}\times\hat{v})\times\hat{v}, \tag{B.1}$$

are supposed to be measured, are easy to answer. They are measured in reference to the center of the gravitational field. In the more general case of a test-particle influenced by the gravitational field of N spherically symmetric bodies that are moving with respect to each other, things get a bit more tricky. Under classical Newtonian gravity the superposition principle is valid. The total gravitational acceleration of a body due to N other point-like massive bodies is found by calculating the gravitational acceleration from each and every of the N bodies, like the others did not exist, and then add them together to get the total acceleration. This procedure does not work using (B.1). To see this we take the example of a body, A, gravitationally accelerated by a system of N point-like massive bodies. We assume that all of the N bodies have the mass M/N and are positioned at B. We also assume that they have no motion relative to each other. The acceleration of A should be the same, no matter the value of N, because the situations are physically equivalent. If we combine (B.1) with the superposition principle, this will not be the case. If we try to generalize (B.1) so it reads:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \sum_{j=1}^{N} -\frac{GM_j}{r_j^2}(\hat{r}_j\cdot\hat{v}_j)\left(1 - 3\frac{v_j^2}{c^2[1-\frac{2G}{c^2}\sum_{i=1}^{N} M_i/r_i]} + \frac{v_j^4}{c^4[1-\frac{2G}{c^2}\sum_{i=1}^{N} M_i/r_i]^2}\right)\hat{v}_j + \frac{GM_j}{r_j^2}(\hat{r}_j\times\hat{v}_j)\times\hat{v}_j, \tag{B.2}$$

the superposition principle will be valid in this example. By r_j and v_j we here mean the position and velocity of the body to be accelerated relative to body j which has the mass M_j. Using (B.2) the maximum achievable velocity (the local velocity of light) will depend on all nearby masses as the the two later summation symbols in (B.2) indicate. Seemingly, (B.2) should give reasonable results at least for the case of a static mass-configuration, where all the N bodies are at rest relative to each other. In the more general case where the N bodies move relative to each other we need further generalization. If we define

$$\bar{v}_e = \frac{\sum_{i=1}^{N} M_i \bar{v}_i / r_i}{\sum_{i=1}^{N} M_i / r_i} \tag{B.3}$$

we get an"effective" value, $\bar{v}_e$, for the velocity of the body whose acceleration we are concerned with, that is determined in relation to the N point-like bodies in the same way that we determined the local maximum velocity. For clarity we could also define an effective limiting velocity as

$$c_e^2 = c^2 \left(1 - \frac{2G}{c^2} \sum_{i=1}^{N} \frac{M_i}{r_i} \right) \tag{B.4}$$

and replace (B.2) with

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \sum_{j=1}^{N} -\frac{GM_j}{r_j^2} (\hat{r}_j \cdot \hat{v}_e) \left(1 - 3\frac{v_e^2}{c_e^2} + \frac{v_e^4}{c_e^4} \right) \hat{v}_e + \frac{GM_j}{r_j^2} (\hat{r}_j \times \hat{v}_e) \times \hat{v}_e. \tag{B.5}$$

This is just an outline on how our theory might be extended to the case of N point-like bodies. At least to account for a scenario where the N point-like bodies accelerate relative to each other and to account for gravitational radiation some further generalization will have to be made.

Acknowledgements The author thanks Carl Brannen for encouragement.

References

1. Agerhäll, H.: Another road to Schwarzschild. Books On Demand, Visby (2007)
2. Blanchet, L. and Iyer, B.: Third post-Newtonian dynamics of compact binaries: equations of motion in the centre-of-mass frame. Class. Quantum Grav. **20** (2003) 755
3. Brumberg, V. A.: On derivation of EIH (Einstein-Infeld-Hoffman) equations of motion from the linearized metric of general relativity theory. Celest. Mech. Dyn. Astr. **99** (2007) 245-252
4. Brumberg, V. A.: Relativistic celestial mechanics on the verge of its 100 year anniversary. Celest. Mech. Dyn. Astr. **106** (2010) 209-234
5. Chicone, C., Mashhoon, B. and Rosquist, K.: Double-Kasner spacetime: peculiar velocities and somic jets. Phys. Rev. D **83** (2011) 124029
6. Elizalde, E., Nojiri, S., Odintsov, S.D., Sebastiani, L. and Zerbini, S.: Nonsingular exponential gravity: a simple theory for early- and late-time accelerated expansion. Phys. Rev. D **83** (2011) 086006
7. Moffat, J.W.: Scalar-tensor-vector gravity theory. J. Cosmol. Astropart. Phys. JCAP03(2006)004
8. Montanus, J. M. C.: Flat space gravitation. Found. Phys. **35** (2005) 1543-62
9. Moyer, T. H.: Formulation for observed and computed values of deep space network data types for navigation. http://descanso.jpl.nasa.gov/Monograph/series2/Descanso2_all.pdf pp 2.6-2.10,4.23-4.24,4.42 (2000). Accessed 15 July 2012
10. Mukhi, S.: String theory: a perspective over the last 25 years. Class. Quantum Grav. **28** (2011) 153001

11. Psaltis, D.: Probes and tests of strong-field gravity with observations in the electromagnetic spectrum. Living Rev. Relativity 2008-9
12. Rovelli, C.: Loop quantum gravity: the first 25 years. Class. Quantum Grav. **28** (2011) 153002
13. Seidelmann, P. K.: Explanatory supplement to the astronomical almanac. pp. 281. University Science Books, Sausalito (2006)
14. Turyshev, S. and Toth, V.: The Pioneer anomaly, Living Rev. Relativity 2010-4
15. Will, C. M.: The confrontation between general relativity and experiment. Living Rev. Relativity 2006-3